arihant

Understanding Physics

JEE Main & Advanced

WAVES AND THERMODYNAMICS

DC PANDEY

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[B.Tech, M.Tech, Pantnagar, ID 15722]

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ARIHANT PRAKASHAN (Series), MEERUT

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PREFACE

The overwhelming response to the previous editions of this book gives me an immense feeling of satisfaction and I take this an opportunity to thank all the teachers and the whole student community who have found this book really beneficial.

In the present scenario of ever-changing syllabus and the test pattern of JEE Main & Advanced, the NEW EDITION of this book is an effort to cater all the difficulties being faced by the students during their preparation of JEE Main & Advanced. The exercises in this book have been divided into two sections viz., JEE Main & Advanced. Almost all types and levels of questions are included in this book. My aim is to present the students a fully comprehensive textbook which will help and guide them for all types of examinations. An attempt has been made to remove all the printing errors that had crept in the previous editions. I am extremely thankful to (Dr.)Mrs. Sarita Pandey, Mr. Anoop Dhyani and Nisar Ahmad for their endless efforts during the project.

Comments and criticism from readers will be highly appreciated and incorporated in the subsequent editions.

DC Pandey

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SYLLABUS

JEE Main

HEAT AND THERMODYNAMICS

Heat, temperature, thermal expansion; specific heat capacity, calorimetry; change of state, latent heat. Heat transfer-conduction, convection and radiation, Newton's law of cooling.

Thermal equilibrium, Zeroth law of thermodynamics, concept of temperature. Heat, work and internal energy. First law of thermodynamics. Second law of thermodynamics: reversible and irreversible processes. Carnot engine and its efficiency.

Equation of state of a perfect gas, work done on compressing a gas. Kinetic theory of gases – assumptions, concept of pressure. Kinetic energy and temperature: rms speed of gas molecules; Degrees of freedom, Law of equipartition of energy, applications to specific heat capacities of gases; Mean free path, Avogadro's number.

WAVES

Wave motion. Longitudinal and transverse waves, speed of a wave. Displacement relation for a progressive wave. Principle of superposition of waves, reflection of waves, Standing waves in strings and organ pipes, fundamental mode and harmonics, Beats, Doppler's effect in sound.

JEE Advanced

GENERAL

Specific heat of a liquid using calorimeter. Speed of sound using resonance column.

WAVES

Wave motion (plane waves only), longitudinal and transverse waves, superposition of waves, Progressive and stationary waves, Vibration of strings and air columns, Resonance, Beats, Speed of sound in gases, Doppler's effect (in sound).

HEAT and THERMODYNAMICS

Thermal expansion of solids, liquids and gases, calorimetry, latent heat, Heat conduction in one dimension, Elementary concepts of convection and radiation, Newton's law of cooling, Ideal gas laws, Specific heats (and for monoatomic and diatomic gases), Isothermal and adiabatic processes, bulk modulus of gases, Equivalence of heat and work, First law of thermodynamics and its applications (only for ideal gases), Black body radiation: absorptive and emissive powers, Kirchhoff's law, Wien's displacement law, Stefan's law.

This book is dedicated to my honourable grandfather

(Late) Sh. Pitamber Pandey

a Kumaoni poet and a resident of Village
Dhaura (Almora), Uttarakhand

17

Wave Motion

Chapter Contents

17.1 Introduction
17.2 Classification of a Wave
17.3 Equation of a Travelling Wave
17.4 Sine Wave
17.5 Two Graphs in Sine Wave
17.6 Wave Speed
17.7 Energy in Wave Motion

17.1 Introduction

In every wave, a physical quantity y is made to oscillate at one point and gradually these oscillations of y propagate to other points also without transport of matter. In general, a wave also transports energy and momentum (in addition to oscillations of y).

Thus, in a wave, the following three physical quantities transfer from one point to another point.

(i) oscillations of y

(ii) energy and

(iii) momentum

In different situations, the physical quantity y may be different. For example:

y is displacement of medium particles (between $+A$ and $-A$) from the mean position in case of string wave, where A is the amplitude of oscillations. Similarly, y is electric and magnetic fields for electromagnetic waves and it is displacement, pressure and density which oscillate in longitudinal (or sound) wave. A wave is said to be **travelling** or **progressive** if it travels from one point to another. A **plane wave** is a wave of constant frequency and amplitude with wavefronts that are infinitely long straight lines. Plane waves travel in the perpendicular direction to the wavefronts. Many physical waves are approximate plane waves far from their sources.

17.2 Classification of a Wave

A wave can be classified in the following three manners :

1-D, 2-D and 3-D Wave

In 1-D wave, oscillations of y (or energy and momentum also) transfer in a straight line or we can say that wave travels in a straight line. String wave is the best example of a 1-D wave. In 2-D wave, it travels in a plane. Wave travelling on the surface of water is an example of a 2-D wave. In 3-D wave, it travels in whole space. Sound or light wave produced by a point source is an example of a 3-D wave.

Transverse and Longitudinal Wave

In transverse waves, oscillations of y are perpendicular to wave velocity and wave velocity is the direction in which oscillations (or energy and momentum) transfer from one point to another.

Electromagnetic waves and string waves are transverse in nature.In longitudinal waves, oscillations are along the wave velocity. Sound wave is a longitudinal wave.

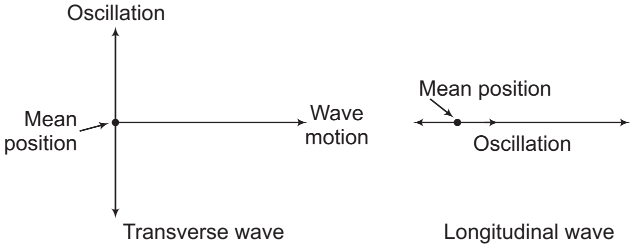


Fig. 17.1

Mechanical and Non-mechanical Wave

The waves which require medium for their propagation from one point to another point are called mechanical waves. Although this medium may be a gas also. Sound wave is a mechanical wave. Over the moon, sound waves cannot travel, as there is no gas on its surface.

Non-mechanical waves can travel with or without medium. Electromagnetic waves are non-mechanical waves.

✓ Extra Points to Remember

- Apart from mechanical and non-mechanical waves, there is also another kind of waves called **"matter waves"**.
- Sound wave is basically a part of longitudinal waves with frequency varying from 20 Hz to 20,000 Hz. Similarly, light wave is a part of electromagnetic waves with wavelength varying from approximately 4000 Å to 7000 Å.
- In the propagation of mechanical waves, elasticity and inertia of the medium play an important role.
- Waves traveling through a solid medium can be either transverse or longitudinal waves. But waves travelling through a fluid (such as a liquid or a gas) are always longitudinal waves. Transverse waves require a relatively rigid medium in order to transmit their energy. As one particle begins to move it must be able to exert a pull on its nearest neighbor. If the medium is not rigid as is the case with fluids, the particles will slide past each other. This sliding action that is characteristic of liquids and gases prevents one particle from displacing its neighbour in a direction perpendicular to wave direction. It is for this reason that only longitudinal waves are observed moving through the bulk of liquids such as our ocean.
- While waves that travel within the depths of the ocean are longitudinal waves, the waves that travel along the surface of the oceans are referred to as surface waves. A **surface wave** is a wave in which particles of the medium undergo a circular motion. Surface waves are neither longitudinal nor transverse. The radius of the circles decreases as the depth into the water increases.
- Earthquakes are capable of producing both types of waves transverse as well as longitudinal. The **P-waves (Primary waves)** in an earthquake are examples of longitudinal waves. The P-waves travel with the fastest velocity and are the first to arrive. The **S-waves (Secondary waves)** in an earthquake are examples of transverse waves. S-waves propagate with a velocity slower than P-waves, arriving several seconds later.

17.3 Equation of a Travelling Wave

As we have already read, in a wave motion a physical quantity y is made to oscillate at one point and these oscillations of y propagate to other places also.

Therefore, in a wave, several particles oscillate (unlike SHM in which normally a single oscillates).

So, to determine the value of y (from its mean position between $+A$ and $-A$) we will have to tell position of the particle and time. Thus,

$$\boxed{y = f\text{ (position of particle, time)}}$$

In three dimensional space, position of the particle can be represented by three variable co-ordinates. Thus, in general, y is a function of four variables, three in co-ordinates and the fourth one is time.

But in physics, we normally keep least number of variables. If the wave is one-dimensional, then position of the particle can be represented by a single variable co-ordinate (say x).

Thus, in a one-dimensional wave y is a function of two variables x and t. Here, x is used for position of the particle and t for time or

$$\boxed{y = f(x, t)}$$

Note *Since, there are two variables in y, therefore whenever required, y is always differentiated partially. For example,*

$$\frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, \frac{\partial^2 y}{\partial x^2} \text{ etc.}$$

Now, the question is, there may be infinite number of functions of x and t, like

$$y = x + t$$

$$y = xt + t^3 x^2 \text{ etc}$$

Obviously, all functions will not represent a wave function. Only those functions of x and t will represent a wave function which satisfies the following three conditions.

Condition 1 The given function must satisfy the following differential equation :

$$\boxed{\left(\frac{\partial^2 y}{\partial t^2}\right) = k\left(\frac{\partial^2 y}{\partial x^2}\right)} \qquad \text{...(i)}$$

Here, k is a constant which is equal to square of the wave velocity or

$$\boxed{k = v^2}$$

Condition 2 The wave function must be single valued. For given values of x and t there should be only one value of y.

Condition 3 The wave function and its first derivative must be continuous. Therefore, there should not be a sudden change in the value of y and its first derivative (in some cases it will be called slope).

Note *Last two conditions are slightly difficult to explain at this stage. So, students need not to go in detail of those.*

The general solution of Eq. (i) discussed above is of the form :

$$\boxed{y(x, t) = f(ax \pm bt)} \qquad \text{...(ii)}$$

Thus, any function of x and t which satisfies Eq. (i) and which can be written as Eq. (ii) represents a wave, provided conditions (2) and (3) are also satisfied.

Further, if these conditions are satisfied, then speed of wave (v) is given by

$$\boxed{v = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{b}{a}}$$

If ax and bt are of same sign (both positive or both negative), then wave will be travelling along negative x-direction. If one is positive and other is negative, then wave travels in negative x-direction.

Alternate Method of Understanding the Wave Equation

Let us consider a long string with one end fixed to a wall and the other held by a person. The person pulls on the string keeping it tight. Suppose the person snaps his hand a little up and down producing a bump in the string near his hand.

Experiments shows that, as time passes the bump travels on the string towards right.

Suppose the man starts snapping his hand at time $t = 0$ and finishes his job at $t = \Delta t$. The vertical displacement y of the left end of the string is a function of time.

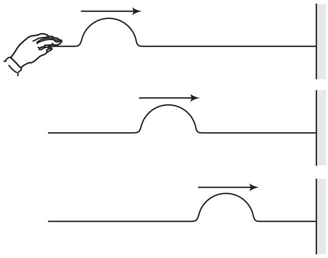


Fig. 17.2

$$y = 0 \text{ for } t < 0 \text{ and } t > \Delta t$$
$$y \neq 0 \text{ for } 0 < t < \Delta t$$

Let us represent this function by $f(t)$.

Thus,

$$y(x = 0, t) = f(t)$$

Here, $x = 0$ is the extreme left end of the string.

The disturbance travels on the string towards right with a constant speed v. Thus, the displacement produced at the left end at time t, reaches the point x at time $t + \frac{x}{v}$.

But the displacement of the particle at point x at time t was originated at $x = 0$ at time $t - \frac{x}{v}$.

∴ $$y(x, t) = y\left(x = 0, t - \frac{x}{v}\right) = f\left(t - \frac{x}{v}\right)$$

Thus, $$\boxed{y(x, t) = f\left(t - \frac{x}{v}\right)} \qquad \text{...(iii)}$$

Eq. (iii) represents a wave travelling in the positive x-direction with a constant speed v. The time t and the position x must appear in the wave equation in the combination $t - \frac{x}{v}$. If the wave travels in the negative x-direction, its general equation may be written as

$$y(x, t) = f\left(t + \frac{x}{v}\right)$$

Note *In the wave equation,*

$$y = (x, t) = f\left(t - \frac{x}{v}\right)$$

Coefficient of t is 1 and coefficient of x is $\frac{1}{v}$.

∴ $$\text{wave speed} = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{1}{(1/v)} = v$$

Example 17.1 *In a wave motion $y = a\sin(kx - \omega t)$, y can represent:* **(JEE 1999)**

(a) *electric field* (b) *magnetic field*
(c) *displacement* (d) *pressure*

Solution (a, b, c, d)

In case of sound wave, *y* can represent pressure and displacement, while in the case of electromagnetic wave, it represents electric and magnetic fields.

Example 17.2 *Show that the equation, $y = a\sin(\omega t - kx)$ satisfies the wave equation $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$. Find speed of wave and the direction in which it is travelling.*

Solution $\frac{\partial^2 y}{\partial t^2} = -\omega^2 a\sin(\omega t - kx)$ and $\frac{\partial^2 y}{\partial x^2} = -k^2 a\sin(\omega t - kx)$

We can write these two equations as

$$\frac{\partial^2 y}{\partial t^2} = \frac{\omega^2}{k^2}\cdot\frac{\partial^2 y}{\partial x^2}$$

Comparing this with,
$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We get wave speed $v = \frac{\omega}{k}$ **Ans.**

The negative sign between ωt and kx implies that wave is travelling along positive *x*-direction.

INTRODUCTORY EXERCISE 17.1

1. Prove that the equation $y = a\sin\omega t$ does not satisfy the wave equation and hence it does not represent a wave.
2. A wave pulse is described by $y(x, t) = ae^{-(bx - ct)^2}$, where a, b and c are positive constants. What is the speed of this wave?
3. You have learnt that a travelling wave in one dimension is represented by a function $y = f(x, t)$ where *x* and *t* must appear in the combination $ax \pm bt$ or $x - vt$ or $x + vt$, i.e. $y = f(x \pm vt)$. Is the converse true? Examine if the following functions for *y* can possibly represent a travelling wave
   (a) $(x - vt)^2$ (b) $\log[(x + vt)/x_0]$ (c) $1/(x + vt)$
4. The equation of a wave travelling on a string stretched along the *X*-axis is given by
$$y = Ae^{-\left(\frac{x}{a} + \frac{t}{T}\right)^2}$$
   (a) Write the dimensions of *A*, *a* and *T*.
   (b) Find the wave speed.
   (c) In which direction is the wave travelling?
   (d) Where is the maximum of the pulse located at $t = T$ and at $t = 2T$?

17.4 Sine Wave

If oscillations of y are simple harmonic in nature, then $y(x, t)$ function is sine or cosine function. Such type of wave is called **sine wave** or **sinusoidal wave**. For better understanding, we can visualize a sine wave travelling on a string.

The general expression of a sine wave is

$$\boxed{y = A\sin(\omega t \pm kx \pm \phi)} \quad \text{and} \quad \boxed{y = A\cos(\omega t \pm kx \pm \phi)}$$

In the above equations,

(i) A is the amplitude of oscillations of y.

(ii) ω is called the angular frequency. Here,

$$\boxed{\omega = 2\pi f = \frac{2\pi}{T} \quad \text{and} \quad f = \frac{1}{T}}$$

where, f is normal frequency of oscillations and T is the time period of oscillations. SI unit of ω is radian/sec while that of f is Hz or s^{-1}.

(iii) k is called the angular wave number, where

$$\boxed{k = \frac{2\pi}{\lambda}}$$

Here, λ is the wavelength of wave.

Wavelength (λ)

In a **transverse wave motion,** the particles of the medium oscillate about their mean or equilibrium position at right angles to the direction of propagation of wave motion itself.

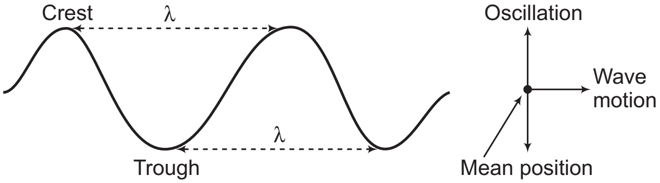


Fig. 17.3

This form of wave motion travels in the form of **crests and troughs**, for example, waves travelling along a stretched string. The distance between two successive crests or troughs is known as **wavelength (λ)** of the wave. In a **longitudinal wave motion,** particles of the medium oscillate about their mean or equilibrium position along the direction of propagation of the wave motion itself. This type of wave motion travels in the form of **compressions** and **rarefactions**.

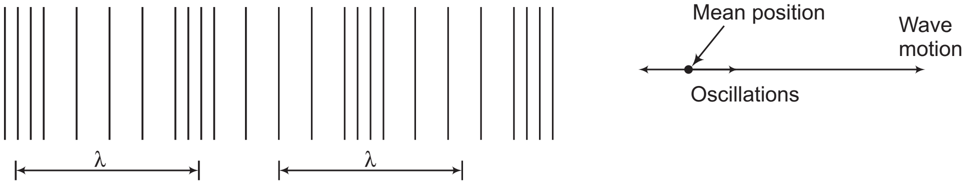


Fig. 17.4

The distance between two successive compressions or rarefactions constitute one wavelength.

Note *Numerically,* $k = \frac{2\pi}{\lambda}$ *or angular wave number is defined as the number of waves in* 2π *length. Similarly, there is one another word called wave number. This is equal to* $\frac{1}{\lambda}$ *and numerically this is equal to number of waves in unit length.*

(iv) Wave speed v,

$$= \frac{\text{coefficient of } t}{\text{coefficient of } x}$$

$$\therefore \qquad \boxed{v = \frac{\omega}{k}}$$

But $\qquad \omega = 2\pi f$

and $\qquad k = \frac{2\pi}{\lambda}$

Substituting in the above equation, we also get

$$\boxed{v = f\lambda}$$

(v) $\frac{\partial y}{\partial t}$ is SHM velocity of the particle executing oscillations (lying between $+\omega A$ and $-\omega A$).

Similarly, $\frac{\partial^2 y}{\partial t^2}$ is SHM acceleration (lying between $+\omega^2 A$ and $-\omega^2 A$).

Let us take an example: suppose a sine wave travelling on a string is

$$y = A \sin(\omega t - kx + \phi) \qquad \text{...(i)}$$

Then,

$$\frac{\partial y}{\partial t} = \omega A \cos(\omega t - kx + \phi) \qquad \text{...(ii)}$$

and

$$\frac{\partial^2 y}{\partial \tau^2} = -\omega^2 A \sin(\omega t - kx + \phi) \qquad \text{...(iii)}$$

From Eq. (i), we can determine y-displacement of SHM of any particle at position x and at time t. Value of y lies between $+A$ and $-A$.

From Eq. (ii), we can determine $\frac{\partial y}{\partial t}$ or SHM velocity and from Eq. (iii), we can find $\frac{\partial^2 y}{\partial t^2}$ or SHM acceleration.

Angle $(\omega t - kx + \phi)$ is same in all three equations which is also called phase angle of three functions. With the help of this phase angle, we can determine values of y, $\frac{\partial y}{\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$ at any position x at time t. If we put $x = 0$ and $t = 0$, then this angle is only ϕ. This is called initial phase angle of the particle at co-ordinate $x = 0$. With the help of this angle, we can find the initial values of y, $\frac{\partial y}{\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$ of this string particle (which is executing SHM at position $x = 0$).

Extra Points to Remember

- Alternate expressions of a sine wave travelling along positive x-direction are

$$y = A\sin k(x - vt) = A\sin(kx - \omega t) = A\sin\frac{2\pi}{\lambda}(x - vt) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

Similarly, the expression, $y = A\sin k(x + vt) = A\sin(kx + \omega t) = A\sin 2\pi\left(\frac{x}{\lambda} + \frac{t}{T}\right)$ etc.

represent a sine wave travelling along negative x-direction.

- **Difference between two equations**

$$y = A\sin(kx - \omega t) \text{ and } y = A\sin(\omega t - kx)$$

It hardly matters whether we write the first or the second equation. Both the equations represent a wave travelling in positive x-direction with speed $v = \frac{\omega}{k}\cdot$ The difference between them is that they are out of phase, i.e. phase difference between them is π. It means, if a particle in position $x = 0$ at time $t = 0$ is in its mean position and moving upwards (represented by first wave) then the same particle will be in its mean position but moving downwards (represented by the second wave). Similarly, the waves

$y = A\sin(kx - \omega t)$ and $y = -A\sin(kx - \omega t)$ are also out of phase.

Example 17.3 *A wave travelling along a string is described by*

$$y(x, t) = 0.005\sin(80.0x - 3.0t).$$

in which the numerical constants are in SI units (0.005 *m*, 80.0 *rad* m^{-1} *and* 3.0 *rad* s^{-1}). *Calculate (a) the amplitude. (b) the wavelength (c) the period and frequency of the wave. Also, calculate the displacement y of the wave at a distance x* = 30.0 *cm and time t* = 20 *s?* **(NCERT Solved Example)**

Solution On comparing the given equation with

$$y(x, t) = A\sin(kx - \omega t), \text{ we find}$$

(a) the amplitude $A = 0.005\text{ m} = 5\text{ mm}$.

(b) the angular wave number k and angular frequency ω are

$$k = 80.0\text{ m}^{-1} \quad\text{and}\quad \omega = 3.0\text{ s}^{-1}$$

$$\therefore \quad \lambda = 2\pi/k = \frac{2\pi}{80.0} = 0.0785\text{ m}$$

$$= 7.85\text{ cm}$$ **Ans.**

(c) $T = 2\pi/\omega = \frac{2\pi}{3.0} = 2.09\text{ s}$ **Ans.**

and frequency, $f = 1/T = 0.48\text{ Hz}$ **Ans.**

The displacement y at $x = 30.0$ cm and time $t = 20$ s is given by

$$y = (0.005\text{ m})\sin(80.0 \times 0.3 - 3.0 \times 20)$$
$$= (0.005\text{ m})\sin(-36)$$
$$= (0.005\text{ m})\sin(-36 + 12\pi)$$
$$= (0.005\text{ m})\sin(1.714)$$
$$= (0.005\text{ m})\sin(98.27^\circ) = 4.94\text{ mm}$$ **Ans.**

Example 17.4 *The equation of a wave is*

$$y(x, t) = 0.05 \sin\left[\frac{\pi}{2}(10\,x - 40\,t) - \frac{\pi}{4}\right] m$$

Find : *(a) the wavelength, the frequency and the wave velocity*
(b) the particle velocity and acceleration at $x = 0.5$ m and $t = 0.05$ s.

Solution (a) The equation may be rewritten as

$$y(x, t) = 0.05 \sin\left(5\pi x - 20\pi t - \frac{\pi}{4}\right) \text{m}$$

Comparing this with equation of plane progressive harmonic wave,

$$y(x, t) = A \sin(kx - \omega t + \phi) \text{ we have,}$$

Wave number, $k = \dfrac{2\pi}{\lambda} = 5\pi$ rad/m

∴ $\lambda = 0.4$ m **Ans.**

The angular frequency is

$$\omega = 2\pi f = 20\pi \text{ rad/s}$$

∴ $f = 10$ Hz **Ans.**

The wave velocity is,

$$v = f\lambda = \frac{\omega}{k} = 4 \text{ m/s in } + x\text{-direction}$$ **Ans.**

(b) The particle velocity and particle acceleration at the given values of x and t are

$$\frac{\partial y}{\partial t} = -(20\pi)(0.05)\cos\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$= 2.22 \text{ m/s}$$ **Ans.**

$$\frac{\partial^2 y}{\partial t^2} = -(20\pi)^2(0.05)\sin\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$= 140 \text{ m/s}^2$$ **Ans.**

INTRODUCTORY EXERCISE 17.2

1. Consider the wave $y = (5\text{ mm})\sin[1\text{cm}^{-1}x - (60\text{s}^{-1})t]$. Find (a) the amplitude, (b) the angular wave number, (c) the wavelength, (d) the frequency, (e) the time period and (f) the wave velocity.
2. A wave is described by the equation $y = (1.0\text{ mm})\sin\pi\left(\dfrac{x}{2.0\text{ cm}} - \dfrac{t}{0.01\text{ s}}\right)$.
   (a) Find time period and wavelength.
   (b) Find the speed of particle at $x = 1.0$ cm and time $t = 0.01$ s.
   (c) What are the speeds of the particles at $x = 3.0$ cm, 5.0 cm and 7.0 cm at $t = 0.01$ s?
   (d) What are the speeds of the particles at $x = 1.0$ cm at $t = 0.011, 0.012$ and 0.013 s?

17.5 Two Graphs in Sine Wave

As we have learned in the above article that $y(x,t)$ equation of a sine wave is either sine or cosine equation. Now, corresponding to this equation we can have two graphs and two simple equations.

First Graph

In $y(x,t)$ equation, if value of t is fixed (or substituted), then the equation left is $y(x)$ equation. So, we can plot y-x graph corresponding to this equation. And obviously the graph (or the equation) will be a sine or cosine graph.

For example Suppose a sine wave is travelling along positive x-direction on a string. At a given time (say at 9 AM), the y-x graph may be as shown in the figure.

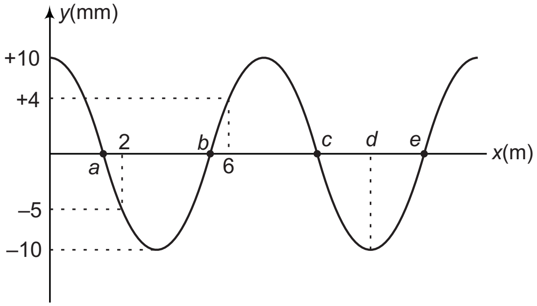


Fig. 17.5

The important points in the above graph are

(i) amplitude of oscillation is 10 mm.

(ii) at 9 AM, y-displacement of the string particle at $x = 2$ m is -5 mm and of the particle at $x = 6$ m is $+4$ mm. Or, we can say that this graph represents SHM y-displacements of different string particles (at different x-coordinates) at 9 AM. Hence, this is a photograph (or snapshot) of the string at 9 AM.

(iii) Slope of this graph at any point is $\frac{\partial y}{\partial x}\left(\text{not } \frac{dy}{dx}\right)$, as y has two variables x and t.

(iv) Two string particles at different locations are in different phases. The phase difference between them is given by

$$\boxed{\Delta\phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x)}$$

Here, Δx is the path difference between them.

For example

Particles 'a' and 'b' have a path difference of $\frac{\lambda}{2}$. So, phase difference between them is π.

Particle 'a' and 'c' have a path difference of λ. So, phase difference is 2π.

Similarly, particles 'c' and 'd' have a path difference of $\frac{\lambda}{4}$. Therefore, the phase difference is $\frac{\pi}{2}$.

(v) From the above graph, we cannot determine the direction of wave velocity. For that, wave equation will be required.

Second Graph

In $y(x, t)$ equation, if the value of x is fixed (or substituted) then the equation left is $y(t)$ equation. So, we can plot y-t graph corresponding to this equation. Again, the graph (or the equation) will be a sine or cosine graph.

For example Suppose a sine wave is travelling along positive x-direction on a string. At a given position (say $x = 4$ m), the y-t graph may be as shown in the figure.

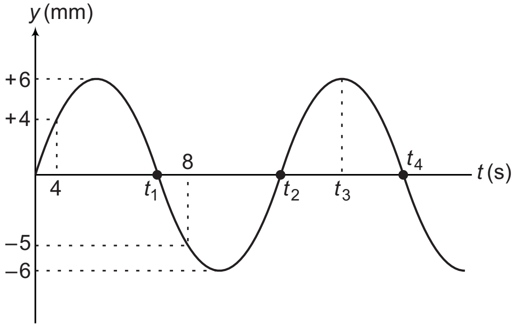


Fig. 17.6

The important points in this graph are

(i) amplitude of oscillations is 6 mm.

(ii) y-displacement of the particle at $x = 4$ m is 4 mm at 4 s. Similarly, y-displacement of the particle at $x = 4$ m is −5 mm at 8 s or we can say that this graph is basically SHM y-t graph of the string particle at $x = 4$ m, which shows different SHM y-displacements of this particle at different times or it is videography of SHM oscillations of this particle.

(iii) Slope of this particle at any time is $\frac{\partial y}{\partial t}$ and this is SHM velocity $(= \pm\omega\sqrt{A^2 - y^2})$ of the particle at $x = 4$ m at that time.

(iv) The same particle at two different times will have different phase angles.

The phase difference in a time interval of Δt is given by

$$\boxed{\Delta\phi = \left(\frac{2\pi}{T}\right)\Delta t}$$

For example,

t_1 and t_2 have a time interval of $\frac{T}{2}\cdot$ So, phase difference is π.

t_2 and t_3 have a time interval of $\frac{T}{4}\cdot$ So, the phase difference is $\frac{\pi}{2}$.

Similarly, t_1 and t_4 have a time interval of T.

Therefore, phase difference is 2π.

Wave velocity (v), particle velocity (v_p) and particle acceleration (a_p) in a Sinusoidal wave

Wave velocity (v) is the velocity by which oscillations of y (or energy) transfer from one point to another point. Later, we will see that this velocity depends on characteristics of medium.

Particle velocity (v_P) and particle acceleration (a_P) are different.

In a sinusoidal wave, particles of the medium oscillate simple harmonically about their mean position. Therefore, all the formulae we have read in SHM apply to the particles here also. For example, maximum particle velocity is $\pm A\omega$ at mean position and it is zero at extreme positions etc. Similarly, maximum particle acceleration is $\pm \omega^2 A$ at extreme positions and zero at mean position.

However, the wave velocity is different from the particle velocity. This depends on certain characteristics of the medium. Unlike the particle velocity which oscillates simple harmonically (between $+A\omega$ and $-A\omega$) the wave velocity is constant for given characteristics of the medium.

Suppose, a sine wave travelling along positive x- axis is

$$y(x, t) = A \sin (kx - \omega t) \qquad \ldots\text{(i)}$$

Let us differentiate this function partially with respect to t and x.

$$\frac{\partial y(x, t)}{\partial t} = -A\omega \cos (kx - \omega t) \qquad \ldots\text{(ii)}$$

$$\frac{\partial y(x, t)}{\partial x} = Ak \cos (kx - \omega t) \qquad \ldots\text{(iii)}$$

Now, these can be written as

$$\frac{\partial y(x, t)}{\partial t} = -\left(\frac{\omega}{k}\right)\frac{\partial y(x, t)}{\partial x}$$

Here,

$$\frac{\partial y(x, t)}{\partial t} = \text{particle velocity } v_P$$

$$\frac{\omega}{k} = \text{wave velocity } v$$

and

$$\frac{\partial y(x, t)}{\partial x} = \text{slope of the wave}$$

Thus,

$$\boxed{v_P = -v\,(\text{slope})} \qquad \ldots\text{(iv)}$$

i.e. particle velocity at a given position and time is equal to negative of the product of wave velocity with slope of the wave at that point at that instant.

Acceleration of the particle is the second partial derivative of $y(x, t)$ with respect to t,

$$\therefore \quad a_P = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A \sin (kx - \omega t) = -\omega^2 y(x, t)$$

i.e. acceleration of the particle equals $-\omega^2$ times its displacement, which is the same result we obtained in SHM. Thus,

$$\boxed{a_P = -\omega^2\,(\text{displacement})} \qquad \ldots\text{(v)}$$

We can also show that,

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \left(\frac{\omega^2}{k^2}\right) \cdot \frac{\partial^2 y(x,t)}{\partial x^2}$$

or
$$\boxed{\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}} \qquad \text{...(vi)}$$

which is also the wave equation.

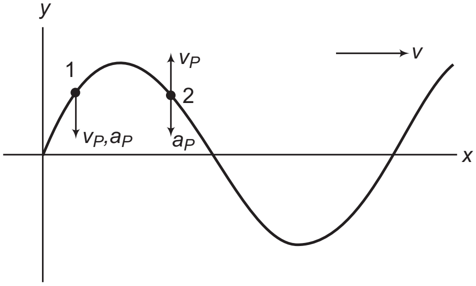


Fig. 17.7

Fig. 17.7 shows velocity (v_P) and acceleration (a_P) given by Eqs. (iv) and (v) for two points 1 and 2 on a string. The sinusoidal wave is travelling along positive x-direction.

At 1 Slope of the curve is positive. Hence, from Eq. (iv) particle velocity (v_P) is negative or downwards. Similarly, displacement of the particle is positive, so from Eq. (v) acceleration will also be negative or downwards.

At 2 Slope is negative while displacement is positive. Hence, v_P will be positive (upwards) and a_P is negative (downwards).

Note *Direction of v_P will change if the wave travels along negative x-direction.*

Example 17.5 *Under what condition, maximum particle velocity is four times the wave velocity corresponding to the equation,*

$$y = A\sin(\omega t - kx)$$

Solution Maximum particle velocity is ωA and wave velocity is $\frac{\omega}{k}$.

Now, given that

$$\text{maximum particle velocity} = 4\,(\text{wave velocity})$$

or
$$\omega A = 4\left(\frac{\omega}{k}\right)$$

$\therefore$
$$A = \frac{4}{k}$$

So, this is the required condition.

Example 17.6 *A transverse sinusoidal wave moves along a string in the positive x-direction at a speed of 10 cm/s. The wavelength of the wave is 0.5 m and its amplitude is 10 cm. At a particular time t, the snapshot of the wave is shown in figure. The velocity of point P when its displacement is 5 cm is*

(JEE 2008)

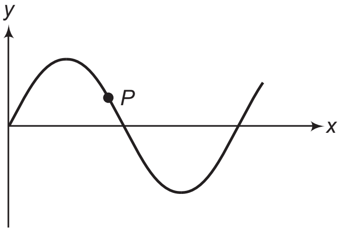


Fig. 17.8

(a) $\dfrac{\sqrt{3}\pi}{50}\hat{\mathbf{j}}$ *m/s* (b) $-\dfrac{\sqrt{3}\pi}{50}\hat{\mathbf{j}}$ *m/s*

(c) $\dfrac{\sqrt{3}\pi}{50}\hat{\mathbf{i}}$ *m/s* (d) $-\dfrac{\sqrt{3}\pi}{50}\hat{\mathbf{i}}$ *m/s*

Solution Particle velocity $v_P = -v$ (slope of y-x graph)

Here, $v = +$ ve, as the wave is travelling in positive x-direction.Slope at P is negative.

$\therefore$ Velocity of particle is in positive y (or $\hat{\mathbf{j}}$) direction.

$\therefore$ Correct option is (a).

Note *We can also find the magnitude of particle velocity by the relation.*

$$v_P = \omega\sqrt{A^2 - y^2}$$

Here, $A = 10$ *cm* $= 0.1 m$, $y = 5$ *cm* $= 0.05$ *m and* $\omega = 2\pi f = 2\pi\left(\dfrac{v}{\lambda}\right) = 2\pi\left(\dfrac{0.1}{0.5}\right) = \dfrac{2\pi}{5}$ *rad* s^{-1}

Example 17.7 *Equation of a transverse wave travelling in a rope is given by*

$$y = 5\sin(4.0t - 0.02x)$$

where y and x are expressed in cm and time in seconds. Calculate

(a) the amplitude, frequency, velocity and wavelength of the wave.

(b) the maximum transverse speed and acceleration of a particle in the rope.

Solution (a) Comparing this with the standard equation of wave motion,

$$y = A\sin(\omega t - kx) = A\sin\left(2\pi ft - \frac{2\pi}{\lambda}x\right)$$

where A, f and λ are amplitude, frequency and wavelength respectively.

Thus, amplitude $A = 5$ cm

$\Rightarrow$ $2\pi f = 4 \Rightarrow$ Frequency, $f = \dfrac{4}{2\pi} = 0.637$ Hz

Again $\dfrac{2\pi}{\lambda} = 0.02$ or Wavelength, $\lambda = \dfrac{2\pi}{0.02} = (100\pi)$ cm

Velocity of the wave, $v = f\lambda = \dfrac{4}{2\pi}\,\dfrac{2\pi}{0.02} = 200$ cm/s **Ans.**

(b) Transverse velocity of the particle,

$$v_P = \frac{\partial y}{\partial t} = 5 \times 4\cos(4.0t - 0.02x) = 20\cos(4.0t - 0.02x)$$

Maximum velocity of the particle = 20 cm/s

Particle acceleration, $a_P = \dfrac{\partial^2 y}{\partial t^2} = -20 \times 4 \sin(4.0t - 0.02x)$

Maximum particle acceleration = 80 cm/s^2

Example 17.8 *In the above example, find phase difference* $\Delta\phi$.

(a) of same particle at two different times with a time interval of 1 *s*

(b) of two different particles located at a distance of 10 *cm at same time*

Solution (a) $\Delta\phi = \left(\dfrac{2\pi}{T}\right)\Delta t$, but $\dfrac{2\pi}{T} = \omega = 4.0$ rad/s

$$\therefore \qquad \Delta\phi = \omega\,\Delta t = (4.0)(1.0)$$

$$= 4 \text{ rad}$$ **Ans.**

(b) $\Delta\phi = \left(\dfrac{2\pi}{\lambda}\right)\Delta x$, but $\dfrac{2\pi}{\lambda} = k = 0.02\ \text{cm}^{-1}$

$$\therefore \qquad \Delta\phi = (0.02)(10)$$

$$= 0.2 \text{ rad}$$ **Ans.**

INTRODUCTORY EXERCISE 17.3

1. The equation of a wave travelling on a string is

$$y = (0.10\ \text{mm}) \sin[(31.4\ \text{m}^{-1})x + (314\ \text{s}^{-1})t]$$

(a) In which direction does the wave travel?

(b) Find the wave speed, the wavelength and the frequency of the wave.

(c) What is the maximum displacement and the maximum speed of a portion of the string?

2. The equation for a wave travelling in x-direction on a string is

$$y = (3.0\ \text{cm}) \sin[(3.14\ \text{cm}^{-1})x - (314\ \text{s}^{-1})t].$$

(a) Find the maximum velocity of a particle of the string.

(b) Find the acceleration of a particle at $x = 6.0$ cm at time $t = 0.11$ s

3. The equation of a travelling wave is

$$y(x, t) = 0.02 \sin\left(\frac{x}{0.05} + \frac{t}{0.01}\right) \text{m}$$

Find

(a) the wave velocity and

(b) the particle velocity at $x = 0.2$ m and $t = 0.3$ s.

Given $\cos\theta = -0.85$, where $\theta = 34$ rad

4. A wave of frequency 500 Hz has a wave velocity of 350 m/s.

(a) Find the distance between two points which are 60° out of phase.

(b) Find the phase difference between two displacements at a certain point at time 10^{-3} s apart.

17.6 Wave Speed

Wave speed (v) depends on the medium or characteristics of the medium. Waves of all frequencies or all wavelengths travel in a given medium with same speed. For examples, all electromagnetic waves travel in vacuum with same speed ($\approx 3\times10^8$ m/s).

In water, this speed will be different. Similarly, all longitudinal waves (including the sound wave) travel in air with the same speed (≈ 330 m/s).

Now, frequency of a wave depends on source and wavelength is self adjusted in a value,

$$\boxed{\lambda = \frac{v}{f}}$$

In the above expression, v is same (obviously in a given medium) for all sources (or frequencies). So, with increase in value of f, wavelength λ automatically decreases.

For example, usually frequency of female voice is more than frequency of male voice.

So, wavelength of female voice will be less.

Example 17.9 *Speed of sound in air is 330 m/s. Find maximum and minimum wavelength of audible sound in air.*

Solution Frequency of audible sound varies from 20 Hz to 20000 Hz.

Speed of all frequencies in air will be same ($= 330\,\text{m/s}$).

$\therefore$
$$\lambda_{\text{min}} = \frac{v}{f_{\text{max}}} = \frac{330}{20000}$$
$$= 0.0165\,\text{m}$$
Ans.

Similarly,
$$\lambda_{\text{max}} = \frac{v}{f_{\text{min}}} = \frac{330}{20}$$
$$= 16.5\ \text{m}$$
Ans.

INTRODUCTORY EXERCISE 17.4

1. Speed of light in vacuum is 3×10^8 m/s. Range of wavelength of visible light is 4000 Å - 7000 Å. Find the range of frequency of visible light.
2. Speed of sound in air is 330 m/s. Frequency of Anoop's voice is 1000 Hz and of Shubham's voice is 2000 Hz. Find the wavelength corresponding to their voice.

Speed of Different Waves

Normally, two wave speeds are required at this stage.

(i) Transverse wave speed on a string.

(ii) Longitudinal wave speed in all three states: solid, liquid and gas.

Transverse Wave Speed on a String

Speed of transverse wave on a string is given by

$$v = \sqrt{\frac{T}{\mu}}$$

Here, μ = mass per unit length of the string

$$= \frac{m}{l} = \frac{mA}{lA} \qquad (A = \text{area of cross-section of the string})$$

$$= \left(\frac{m}{V}\right) A \qquad (V = \text{volume of string})$$

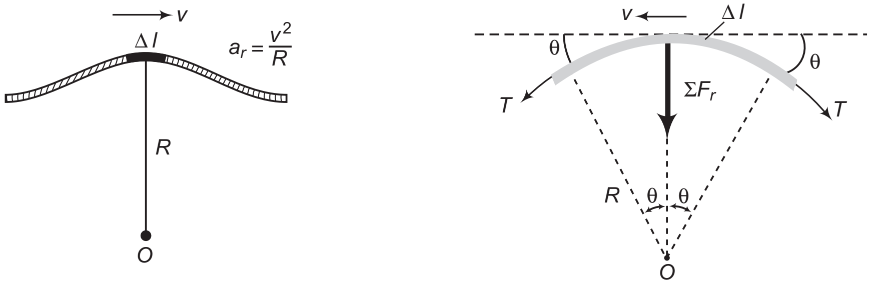
$$= \rho A \qquad (\rho = \text{density of string})$$

Hence, the above expression can also be written as

$$\boxed{v = \sqrt{\frac{T}{\rho A}}}$$

Proof

Consider a pulse travelling along a string with a speed v to the right. If the amplitude of the pulse is small compared to the length of the string, the tension T will be approximately constant along the string. In the reference frame moving with speed v to the right, the pulse is stationary and the string moves with a speed v to the left.



(a) (b)

Fig. 17.9 (a) To obtain the speed v of a wave on a stretched string, it is convenient to describe the motion of a small segment of the string in a moving frame of reference. (b) In the moving frame of reference, the small segment of length Δl moves to the left with speed v. The net force on the segment is in the radial direction because the horizontal components of the tension forces are cancelled.

Figure shows a small segment of the string of length Δl. This segment forms a part of a circular arc of radius R. Instantaneously, the segment is moving with speed v in a circular path, so it has a centripetal acceleration $\frac{v^2}{R}$. The forces acting on the segment are the tension T at each end. The horizontal components of these forces are equal and opposite and thus cancel. The vertical components of these forces point radially inward toward the centre of the circular arc. These radial forces provide the centripetal acceleration. Let the angle subtended by the segment at centre be 2θ. The net radial force acting on the segment is

$$\Sigma F_r = 2T \sin\theta = 2T\theta$$

Where we have used the approximation $\sin\theta \approx \theta$ for small θ.

If μ is the mass per unit length of the string, the mass of the segment of length Δl is

$$m = \mu\Delta l = 2\mu R\theta \qquad (\text{as } \Delta l = 2R\theta)$$

From Newton's second law, $\Sigma F_r = ma = \dfrac{mv^2}{R}$ or $2T\theta = (2\mu R\theta)\left(\dfrac{v^2}{R}\right)$

$\therefore$ $$v = \sqrt{\frac{T}{\mu}}$$

Longitudinal Wave Speed in Three States

Speed of longitudinal wave through a gas (or a liquid) is given by

$$\boxed{v = \sqrt{\frac{B}{\rho}}}$$

Here, B = Bulk modulus of the gas (or liquid)

and ρ = density of the gas (or liquid)

Now Newton, who first deduced this relation for v, assumed that during the passage of a sound wave through a gas (or air), the temperature of the gas remains constant, i.e. sound wave travels under isothermal conditions and hence took B to be the isothermal elasticity of the gas and which is equal to its pressure p. So, **Newton's** formula for the velocity of a sound wave (or a longitudinal wave) in a gaseous medium becomes

$$\boxed{v = \sqrt{\frac{p}{\rho}}}$$

If, however, we calculate the velocity of sound in air at NTP with the help of this formula by substituting.

$$p = 1.01 \times 10^5 \ \text{N/m}^2 \quad \text{and} \quad \rho = 1.29 \times 10^{-3} \ \text{kg/m}^3$$

then v comes out to be nearly 280 m/s. Actually, the velocity of sound in air at NTP as measured by Newton himself, is found to be 332 m/s. Newton could not explain this large discrepancy between his theoretical and experimental results.

Laplace after 140 years correctly argued that a sound wave passes through a gas (or air) very rapidly. So, adiabatic conditions are developed. So, he took B to be the adiabatic elasticity of the gas, which is equal to γp where γ is the ratio of C_p (molar heat capacity at constant pressure) and C_V (molar heat capacity at constant volume). Thus, Newton's formula as corrected by Laplace becomes

$$\boxed{v = \sqrt{\frac{\gamma p}{\rho}}}$$

For air, $\gamma = 1.41$, so that in air, $$v = \sqrt{\frac{1.41\,p}{\rho}}$$

which gives 331.6 m/s as the velocity of sound (in air) at NTP which is in agreement with Newton's experimental result.

Speed of longitudinal wave in a thin rod or wire is given by

$$\boxed{v = \sqrt{\frac{Y}{\rho}}}$$

Here, Y is the Young's modulus of elasticity.

Note *In the chapter of sound wave (Chapter-19), we will discuss longitudinal wave speed in detail (with proof and examples). In the present chapter, we are taking examples of only transverse wave speed on a string.*

Example 17.10 *One end of 12.0 m long rubber tube with a total mass of 0.9 kg is fastened to a fixed support. A cord attached to the other end passes over a pulley and supports an object with a mass of 5.0 kg. The tube is struck a transverse blow at one end. Find the time required for the pulse to reach the other end. ($g = 9.8\ m/s^2$)*

Solution Tension in the rubber tube $AB, T = mg$

or $$T = (5.0)(9.8) = 49\ \text{N}$$

Mass per unit length of rubber tube, $\mu = \dfrac{0.9}{12} = 0.075\ \text{kg/m}$

$\therefore$ Speed of wave on the tube, $v = \sqrt{\dfrac{T}{\mu}} = \sqrt{\dfrac{49}{0.075}} = 25.56\ \text{m/s}$

$\therefore$ The required time is, $t = \dfrac{AB}{v} = \dfrac{12}{25.56} = 0.47\ \text{s}$ **Ans.**

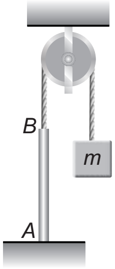


Fig. 17.10

Example 17.11 *A wire of uniform cross-section is stretched between two points 100 cm apart. The wire is fixed at one end and a weight is hung over a pulley at the other end. A weight of 9 kg produces a fundamental frequency of 750 Hz.*

(a) What is the velocity of the wave in wire?

(b) If the weight is reduced to 4 kg, what is the velocity of wave?

Note *Fundamental frequency is given by*

$$f = \frac{v}{2L}$$

Solution (a) $L = 100\ \text{cm}, \quad f_1 = 750\ \text{Hz}$

$$v_1 = 2Lf_1 = 2 \times 100 \times 750$$
$$= 150000\ \text{cms}^{-1} = 1500\ \text{ms}^{-1}$$ **Ans.**

(b) $v_1 = \sqrt{\dfrac{T_1}{\mu}}$ and $v_2 = \sqrt{\dfrac{T_2}{\mu}}$

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$$

$\therefore$ $$\frac{v_2}{1500} = \sqrt{\frac{4}{9}}$$

$\therefore$ $$v_2 = 1000\ \text{ms}^{-1}$$ **Ans.**

INTRODUCTORY EXERCISE 17.5

1. Figure shows a string of linear mass density 1.0 g cm^{-1} on which a wave pulse is travelling. Find the time taken by the pulse in travelling through a distance of 50 cm on the string. Take $g = 10 \text{ ms}^{-2}$.

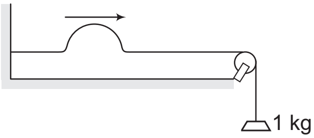


Fig. 17.11

2. A steel wire of length 64 cm weighs 5 g. If it is stretched by a force of 8 N, what would be the speed of a transverse wave passing on it?

3. Two blocks each having a mass of 3.2 kg are connected by a wire *CD* and the system is suspended from the ceiling by another wire *AB*. The linear mass density of the wire *AB* is 10 gm^{-1} and that of *CD* is 8 g m^{-1}. Find the speed of a transverse wave pulse produced in *AB* and in *CD*.

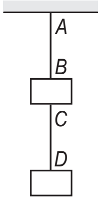


Fig. 17.12

4. In the arrangement shown in figure, the string has a mass of 4.5 g. How much time will it take for a transverse disturbance produced at the floor to reach the pulley? Take $g = 10 \text{ ms}^{-2}$.

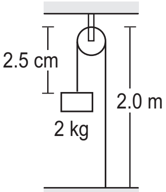


Fig. 17.13

5. A copper wire 2.4 mm in diameter is 3 m long and is used to suspend a 2 kg mass from a beam. If a transverse disturbance is sent along the wire by striking it lightly with a pencil, how fast will the disturbance travel? The density of copper is 8920 kg/m^3.

6. One end of a horizontal rope is attached to a prong of an electrically driven tuning fork that vibrates at 120 Hz. The other end passes over a pulley and supports a 1.50 kg mass. The linear mass density of the rope is 0.0550 kg/m.
   (a) What is the speed of a transverse wave on the rope?
   (b) What is the wavelength?
   (c) How would your answers to parts (a) and (b) change if the mass were increased to 3.00 kg?

17.7 Energy in Wave Motion

In a wave, many particles oscillate. In a sinusoidal wave, these oscillations are simple harmonic in nature. Each particle has some energy of oscillation. At the same time, energy transfer also takes place. Related to energy of oscillation and energy transfer, there are three terms, namely, energy density (u), power (P) and intensity (I).

Energy density (u)

Energy of oscillation per unit volume is called energy density (u). Its SI unit is J/m^3. In case of SHM, energy of oscillation (of a single particle) is $E = \frac{1}{2} m\omega^2 A^2$. In a sinusoidal wave, each particle of the string oscillates simple harmonically.

Therefore,

$$\text{Energy density} = \frac{\text{energy of oscillation}}{\text{volume}} \quad \text{or} \quad u = \frac{E}{V} = \frac{\frac{1}{2} m\omega^2 A^2}{V}$$

But, $$\frac{m}{V} = \text{density or } \rho$$

∴ $$\boxed{u = \frac{1}{2}\rho\omega^2 A^2}$$

Power (P)

Energy transferred per unit time is called power. The expression of power is

$$\boxed{P = \frac{1}{2}\rho\omega^2 A^2 Sv}$$

Here, S is area of cross-section of the medium in which wave is travelling and v is the wave speed. SI unit of power is J/s or Watt. Now, let us derive the above expression for a sinusoidal travelling wave on a string. Area of cross-section of string is S and the wave speed is v.

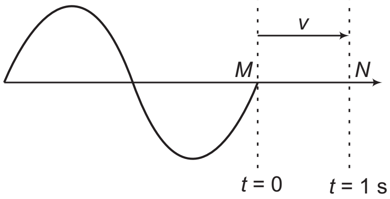


Fig. 17.14

Suppose at time $t = 0$, wave is at point M. In 1 s, it will travel a distance v and it will reach at point N. Or, we can say that v new length of string will start oscillating. Area of cross-section of string is S. Therefore, in 1 s, Sv new volume of string will start oscillating. Energy of oscillation per unit volume is called energy density (u). So, in 1 s, (uSv) new energy will be added or (uSv) energy will be transferred from the source. Energy transferred in one second is called power.

∴ $$P = uSv$$

$$= \frac{1}{2}\rho\omega^2 A^2 Sv \qquad \left(\text{as } u = \frac{1}{2}\rho\omega^2 A^2\right)$$

or $$P = \frac{1}{2}\rho\omega^2 A^2 Sv$$

This is the desired expression of power.

Intensity (I)

Energy transferred per unit cross-sectional area per unit time is called intensity.

Thus,

$$I = \frac{\text{Energy transferred}}{(\text{time})(\text{cross-sectional area})} = \frac{\text{Power}}{\text{cross-sectional area}} = \frac{P}{S}$$

$$= \frac{\frac{1}{2}\rho\omega^2 A^2 S v}{S} \quad \text{or} \quad \boxed{I = \frac{1}{2}\rho\omega^2 A^2 v}$$

The SI unit of intensity is J/s-m^2 or Watt/m^2.

Extra Points to Remember

- Although the above relations for power and intensity have been discussed for a transverse wave on a string, they hold good for other waves also.
- In SHM, potential energy is maximum (and kinetic energy is zero) at the extreme positions.
- For a string segment, the potential energy depends on the slope of the string and is maximum when the slope is maximum, which is at the equilibrium position of the segment, the same position for which the kinetic energy is maximum.

  At A : Kinetic energy and potential energy both are zero.

  At B : Kinetic energy and potential energy both are maximum.

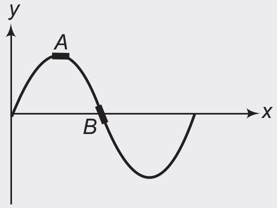


Fig. 17.15

- **Intensity due to a point source** If a point source emits wave uniformly in all directions, the energy at a distance r from the source is distributed uniformly on a spherical surface of radius r and area $S = 4\pi r^2$. If P is the power emitted by the source, the power per unit area at a distance r from the source is $\frac{P}{4\pi r^2}$. The average power per unit area that is incident perpendicular to the direction of propagation is called the intensity. Therefore,

$$I = \frac{P}{4\pi r^2} \quad \text{or} \quad I \propto \frac{1}{r^2}$$

Example 17.12 *A stretched string is forced to transmit transverse waves by means of an oscillator coupled to one end. The string has a diameter of 4 mm. The amplitude of the oscillation is 10^{-4} m and the frequency is 10 Hz. Tension in the string is 100 N and mass density of wire is 4.2×10^3 kg/m^3. Find*

(a) the equation of the waves along the string

(b) the energy per unit volume of the wave

(c) the average energy flow per unit time across any section of the string and

(d) power required to drive the oscillator.

Solution (a) Speed of transverse wave on the string is

$$v = \sqrt{\frac{T}{\rho S}} \qquad (\text{as } \mu = \rho S)$$

Substituting the values, we have

$$v = \sqrt{\frac{100}{(4.2\times 10^3)\left(\frac{\pi}{4}\right)(4.0\times 10^{-3})^2}} = 43.53 \text{ m/s}$$

$$\omega = 2\pi f = 20\pi = 62.83 \text{ rad/s}$$

$$k = \frac{\omega}{v} = 1.44 \text{ m}^{-1}$$

$\therefore$ Equation of the waves along the string, $y(x, t) = A \sin(kx - \omega t)$

$$= (10^{-4} \text{ m}) \sin [(1.44 \text{ m}^{-1})x - (62.83 \text{ rad/s})t]$$ **Ans.**

(b) Energy per unit volume of the string, $u =$ energy density $= \frac{1}{2}\rho\omega^2 A^2$

Substituting the values, we have

$$u = \left(\frac{1}{2}\right)(4.2\times 10^3)(62.83)^2 (10^{-4})^2$$

$$= 8.29\times 10^{-2} \text{ J/m}^3$$ **Ans.**

(c) Average energy flow per unit time, $P =$ power $= \left(\frac{1}{2}\rho\omega^2 A^2\right)(Sv) = (u)(Sv)$

Substituting the values, we have $P = (8.29\times 10^{-2})\left(\frac{\pi}{4}\right)(4.0\times 10^{-3})^2 (43.53)$

$$= 4.53\times 10^{-5} \text{ J/s}$$ **Ans.**

(d) Therefore, power required to drive the oscillator is 4.53×10^{-5} W. **Ans.**

INTRODUCTORY EXERCISE 17.6

1. Spherical waves are emitted from a 1.0 W source in an isotropic non-absorbing medium. What is the wave intensity 1.0 m from the source?
2. A line source emits a cylindrical expanding wave. Assuming the medium absorbs no energy, find how the amplitude and intensity of the wave depend on the distance from the source?
3. A certain 120 Hz wave on a string has an amplitude of 0.160 mm. How much energy exists in an 80 g length of the string?
4. A taut string for which $\mu = 5.00\times 10^{-2}$ kg/m is under a tension of 80.0 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0 Hz and an amplitude of 6.00 cm?
5. A 200 Hz wave with amplitude 1 mm travels on a long string of linear mass density 6 g/m kept under a tension of 60 N.
   (a) Find the average power transmitted across a given point on the string.
   (b) Find the total energy associated with the wave in a 2.0 m long portion of the string.
6. A transverse wave of amplitude 0.50 mm and frequency 100 Hz is produced on a wire stretched to a tension of 100 N. If the wave speed is 100 m/s. What average power is the source transmitting to the wire?

List of Formulae

1. **Wave Equation** Only those functions of x and t or $y(x, t)$ represent a wave function which satisfies the following three conditions

   Condition 1 $$\left(\frac{\partial^2 y}{\partial t^2}\right) = k\left(\frac{\partial^2 y}{\partial x^2}\right)$$

   Here, $$k = v^2 \qquad (v = \text{wave speed})$$

   Condition 2 $y(x, t)$ should be a single valued function for all values of x and t.

   Condition 3 The wave function and its first derivative must be continuous.

2. If $y(x, t)$ is of type $f(ax \pm bt)$, then

   $$\text{wave speed } v = \frac{\text{coefficient of } t}{\text{coefficient of } x}$$

   Further, wave travels in positive x-direction if ax and bt are of opposite sign and it travels along negative x-direction if ax and bt are of same sign.

3. **Sine wave** General equation of this wave is

   $$y = A\sin(\omega t \pm kx \pm \phi)$$

   or $$y = A\cos(\omega t \pm kx \pm \phi)$$

   In these equations,

   (i) A is amplitude of oscillation,

   (ii) ω is angular frequency,

   $$T = \frac{2\pi}{\omega} \Rightarrow \omega = 2\pi f$$

   and $$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

   (iii) k is angular wave number,

   $$k = \frac{2\pi}{\lambda} \qquad (\lambda \rightarrow \text{wavelength})$$

   (iv) Wave speed, $v = \dfrac{\omega}{k} = f\lambda$

   (v) ϕ is initial phase angle at $x = 0$ and

   (vi) $(\omega t \pm kx \pm \phi)$ is phase angle at time t at coordinate x.

4. **Particle velocity (v_P) and wave velocity (v) in sine wave**

   (i) $y = f(x, t)$

   Then, $$v_P = \frac{\partial y}{\partial x}$$

   (ii) In sine wave, particles are executing SHM. Therefore, all equations of SHM can be applied for particles also.

   (iii) Relation between v_P and v

   $$v_P = -v \cdot \frac{\partial y}{\partial x}$$

   Here, $\partial y / \partial x$ is the slope of y - x graph when t is kept constant.

5. Phase Difference ($\Delta\phi$)

Case I $\Delta\phi = \omega(t_1 - t_2)$

or $\Delta\phi = \frac{2\pi}{T} \cdot \Delta t$

= phase difference of one particle at a time interval of Δt.

Case II $\Delta\phi = k(x_1 \sim x_2)$

$= \frac{2\pi}{\lambda} \cdot \Delta x$

= phase difference at one time between two particles at a path difference of Δx.

6. Wave Speed

(i) Speed of transverse wave on a stretched wire

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho S}}$$

(ii) Speed of longitudinal wave

$$v = \sqrt{\frac{E}{\rho}}$$

(a) In solids, $E = Y$ = Young's modulus of elasticity

$\therefore$ $$v = \sqrt{\frac{Y}{\rho}}$$

(b) In liquids, $E = B$ = Bulk modulus of elasticity

$\therefore$ $$v = \sqrt{\frac{B}{\rho}}$$

(c) In gases, $E =$ adiabatic bulk modulus $= \gamma p$

$\therefore$ $$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

7. Energy Density (u), Power (P) and Intensity (I) in sine Wave

(i) Energy density, $u = \frac{1}{2}\rho\omega^2 A^2$ = energy of oscillation per unit volume

(ii) Power, $P = \frac{1}{2}\rho\omega^2 A^2 S v$ = energy transferred per unit time

(iii) Intensity, $I = \frac{1}{2}\rho\omega^2 A^2 v$ = energy transferred per unit time per unit area

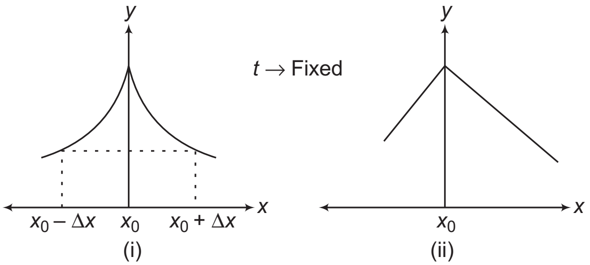
Solved Examples

TYPED PROBLEMS

Type 1. *Based on symmetry of a wave pulse*

Concept

To check symmetry of a wave pulse or wave velocity, always concentrate on maximum or minimum value of y. For example,

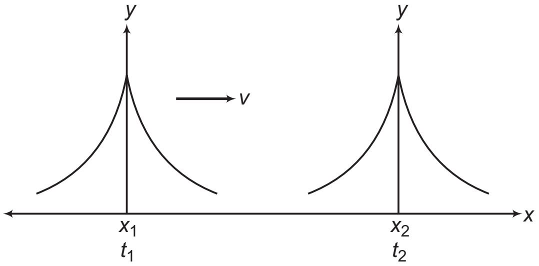


Wave pulse shown in Fig. (i) is symmetric but the wave pulse shown in Fig. (ii) is asymmetric. Now, the question is how will you check whether the pulse is symmetric or not.

The answer is

In both cases maximum value of y at the given time (say t_0) is at $x = x_0$.

Now, if $y(t = t_0, x = x_0 + \Delta x) = y(t = t_0, x = x_0 - \Delta x)$, then the pulse is symmetric otherwise not. So, at a given time (or any other time of your choice) first of all you have to find that x coordinate where you are getting the maximum value of y.



Further, suppose that maximum value of y is at x_1 at time t_1 and at x_2 at time $t_2 (> t_1)$. Then, from the figure we can see that peak of the wave pulse is travelling towards positive x-direction. In time $(t_2 - t_1)$, it has travelled a distance $(x_2 - x_1)$ along positive x-direction. Hence, the wave velocity is

$$v = +\frac{(x_2 - x_1)}{(t_2 - t_1)}$$

Example 1 *$y(x, t) = \dfrac{0.8}{[(4x + 5t)^2 + 5]}$ represents a moving pulse where x and y are in metre and t in second. Then, choose the correct alternative(s):* (JEE 1999)

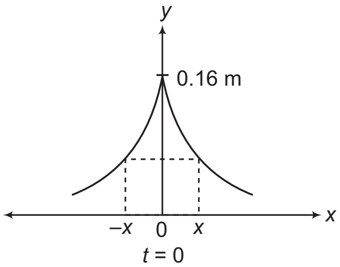
(a) pulse is moving in positive x-direction

(b) in 2 s it will travel a distance of 2.5 m

(c) its maximum displacement is 0.16 m

(d) it is a symmetric pulse

Solution (b), (c) and (d) are correct options.



The shape of pulse at $x = 0$ and $t = 0$ would be as shown in figure

$$y(0, 0) = \frac{0.8}{5} = 0.16 \text{ m}$$

From the equation it is clear that $y_{\max} = 0.16$ m

Pulse will be symmetric (symmetry is checked about $y_{\max}$) if

At $t = 0$; $\quad y(x) = y(-x)$

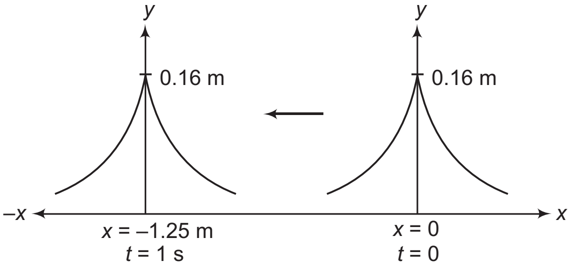
From the given equation

$$\left.\begin{aligned} y(x) &= \frac{0.8}{16x^2 + 5} \\ \text{and} \quad y(-x) &= \frac{0.8}{16x^2 + 5} \end{aligned}\right\} \text{at } t = 0$$

or $$y(x) = y(-x)$$

Therefore, pulse is symmetric.

Speed of pulse



At $t = 1$ s and $x = -1.25$ m, value of y is again 0.16 m, i.e. pulse has travelled a distance of 1.25 m in 1 second in negative x-direction or we can say that the speed of pulse is 1.25 m/s and it is travelling in negative x-direction. Therefore, it will travel a distance of 2.5 m in 2 seconds.

Type 2. *To find wave velocity from two $y(x)$ equations given at two different times.*

Concept

In $y(x,t)$ equation, if value of t is substituted then the equation left is $y(x)$ equation.

In this type of problem, $y(x)$ equation will be given at two different times and we have to find the wave velocity.

How to Solve?

- At the given time find the x-coordinates where you are getting the maximum (or minimum) value of y. From these two x-coordinates and two times we can calculate the wave velocity by the method discussed in Type 1.

Example 2 *At time $t=0$, $y(x)$ equation of a wave pulse is*

$$y=\frac{10}{2+(x-2)^2}$$

and at $t=2\,\text{s}$, $y(x)$ equation of the same wave pulse is

$$y=\frac{10}{2+(x+4)^2}$$

Here, y is in mm and x in metres. Find the wave velocity.

Solution From the given $y(x)$ equations at two different times we can see that value of y is maximum $\left(=\frac{10}{2}\text{ or }5\text{ mm}\right)$ at $x=2\,\text{m}$ at time $t=0$ and at $x=-4\,\text{m}$ at time $t=2\,\text{s}$.

So, peak of the wave pulse has travelled a distance of 6 m (from $x=2$ m to $x=-4$ m) in 2 s along negative x-direction.

Hence, the wave velocity is

$$v=-\frac{6}{2}=-3\text{ m/s}$$ **Ans.**

Type 3. *To make complete $y(x,t)$ function if $y(x)$ function at some given time and wave velocity are given*

Concept

(i) Wave speed, $v=\dfrac{\text{coefficient of } t}{\text{coefficient of } x}$

$\therefore$ Coefficient of $t=(v)(\text{coefficient of } x)$

(ii) Sign of coefficient of t will be opposite to the sign of coefficient of x if wave travels along positive x-direction and they are of same sign if the wave travels along negative x-direction.

Example 3 *A wave is travelling along positive x-direction with velocity 2 m/s. Further, y(x) equation of the wave pulse at t = 0 is*

$$y = \frac{10}{2 + (2x + 4)^2}$$

(a) From the given information make complete y(x, t) equation.

(b) Find y(x) equation at t = 1 s

Solution (a) Here, coefficient to x is 2.

Wave speed is 2 m/s. Therefore, coefficient of $t = v$ (coefficient of x) $= 2 \times 2 = 4$ units.

Further, coefficient of x is positive and the wave is travelling along positive x-direction. Hence, coefficient of t must be negative.

Now, suppose the $y(x, t)$ function is

$$y = \frac{10}{2 + (2x - 4t + \alpha)^2} \qquad \text{...(i)}$$

Here, α is a constant.

At time $t = 0$, Eq. (i) becomes

$$y = \frac{10}{2 + (2x + \alpha)^2}$$

and the given function is

$$y = \frac{10}{2 + (2x + 4)^2}$$

Therefore, the value of α is 4.

Substituting in Eq. (i), we have

$$y = \frac{10}{2 + (2x - 4t + 4)^2}$$

(b) At $t = 1$ s

$$y = \frac{10}{2 + (2x - 4 \times 1 + 4)^2}$$

or $$y = \frac{10}{2 + 4x^2}$$ **Ans.**

Type 4. *Based on transverse wave speed on a string*

Concept

(i) We know that transverse wave speed is given by

$$v = \sqrt{\frac{T}{\mu}} \quad \text{or} \quad \sqrt{\frac{T}{\rho S}}$$

(ii) If tension is uniform, then v is also uniform and we can calculate the time taken by the wave pulse in travelling from one point to another point by the direct relation

$$t = \frac{\text{distance}}{\text{speed}}$$

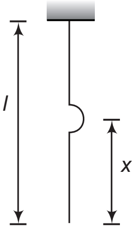
If tension is non-uniform, then v will be non-uniform and in that case time can be obtained by integration.

Example 4 *A uniform rope of mass 0.1 kg and length 2.45 m hangs from a ceiling.*

(a) Find the speed of transverse wave in the rope at a point 0.5 m distant from the lower end.

(b) Calculate the time taken by a transverse wave to travel the full length of the rope.

Solution (a) As the string has mass and it is suspended vertically, tension in it will be different at different points. For a point at a distance x from the free end, tension will be due to the weight of the string below it. So, if m is the mass of string of length l, the mass of length x of the string will be $\left(\frac{m}{l}\right)x$.



$$\therefore \qquad T = \left(\frac{m}{l}\right)xg = \mu xg \qquad \left(\frac{m}{l} = \mu\right)$$

$$\therefore \qquad \frac{T}{\mu} = xg$$

$$\text{or} \qquad v = \sqrt{\frac{T}{\mu}} = \sqrt{xg} \qquad \text{...(i)}$$

$$\text{At } x = 0.5 \text{ m}, \qquad v = \sqrt{0.5 \times 9.8}$$

$$= 2.21 \text{ m/s}$$

(b) From Eq. (i), we can see that velocity of the wave is different at different points. So, if at point x the wave travels a distance dx in time dt, then

$$dt = \frac{dx}{v} = \frac{dx}{\sqrt{gx}}$$

$$\therefore \qquad \int_0^t dt = \int_0^l \frac{dx}{\sqrt{gx}}$$

$$\text{or} \qquad t = 2\sqrt{\frac{l}{g}} = 2\sqrt{\frac{2.45}{9.8}}$$

$$= 1.0 \text{ s}$$

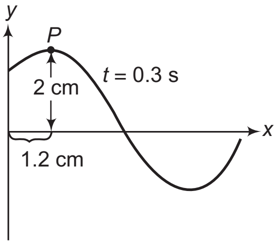
Ans.

Type 5. *To write the equation corresponding to given y - x graph (or a snapshot) at a given time*

Concept

From the given y - x graph, we can easily determine A, ω and k. Secondly, we have to check whether the wave is travelling along positive x -direction or negative x -direction. Because this factor will decide whether, ωt and kx should be of same sign or opposite sign.

Example 5 *Figure shows a snapshot of a sinusoidal travelling wave taken at* $t = 0.3$ *s. The wavelength is 7.5 cm and the amplitude is 2 cm. If the crest P was at* $x = 0$ *at* $t = 0$, *write the equation of travelling wave.*



Solution Given, $A = 2$ cm, $\lambda = 7.5$ cm

$$\therefore \qquad k = \frac{2\pi}{\lambda} = 0.84 \text{ cm}^{-1}$$

The wave has travelled a distance of 1.2 cm in 0.3 s. Hence, speed of the wave,

$$v = \frac{1.2}{0.3} = 4 \text{ cm/s}$$

$\therefore$ Angular frequency $\omega = (v)\,(k) = 3.36$ rad/s

Since the wave is travelling along positive x-direction and crest (maximum displacement) is at $x = 0$ at $t = 0$, we can write the wave equation as

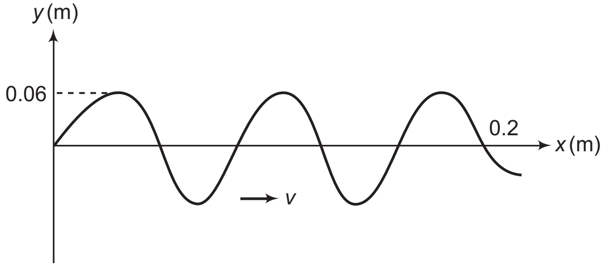
$$y\,(x, t) = A \cos\,(kx - \omega t)$$

or
$$y\,(x, t) = A \cos\,(\omega t - kx) \quad \text{as} \quad \cos\,(-\theta) = \cos\theta$$

Therefore, the desired equation is

$$y\,(x, t) = (2 \text{ cm}) \cos\,[(0.84 \text{ cm}^{-1})\,x - (3.36 \text{ rad/s})\,t] \text{ cm}$$ **Ans.**

Example 6 *For the wave shown in figure, write the equation of this wave if its position is shown at* $t = 0$. *Speed of wave is* $v = 300$ *m/s.*



Solution The amplitude $A = 0.06$ m

$$\frac{5}{2}\lambda = 0.2 \text{ m}$$

$$\therefore \qquad \lambda = 0.08 \text{ m}$$

$$f = \frac{v}{\lambda} = \frac{300}{0.08} = 3750 \text{ Hz}$$

$$k = \frac{2\pi}{\lambda} = 78.5 \text{ m}^{-1}$$

and $\omega = 2\pi f = 23562$ rad/s

At $t = 0$, $x = 0$, $\frac{\partial y}{\partial x}$ = positive

and the given curve is a sine curve.

Hence, equation of wave travelling in positive x-direction should have the form,

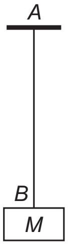
$$y(x, t) = A \sin (kx - \omega t)$$

Substituting the values, we have

$$y(x, t) = (0.06 \text{ m}) \sin [(78.5 \text{ m}^{-1})\, x - (23562 \text{ s}^{-1})\, t] \text{ m}$$ **Ans.**

Miscellaneous Examples

Example 7 *A block of mass $M = 2$ kg is suspended from a string AB of mass 6 kg as shown in figure. A transverse wave pulse of wavelength λ_0 is produced at point B. Find its wavelength while reaching at point A.*



Concept

While moving from B to A, tension will increase. So, wave speed will also increase (as $v = \sqrt{T/\mu}$). Frequency will remain unchanged because it depends on source. Therefore, wavelength will also increase (as $\lambda = \frac{v}{f}$ or $\lambda \propto v$).

Solution $\lambda = \frac{v}{f} = \frac{\sqrt{T/\mu}}{f}$

or $\lambda \propto \sqrt{T}$ (as μ and f are constants)

$\therefore$ $$\frac{\lambda_B}{\lambda_A} = \sqrt{\frac{T_B}{T_A}}$$

$\therefore$ $$\lambda_B = \left(\sqrt{\frac{T_B}{T_A}}\right) \lambda_A$$

$$= \left[\sqrt{\frac{(2+6)g}{2g}}\right]$$

$$= 2\lambda$$ **Ans.**

Example 8 *A wave moves with speed 300 m/s on a wire which is under a tension of 500 N. Find how much tension must be changed to increase the speed to 312 m/s?*

Solution Speed of a transverse wave on a wire is,

$$v = \sqrt{\frac{T}{\mu}} \qquad \text{...(i)}$$

Differentiating with respect to tension, we have

$$\frac{dv}{dT} = \frac{1}{2\sqrt{\mu T}} \qquad \text{...(ii)}$$

Dividing Eq. (ii) by Eq. (i), we have

$$\frac{dv}{v} = \frac{1}{2}\frac{dT}{T} \quad \text{or} \quad dT = (2T)\frac{dv}{v}$$

Substituting the proper values, we have

$$dT = \frac{(2)(500)(312-300)}{300}$$

$$= 40 \text{ N}$$ **Ans.**

i.e. tension should be increased by 40 N.

Example 9 *For a wave described by $y = A\sin(\omega t - kx)$, consider the following points (a) $x = 0$ (b) $x = \frac{\pi}{4k}$ (c) $x = \frac{\pi}{2k}$ and (d) $x = \frac{3\pi}{4k}$.*

For a particle at each of these points at $t = 0$, describe whether the particle is moving or not and in what direction and describe whether the particle is speeding up, slowing down or instantaneously not accelerating?

Solution $y = A\sin(\omega t - kx)$

Particle velocity $v_P(x, t) = \frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx)$ and particle acceleration

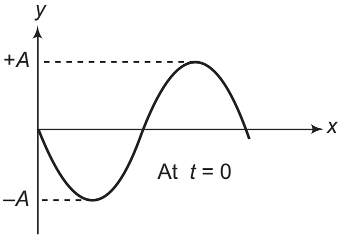
$$a_P(x, t) = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx)$$

(a) $t = 0, \ x = 0: \ v_P = +\omega A$ and $a_P = 0$

i.e. particle is moving **upwards** but its **acceleration is zero**.

Note *Direction of velocity can also be obtained in a different manner as under,*

At $t = 0, \ y = A\sin(-kx) = -A\sin kx$



i.e. y-x graph is as shown in figure. At $x = 0$, slope is negative. Therefore, particle velocity is positive ($v_P = -v \times$ slope), as the wave is travelling along positive x-direction.

(b) $\boldsymbol{t = 0, x = \frac{\pi}{4k}}$ $\qquad x = \frac{\pi}{4k}$

$\therefore$ $$kx = \frac{\pi}{4}$$

$$v_P = \omega A \cos\left(-\frac{\pi}{4}\right) = +\frac{\omega A}{\sqrt{2}}$$

and $$a_P = -\omega^2 A \sin\left(-\frac{\pi}{4}\right) = +\frac{\omega^2 A}{\sqrt{2}}$$

Velocity of particle is positive, i.e. the particle is moving **upwards** (along positive y-direction). Further v_P and a_P are in the same direction (both are positive). Hence, the particle is **speeding up**.

(c) $\boldsymbol{t = 0, x = \frac{\pi}{2k}}$ $\qquad x = \frac{\pi}{2k} \quad \therefore \quad kx = \frac{\pi}{2}$

$$v_P = \omega A \cos(-\pi/2) = 0$$

$$a_P = -\omega^2 A \sin(-\pi/2) = \omega^2 A$$

i.e. particle is **stationary** or at its extreme position ($y = -A$). So, it is **speeding up** at this instant.

(d) $\boldsymbol{t = 0, \; x = \frac{3\pi}{4k}}$ $\qquad x = \frac{3\pi}{4k} \quad \therefore \quad kx = \frac{3\pi}{4}$

$\therefore$ $$v_P = \omega A \cos\left(-\frac{3\pi}{4}\right) = -\frac{\omega A}{\sqrt{2}}$$

$$a_P = -\omega^2 A \sin\left(-\frac{3\pi}{4}\right) = +\frac{\omega^2 A}{\sqrt{2}}$$

Velocity of particle is negative, i.e. the particle is moving **downwards**. Further v_P and a_P are in opposite directions, i.e. the particle is **slowing down**.

Example 10 *A thin string is held at one end and oscillates vertically so that,*

$$y\,(x = 0, t) = 8 \sin 4t \; (cm)$$

Neglect the gravitational force. The string's linear mass density is 0.2 kg/m and its tension is 1 N. The string passes through a bath filled with 1 kg water. Due to friction heat is transferred to the bath. The heat transfer efficiency is 50%. Calculate how much time passes before the temperature of the bath rises one degree kelvin?

Solution Comparing the given equation with equation of a travelling wave,

$$y = A \sin(kx \pm \omega t) \quad \text{at} \quad x = 0 \quad \text{we find,}$$

$$A = 8 \text{ cm} = 8 \times 10^{-2} \text{ m}$$

$$\omega = 4 \text{ rad/s}$$

Speed of travelling wave, $$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{1}{0.2}} = 2.236 \text{ m/s}$$

Further, $\rho S = \mu = 0.2$ kg/m

The average power over a period is

$$P = \frac{1}{2}(\rho S)\,\omega^2 A^2 v$$

Substituting the values, we have

$$P = \frac{1}{2}(0.2)(4)^2(8\times10^{-2})^2(2.236)$$
$$= 2.29\times10^{-2}\text{ J/s}$$

The power transferred to the bath is,

$$P' = 0.5\,P = 1.145\times10^{-2}\text{ J/s}$$

Now let, it takes t second to raise the temperature of 1 kg water by 1 degree kelvin. Then

$$P't = ms\Delta t$$

Here, s = specific heat of water $= 4.2\times10^3$ J/kg-° K

$$\therefore \qquad t = \frac{ms\Delta t}{P'} = \frac{(1)(4.2\times10^3)(1)}{1.145\times10^{-2}}$$
$$= 3.6\times10^5\text{ s} \approx 4.2\text{ day}$$ **Ans.**

Example 11 *Consider a wave propagating in the negative x-direction whose frequency is 100 Hz. At $t = 5$ s, the displacement associated with the wave is given by*

$$y = 0.5\cos(0.1\,x)$$

where x and y are measured in centimetres and t in seconds. Obtain the displacement (as a function of x) at $t = 10$ s. What is the wavelength and velocity associated with the wave?

Solution A wave travelling in negative x-direction can be represented as

$$y(x,t) = A\cos(kx + \omega t + \phi)$$

At $t = 5$ s,

$$y(x, t = 5) = A\cos(kx + 5\omega + \phi)$$

Comparing this with the given equation,

We have,

$$A = 0.5\text{ cm},\ k = 0.1\text{ cm}^{-1}$$

and $$5\omega + \phi = 0 \qquad \text{...(i)}$$

Now, $$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.1} = (20\pi)\text{ cm}$$ **Ans.**

$$\omega = 2\pi f = (200\pi)\text{ rad/s}$$

$$\therefore \qquad v = \frac{\omega}{k} = \frac{200\pi}{0.1} = (2000\,\pi)\text{ cm/s}$$ **Ans.**

From Eq. (i), $$\phi = -5\omega$$

At $t = 10$ s,

$$y(x, t = 10) = 0.5\cos(0.1\,x + 10\,\omega - 5\,\omega)$$
$$= 0.5\cos(0.1\,x + 5\,\omega)$$

Substituting $\omega = 200\,\pi$,

$$y(x, t = 10) = 0.5\cos(0.1\,x + 1000\pi)$$
$$= 0.5\cos(0.1\,x)$$ **Ans.**

Example 12 *A simple harmonic wave of amplitude 8 units travels along positive x-axis. At any given instant of time, for a particle at a distance of 10 cm from the origin, the displacement is + 6 units, and for a particle at a distance of 25 cm from the origin, the displacement is + 4 units. Calculate the wavelength.*

Solution $y = A\sin\frac{2\pi}{\lambda}(vt - x)$

or
$$\frac{y}{A} = \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

In the first case,
$$\frac{y_1}{A} = \sin 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$$

Here, $y_1 = +6, \quad A = 8, \quad x_1 = 10$ cm

$$\therefore \qquad \frac{6}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) \qquad \text{...(i)}$$

Similarly in the second case,
$$\frac{4}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) \qquad \text{...(ii)}$$

From Eq. (i),
$$2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) = \sin^{-1}\left(\frac{6}{8}\right) = 0.85 \text{ rad}$$

or
$$\frac{t}{T} - \frac{10}{\lambda} = 0.14 \qquad \text{...(iii)}$$

Similarly from Eq. (ii),
$$2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) = \sin^{-1}\left(\frac{4}{8}\right) = \frac{\pi}{6} \text{ rad}$$

or
$$\frac{t}{T} - \frac{25}{\lambda} = 0.08 \qquad \text{...(iv)}$$

Subtracting Eq. (iv) from Eq. (iii), we get
$$\frac{15}{\lambda} = 0.06$$

$$\therefore \qquad \lambda = 250 \text{ cm}$$ **Ans.**

Example 13 *A wave pulse on a horizontal string is represented by the function*
$$y(x, t) = \frac{5.0}{1.0 + (x - 2t)^2} \qquad \textit{(CGS units)}$$

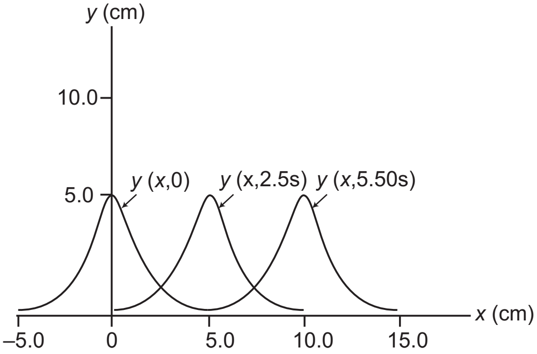
Plot this function at $t = 0, 2.5$ and 5.0 s.

Solution At the given times the function representing the wave pulse is

$$y(x, 0) = \frac{5.0}{1.0 + x^2}$$

$$y(x, 2.5 \text{ s}) = \frac{5.0}{1.0 + (x - 5.0)^2}$$

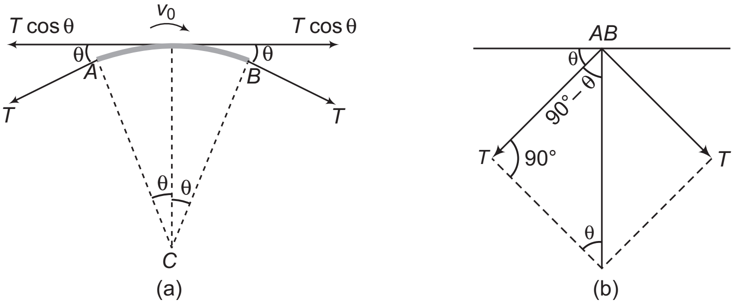
$$y\,(x, 5.0\text{ s}) = \frac{5.0}{1.0 + (x - 10.0)^2}$$



The maximum of $y(x, 0)$ is 5.0 cm; at $t = 0$, it is located at $x = 0$. At $t = 2.5$ and 5.0 s, the maximum of the pulse has moved to $x = 5.0$ and 10.0 cm, respectively. So, in each 2.5 s time interval, the pulse moves 5.0 cm in the positive x-direction. Its velocity is therefore +2.0 cm/s.

Example 14 *A uniform circular hoop of string is rotating clockwise in the absence of gravity. The tangential speed is v_0. Find the speed of the wave travelling on this string.*

Solution



(a) (b)

Let T be the tension in the string. Consider a small circular element AB of the string of length,

$$\Delta l = R\,(2\theta) \qquad (R = \text{radius of hoop})$$

The components of tension $T\cos\theta$ are equal and opposite and thus cancel out. The components towards centre C (i.e. $T\sin\theta$) provides the necessary centripetal force to element AB.

$$\therefore \qquad 2T\sin\theta = \frac{mv_0^2}{R} \qquad \text{...(i)}$$

Here, $$m = \mu\Delta l = 2\mu R\theta \qquad \left(\mu = \frac{\text{mass}}{\text{length}}\right)$$

As θ is small, $\sin\theta \approx \theta$

Substituting in Eq. (i), we get

$$2T\theta = \frac{2\mu R\theta\, v_0^2}{R}$$

or
$$\frac{T}{\mu} = v_0^2$$

or
$$\sqrt{\frac{T}{\mu}} = v_0 \qquad \text{...(ii)}$$

Speed of wave travelling on this string,
$$v = \sqrt{\frac{T}{\mu}} = v_0 \qquad \text{[from Eq. (ii)]}$$

i.e. the velocity of the transverse wave along the hoop of string is the same as the velocity of rotation of the hoop, *viz.* v_0. **Ans.**

Example 15 *A sinusoidal wave travelling in the positive direction on a stretched string has amplitude* 2.0 *cm, wavelength* 1.0 *m and wave velocity* 5.0 *m/s. At* $x = 0$ *and* $t = 0$, *it is given that* $y = 0$ *and* $\frac{\partial y}{\partial t} < 0$. *Find the wave function* $y(x, t)$.

Solution We start with a general equation for a rightward moving wave,
$$y(x, t) = A \sin (kx - \omega t + \phi)$$

The amplitude given is
$$A = 2.0 \text{ cm} = 0.02 \text{ m}$$

The wavelength is given as
$$\lambda = 1.0 \text{ m}$$

∴ Angular wave number,
$$k = \frac{2\pi}{\lambda} = (2\pi)\,\text{m}^{-1}$$

Angular frequency,
$$\omega = vk = (10\,\pi) \text{ rad/s}$$

∴
$$y(x, t) = (0.02) \sin [2\pi\,(x - 5.0\,t) + \phi]$$

We are given that for $x = 0$, $t = 0$,
$$y = 0$$

and
$$\frac{\partial y}{\partial t} < 0$$

i.e.
$$0.02 \sin \phi = 0 \qquad (\text{as } y = 0)$$

and
$$-0.2\,\pi \cos \phi < 0$$

From these conditions, we may conclude that
$$\phi = 2n\pi \qquad \text{where } n = 0, 2, 4, 6 \ldots$$

Therefore,
$$y(x, t) = (0.02 \text{ m}) \sin [(2\pi \text{ m}^{-1})\,x - (10\pi \text{ rad s}^{-1})\,t] \text{ m}$$
Ans.

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Mechanical transverse waves can't travel in gaseous medium.

Reason : They do not possess modulus of rigidity.

2. Assertion : Surface waves are neither transverse nor longitudinal.

Reason : In surface wave particles undergo circular motion.

3. Assertion : Two equations of wave are $y_1 = A\sin(\omega t - kx)$ and $y_2 = A\sin(kx - \omega t)$. These two waves have a phase difference of π.

Reason : They are travelling in opposite directions.

4. Assertion : Wave speed is given by $v = f\lambda$. If frequency f is doubled, v will become two times.

Reason : For given conditions of medium wave speed remains constant.

5. Assertion : On moon you cannot hear your friend standing at some distance from you.

Reason : There is a vacuum on moon.

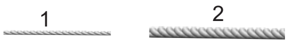
6. Assertion : Wave number is the number of waves per unit length.

Reason : Wave number $= \dfrac{1}{\lambda}$.

7. Assertion : Electromagnetic waves do not require medium for their propagation.

Reason : They can't travel in a medium.

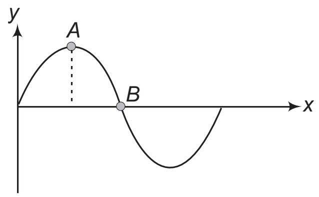
8. Assertion : Two strings shown in figure have the same tension. Speed of transverse waves in string-1 will be more.



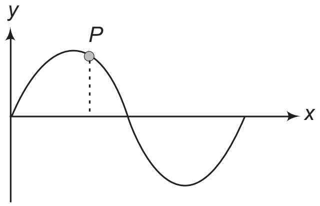
Reason : $v \propto \dfrac{1}{\sqrt{\mu}}$, Here μ is mass per unit length of string.

9. Assertion : y-x graph of a transverse wave on a string is as shown in figure. At point A potential energy and kinetic energy both are minimum.

Reason : At point B kinetic energy and potential energy both are maximum.



10. **Assertion :** y-x graph of a transverse wave on a string is as shown in figure. At the given instant point P is moving downwards. Hence, we can say that wave is moving towards positive y-direction.



Reason : Particle velocity is given by

$$v_P = -v\frac{\partial y}{\partial x}$$

Objective Questions

1. Equation of progressive wave is given by $y = 4\sin\left[\pi\left(\frac{t}{5} - \frac{x}{9}\right) + \frac{\pi}{6}\right]$, where x and y are in metre.

Then,

(a) $v = 5$ m/s (b) $\lambda = 18$ m
(c) $A = 0.04$ m (d) $f = 50$ Hz

2. The equation of a wave is given by $Y = 5\sin 10\pi(t - 0.01x)$ along the x-axis. (All the quantities are expressed in SI units). The phase difference between the points separated by a distance of 10 m along x-axis is

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) $\frac{\pi}{4}$

3. The displacement function of a wave travelling along positive x-direction is $y = \frac{1}{2 + 3x^2}$ at $t = 0$ and by $y = \frac{1}{2 + 3(x-2)^2}$ at $t = 2$ s, where y and x are in metre. The velocity of the wave is

(a) 2 m/s (b) 0.5 m/s
(c) 1 m/s (d) 3 m/s

4. The angle between wave velocity and particle velocity in a travelling wave may be

(a) zero (b) $\frac{\pi}{2}$ (c) π (d) All of these

5. A source oscillates with a frequency 25 Hz and the wave propagates with 300 m/s. Two points A and B are located at distances 10 m and 16 m away from the source. The phase difference between A and B is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

6. The equation of a transverse wave propagating in a string is given by

$$y = 0.02\sin(x + 30t)$$

where, x and y are in metre and t is in second.
If linear density of the string is 1.3×10^{-4} kg/m, then the tension in the string is

(a) 0.12 N (b) 1.2 N (c) 12 N (d) 120 N

7. A harmonic oscillator vibrates with amplitude of 4 cm and performs 150 oscillations in one minute. If the initial phase is 45° and it starts moving away from the origin, then the equation of motion is

(a) $0.04 \sin\left(5\pi t + \frac{\pi}{4}\right)$
(b) $0.04 \sin\left(5\pi t - \frac{\pi}{4}\right)$
(c) $0.04 \sin\left(4\pi t + \frac{\pi}{4}\right)$
(d) $0.04 \sin\left(4\pi t - \frac{\pi}{4}\right)$

Subjective Questions

1. A certain transverse wave is described by

$$y(x, t) = (6.50 \text{ mm}) \cos 2\pi \left(\frac{x}{28.0 \text{ cm}} - \frac{t}{0.0360 \text{ s}}\right).$$

Determine the wave's

(a) amplitude
(b) wavelength
(c) frequency
(d) speed of propagation and
(e) direction of propagation.

2. For the wave $y = 5 \sin 30\pi\, [t - (x/240)]$, where x and y are in cm and t is in seconds, find the

(a) displacement when $t = 0$ and $x = 2$ cm
(b) wavelength
(c) velocity of the wave and
(d) frequency of the wave

3. The displacement of a wave disturbance propagating in the positive x-direction is given by

$$y = \frac{1}{1 + x^2} \text{ at } t = 0 \quad \text{and} \quad y = \frac{1}{1 + (x - 1)^2} \quad \text{at} \quad t = 2 \text{ s}$$

where, x and y are in metre. The shape of the wave disturbance does not change during the propagation. What is the velocity of the wave?

4. A travelling wave pulse is given by $y = \dfrac{10}{5 + (x + 2t)^2}$

Here, x and y are in metre and t in second. In which direction and with what velocity is the pulse propagating? What is the amplitude of pulse?

5. Is there any relationship between wave speed and the maximum particle speed for a wave travelling on a string? If so, what is it?

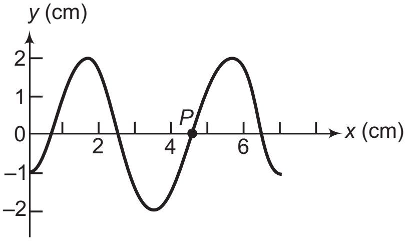
6. Calculate the velocity of a transverse wave along a string of length 2 m and mass 0.06 kg under a tension of 500 N.

7. Calculate the speed of a transverse wave in a wire of 1.0 mm^2 cross-section under a tension of 0.98 N. Density of the material of wire is $9.8 \times 10^3 \text{ kg/m}^3$

8. If at $t = 0$, a travelling wave pulse on a string is described by the function,

$$y = \frac{10}{(x^2 + 2)}$$

Here, x and y are in metre and t in second. What will be the wave function representing the pulse at time t, if the pulse is propagating along positive x-axis with speed 2 m/s?



9. Consider a sinusoidal travelling wave shown in figure. The wave velocity is + 40 cm/s.
Find
(a) the frequency
(b) the phase difference between points 2.5 cm apart
(c) how long it takes for the phase at a given position to change by 60°
(d) the velocity of a particle at point P at the instant shown.

10. The equation of a travelling wave is

$$y(x, t) = 0.02 \sin\left(\frac{x}{0.05} + \frac{t}{0.01}\right) \text{ m}$$

Find
(a) The wave velocity and
(b) the particle velocity at $x = 0.2$ m and $t = 0.3$ s.

Given, $\cos\theta = -0.85$, where, $\theta = 34$ rad

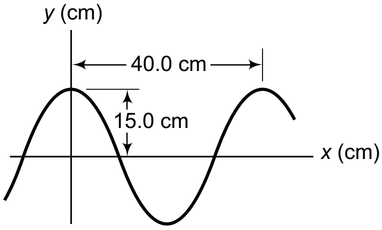
11. Transverse waves on a string have wave speed 12.0 m/s, amplitude 0.05 m and wavelength 0.4 m. The waves travel in the $+x$-direction and at $t = 0$ the $x = 0$ end of the string has zero displacement and is moving upwards.
(a) Write a wave function describing the wave.
(b) Find the transverse displacement of a point at $x = 0.25$ m at time $t = 0.15$ s.
(c) How much time must elapse from the instant in part (b) until the point at $x = 0.25$ m has zero displacement?

12. A wave is described by the equation

$$y = (1.0 \text{ mm}) \sin\pi\left(\frac{x}{2.0 \text{ cm}} - \frac{t}{0.01 \text{ s}}\right)$$

(a) Find the time period and the wavelength.
(b) Write the equation for the velocity of the particles. Find the speed of the particle at $x = 1.0$ cm at time $t = 0.01$ s.
(c) What are the speeds of the particles at $x = 3.0$ cm, 5.0 cm and 7.0 cm at $t = 0.01$ s?
(d) What are the speeds of the particles at $x = 1.0$ cm at $t = 0.011$, 0.012 and 0.013 s?

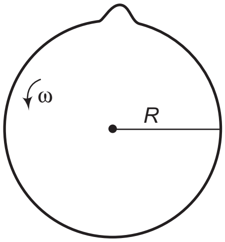
13. A sinusoidal wave travelling in the positive x-direction has an amplitude of 15.0 cm, a wavelength of 40.0 cm and a frequency of 8.00 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15.0 cm as shown in figure.



(a) Find the angular wave number k, period T, angular frequency ω and speed v of the wave.
(b) Write a general expression for the wave function.

14. A flexible steel cable of total length L and mass per unit length μ hangs vertically from a support at one end. (a) Show that the speed of a transverse wave down the cable is $v = \sqrt{g(L-x)}$, where x is measured from the support. (b) How long will it take for a wave to travel down the cable?

15. A loop of rope is whirled at a high angular velocity ω, so that it becomes a taut circle of radius R. A kink develops in the whirling rope.



(a) Show that the speed of the kink in the rope is $v = \omega R$.
(b) Under what conditions does the kink remain stationary relative to an observer on the ground?

16. A non-uniform wire of length L and mass M has a variable linear mass density given by $\mu = kx$, where x is distance from one end of wire and k is a constant. Find the time taken by a pulse starting at one end to reach the other end when the tension in the wire is T.

LEVEL 2

Single Correct Option

1. The speed of propagation of a wave in a medium is 300 m/s. The equation of motion of point at $x = 0$ is given by $y = 0.04 \sin 600 \pi t$ (metre). The displacement of a point $x = 75$ cm at $t = 0.01$ s is
(a) 0.02 m (b) 0.04 m (c) zero (d) 0.028 m

2. A 100 Hz sinusoidal wave is travelling in the positive x-direction along a string with a linear mass density of 3.5×10^{-3} kg/m and a tension of 35 N. At time $t = 0$, the point $x = 0$, has maximum displacement in the positive y-direction. Next when this point has zero displacement, the slope of the string is $\pi/20$. Which of the following expression represent (s) the displacement of string as a function of x (in metre) and t (in second)
(a) $y = 0.025 \cos (200\pi t - 2\pi x)$ (b) $y = 0.5 \cos (200\pi t - 2\pi x)$
(c) $y = 0.025 \cos (100\pi t - 10\pi x)$ (d) $y = 0.5 \cos (100\pi t - 10\pi x)$

3. Vibrations of period 0.25 s propagate along a straight line at a velocity of 48 cm/s. One second after the emergence of vibrations at the initial point, displacement of the point, 47 cm from it is found to be 3 cm. [Assume that at initial point particle is in its mean position at $t = 0$ and moving upwards]. Then,
(a) amplitude of vibrations is 6 cm (b) amplitude of vibrations is $3\sqrt{2}$ cm
(c) amplitude of vibrations is 3 cm (d) None of these

4. Transverse waves are generated in two uniform steel wires A and B by attaching their free ends to a fork of frequency 500 Hz. The diameter of wire A is half that of B and tension in wire A is half the tension in wire B. What is the ratio of velocities of waves in A and B?
(a) 1 : 2 (b) $\sqrt{2} : 1$ (c) 2 : 1 (d) $1 : \sqrt{2}$

5. The frequency of A note is 4 times that of B note. The energies of two notes are equal. The amplitude of B note as compared to that of A note will be

(a) double
(b) equal
(c) four times
(d) eight times

6. If at $t = 0$, a travelling wave pulse on a string is described by the function.

$$y = \frac{6}{x^2 + 3}$$

What will be the wave function representing the pulse at time t, if the pulse is propagating along positive x-axis with speed 4 m/s?

(a) $y = \dfrac{6}{(x + 4t)^2 + 3}$
(b) $y = \dfrac{6}{(x - 4t)^2 + 3}$
(c) $y = \dfrac{6}{(x - t)^2}$
(d) $y = \dfrac{6}{(x - t)^2 + 12}$

More than One Correct Options

1. A transverse wave travelling on a stretched string is represented by the equation $y = \dfrac{2}{(2x - 6.2\,t)^2 + 20}$. Then,

(a) velocity of the wave is 3.1 m/s
(b) amplitude of the wave is 0.1 m
(c) frequency of the wave is 20 Hz
(d) wavelength of the wave is 1 m

2. For energy density, power and intensity of any wave choose the correct options.

(a) u = energy density $= \dfrac{1}{2}\rho\omega^2A^2$
(b) P = power $= \dfrac{1}{2}\rho\,\omega^2A^2v$
(c) I = intensity $= \dfrac{1}{2}\rho\,\omega^2A^2Sv$
(d) $I = \dfrac{P}{S}$

3. For the transverse wave equation $y = A\sin(\pi x + \pi t)$, choose the correct options at $t = 0$

(a) points at $x = 0$ and $x = 1$ are at mean positions
(b) points at $x = 0.5$ and $x = 1.5$ have maximum accelerations
(c) points at $x = 0.5$ and $x = 1.5$ are at rest
(d) the given wave is travelling in negative x- direction

4. In the wave equation,

$$y = A\sin\frac{2\pi}{a}(x - bt)$$

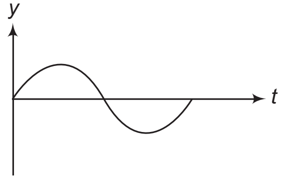
(a) speed of wave is a
(b) speed of wave is b
(c) wavelength of wave is a/b
(d) wavelength of wave is a

5. In the wave equation,

$$y = A\sin 2\pi\left(\frac{x}{a} - \frac{t}{b}\right)$$

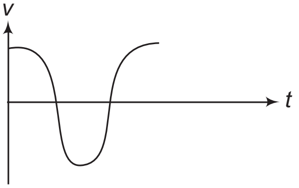
(a) speed of wave is a/b
(b) speed of wave is b/a
(c) wavelength of wave is a
(d) time period of wave is b

6. Corresponding to y-t graph of a transverse harmonic wave shown in figure,

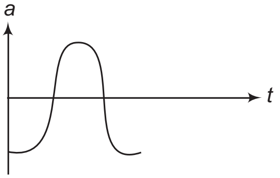


choose the correct options at same position.

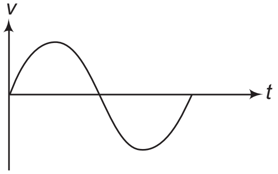
(a)



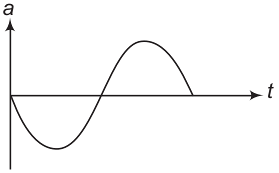
(b)



(c)



(d)



Match the Columns

1. For the wave equation, $y = a \sin (bt - cx)$

Match the following two columns.

Column I	Column II
(a) wave speed	(p) $\frac{b}{2\pi}$
(b) maximum particle speed	(q) $\frac{c}{2\pi}$
(c) wave frequency	(r) $\frac{b}{c}$
(d) wavelength	(s) None of these

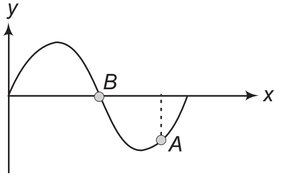
2. For the wave equation,

$$y = (4 \text{ cm}) \sin [\pi t + 2\pi x]$$

Here, t is in second and x in metres.

Column I	Column II
(a) at $x = 0$, particle velocity is maximum at $t =$	(p) 0.5 s
(b) at $x = 0$, particle acceleration is maximum at $t =$	(q) 1.0 s
(c) at $x = 0.5$ m, particle velocity is maximum at $t =$	(r) zero
(d) at $x = 0.5$ m, particle acceleration is maximum at $t =$	(s) 1.5 s

3. y-x graph of a transverse wave at a given instant is shown in figure. Match the following two columns.



Column I	Column II
(a) velocity of particle A	(p) positive
(b) acceleration of particle A	(q) negative
(c) velocity of particle B	(r) zero
(d) acceleration of particle B	(s) can't tell

4. For a travelling wave, match the following two columns.

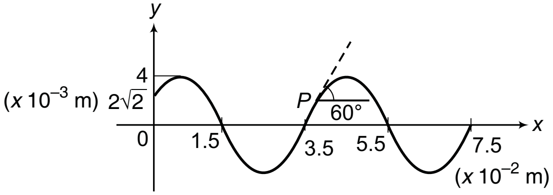
Column I	Column II
(a) energy density	(p) $[ML^2T^{-3}]$
(b) power	(q) $\frac{1}{2}\rho\omega^2A^2Sv$
(c) intensity	(r) $[M^0L^{-1}T]$
(d) wave number	(s) None

5. Match the following two columns.

Column I	Column II
(a) $y = A\sin(\omega t - kx)$	(p) travelling in positive x-direction
(b) $y = A\sin(kx - \omega t)$	(q) travelling in negative x-direction
(c) $y = -A\cos(\omega t + kx)$	(r) at $t = 0$, velocity of particle is positive at $x = 0$
(d) $y = -A\cos(kx - \omega t)$	(s) at $t = 0$ acceleration of particle is positive at $x = 0$

Subjective Questions

1. The figure shows a snap photograph of a vibrating string at $t = 0$. The particle P is observed moving up with velocity $20\sqrt{3}$ cm/s. The tangent at P makes an angle 60° with x-axis.



(a) Find the direction in which the wave is moving.

(b) Write the equation of the wave.

(c) The total energy carried by the wave per cycle of the string. Assuming that the mass per unit length of the string is 50 g/m.

2. A long string having a cross-sectional area 0.80 mm^2 and density 12.5 g/cm^3 is subjected to a tension of 64 N along the positive x-axis. One end of this string is attached to a vibrator at $x = 0$ moving in transverse direction at a frequency of 20 Hz. At $t = 0$, the source is at a maximum displacement $y = 1.0$ cm.

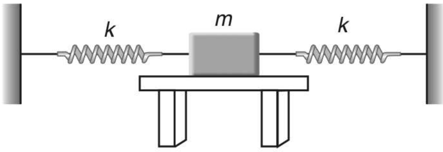
(a) Find the speed of the wave travelling on the string.

(b) Write the equation for the wave.

(c) What is the displacement of the particle of the string at $x = 50$ cm at time $t = 0.05$ s?

(d) What is the velocity of this particle at this instant?

3. One end of each of two identical springs, each of force constant 0.5 N/m are attached on the opposite sides of a wooden block of mass 0.01 kg. The other ends of the springs are connected to separate rigid supports such that the springs are unstretched and are collinear in a horizontal plane. To the wooden piece is fixed a pointer which touches a vertically moving plane paper. The wooden piece kept on a smooth horizontal table is now displaced by 0.02 m along the line of springs and released. If the speed of paper is 0.1 m/s, find the equation of the path traced by the pointer on the paper and the distance between two consecutive maximas on this path.

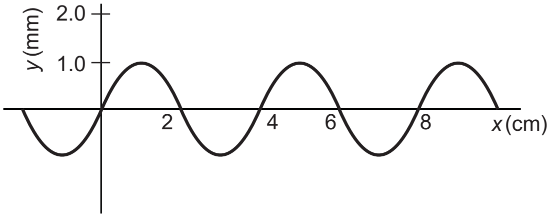


4. A wave pulse is travelling on a string with a speed v towards the positive x- axis. The shape of the string at $t = 0$ is given by $y(x) = A\sin(x/a)$, where A and a are constants.

(a) What are the dimensions of A and a?

(b) Write the equation of the wave for a general time t, if the wave speed is v.

5. Figure shows a plot of the transverse displacement of the particle of a string at $t = 0$ through which a travelling wave is passing in the positive x-direction. The wave speed is 20 cm/s. Find (a) the amplitude (b) the wavelength (c) the wave number and (d) the frequency of the wave.



6. Two wires of different densities but same area of cross-section are soldered together at one end and are stretched to a tension T. The velocity of a transverse wave in the first wire is double of that in the second wire. Find the ratio of the density of the first wire to that of the second wire.

7. Two long strings A and B, each having linear mass density 1.2×10^{-2} kg/m are stretched by different tensions 4.8 N and 7.5 N respectively and are kept parallel to each other with their left ends at $x = 0$. Wave pulses are produced on the strings at the left ends at $t = 0$ on string A and at $t = 20$ ms on string B. When and where will the pulse on B overtake that on A?

8. A sinusoidal transverse wave travels on a string. The string has length 8.00 m and mass 6.00 g. The wave speed is 30.0 m/s and the wavelength is 0.200 m. (a) If the wave is to have an average power of 50.0 W, what must be the amplitude of the wave? (b) For this same string, if the amplitude and wavelength are the same as in part (a) what is the average power for the wave if the tension is increased such that the wave speed is doubled?

9. A uniform rope with length L and mass m is held at one end and whirled in a horizontal circle with angular velocity ω. You can ignore the force of gravity on the rope. Find the time required for a transverse wave to travel from one end of the rope to the other.

Hint : $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

Answers

Introductory Exercise 17.1

2. c/b

3. The converse is not true. Only function (c) satisfies this condition.

4. (a) L, L, T (b) a/T (c) negative x-direction (d) $x = -a$ and $x = -2a$

Introductory Exercise 17.2

1. (a) 5 mm (b) 1 cm^{-1} (c) 2π cm (d) $\frac{30}{\pi}$ Hz (e) $\left(\frac{\pi}{30}\right)$ sec, (f) 60 cm/s

2. (a) 20 ms 4.0 cm (b) zero (c) zero (d) 9.7 cm s^{-1}, 18 cm s^{-1}, 25 cm s^{-1}

Introductory Exercise 17.3

1. (a) negative x-direction (b) 10 ms^{-1}, 20 cm, 50 Hz (c) 0.10 mm, 3.14 cm s^{-1} **2.** (a) 9.4 m/s (b) zero

3. (a) –5 m/s (b) –1.7 m/s **4.** (a) 0.116 m (b) π

Introductory Exercise 17.4

1. 7.5×10^{14} Hz – 4.3×10^{14} Hz **2.** 0.33 m, 0.165 m

Introductory Exercise 17.5

1. 0.05 s **2.** 32 ms^{-1} **3.** 79 ms^{-1} and 63 ms^{-1} **4.** 0.02 s

5. 22 m/s **6.** (a) 16.3 m/s (b) 0.136 m (c) both increase by $\sqrt{2}$ times

Introductory Exercise 17.6

1. $\frac{1}{4\pi}$ W/m^2 **2.** $A \propto \frac{1}{\sqrt{r}}, I \propto \frac{1}{r}$ **3.** 0.58 mJ **4.** 512 W

5. (a) 0.47 W (b) 9.4 mJ **6.** 49 mW

Exercises

LEVEL 1

Assertion and Reason

1. (a) **2.** (b) **3.** (c) **4.** (d) **5.** (a) **6.** (a) **7.** (c) **8.** (d) **9.** (b) **10.** (d)

Objective Questions

1. (b) **2.** (b) **3.** (c) **4.** (d) **5.** (c) **6.** (a) **7.** (a)

Subjective Questions

1. (a) 6.50 mm (b) 28.0 cm (c) 27.8 Hz (d) 7.8 m/s (e) positive x

2. (a) –3.535 cm (b) 16 cm (c) 240 cm/s (d) 15 Hz **3.** 0.5 m/s

4. The pulse is travelling along negative x-axis with velocity 2 m/s. The amplitude of the pulse is 2 m.

5. Yes, $(v_P)_{\max} = (kA)v$ **6.** 129.1 m/s **7.** 10 m/s

8. $y = \dfrac{10}{(x - 2t)^2 + 2}$

9. (a) 10 Hz (b) $\dfrac{5\pi}{4}$ rad (c) $\dfrac{1}{60}$ s (d) –1.26 m/s

10. (a) –5 m/s (b) –1.7 m/s

11. (a) $y(x, t) = (0.05\text{ m}) \sin[(60\pi\text{ s}^{-1})t - (5\pi\text{ m}^{-1})x]$ (b) – 3.54 cm (c) 4.2 ms

12. (a) 20 ms, 4.0 cm (b) $v = -(\pi/10\text{ m/s}) \cos \pi\left[\dfrac{x}{2.0\text{ cm}} - \dfrac{t}{0.01\text{ s}}\right]$, zero

(c) 0 m/s, 0 m/s, 0 m/s (d) 9.7 cm/s, 18 cm/s, 25 cm/s

13. (a) 0.157 rad/cm, 0.125 s, 50.3 rad/s, 320 cm/s (b) $y = (15.0\text{ cm}) \cos(0.157x - 50.3t)$

14. (b) $t = 2\sqrt{\dfrac{L}{g}}$

15. (b) The kink will be stationary with respect to the ground if it moves clockwise with respect to the rope.

16. $\dfrac{2}{3}\sqrt{\dfrac{2ML}{T}}$

LEVEL 2

Single Correct Option

1.(b) **2.**(a) **3.**(a) **4.**(b) **5.**(c) **6.**(b)

More than One Correct Options

1.(a, b) **2.**(a, d) **3.**(all) **4.**(b, d) **5.**(a, c, d) **6.**(a,d)

Match the Columns

1. (a) → r (b) → s (c) → p (d) → s

2. (a) → q,r (b) → p,s (c) → q,r (d) → p,s

3. (a) → s (b) → p (c) → s (d) → r

4. (a) → s (b) → p,q (c) → s (d) → s

5. (a) → p,r (b) → p (c) → q,s (d) → p,s

Subjective Questions

1. (a) Negative x (b) $y = (0.4\text{ cm}) \sin\left(10\pi t + \dfrac{\pi}{2}x + \dfrac{\pi}{4}\right)$ s (c) 1.6×10^{-5} J

2. (a) 80 m/s (b) $y = (1.0\text{ cm}) \cos\left[(40\pi\text{ s}^{-1})t - \left(\dfrac{\pi}{2}\text{ m}^{-1}\right)x\right]$ (c) $\dfrac{1}{\sqrt{2}}$ cm (d) 89 cm/s

3. $y = 0.02 \cos(10t - 100x)$ m, 0.0628 m

4. (a) L, L (b) $y(x, t) = A \sin\left(\dfrac{x - vt}{a}\right)$

5. (a) 1.0 mm (b) 4 cm (c) 1.6 cm^{-1} (d) 5 Hz

6. 0.25

7. at $t = 0.1$ s, at $x = 2.0$ m

8. (a) 7.07 cm (b) 400.0 W

9. $\dfrac{\pi}{\sqrt{2}\omega}$

18

Superposition of Waves

Chapter Contents

18.1 Principle of Superposition
18.2 Resultant Amplitude and Intensity due to Coherent Sources
18.3 Interference
18.4 Standing Wave
18.5 Normal Modes of a String
18.6 Reflection and Transmission of a Wave

18.1 Principle of Superposition

Suppose there are two sources of waves S_1 and S_2.

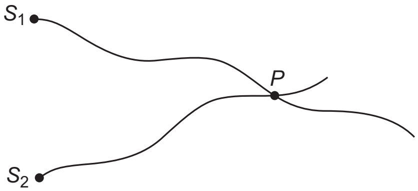


Fig. 18.1

Now, the two waves from S_1 and S_2 meet at some point (say P). Then, according to principle of superposition net displacement at P (from its mean position) at any time is given by

$$\boxed{y = y_1 + y_2}$$

Here, y_1 and y_2 are the displacements of P due to two waves individually.

For example, suppose at 9AM, displacement of P above its mean position should be 6 mm accordingly to wave-1 and the same time its displacement should be 2 mm below its mean position accordingly to wave-2, then at 9AM net displacement of P will be 4 mm above its mean position.

Now, based upon the principle of superposition we have two phenomena in physics, interference and beats. Stationary waves (or standing waves) and Young's double slit experiment (or YDSE) are two examples of interference.

Based on principle of superposition means two or more than two waves meet at one point or several points and at every point net displacement is $y = y_1 + y_2$ or $y = y_1 + y_2 + y_3$ etc.

18.2 Resultant Amplitude and Intensity due to Coherent Sources

In article 18.1, we have seen that the two waves from two sources S_1 and S_2 were meeting at point P. Suppose they meet at P in a phase difference $\Delta\phi$ (or ϕ). If this phase difference remains constant with time, then sources are called coherent, otherwise incoherent.

For sources to be coherent the frequencies (f, ω or T) of the two sources must be same. This can be explained by the following example.

Suppose the phase difference is 0°. It means they are in same phase. Both reach their extremes ($+A$ or $-A$), simultaneously. They cross their mean positions (in the same direction) simultaneously. Now, if we want their phase difference to remain constant or we want that the above situation is maintained all the time, then obviously their time periods (or frequencies) must be same.

Resultant Amplitude

1. Consider the superposition of two sinusoidal waves of same frequency (means sources are coherent) at a point. Let us assume that the two waves are travelling in the same direction with same velocity. The equation of the two waves reaching at a point can be written as

$$y_1 = A_1 \sin (kx - \omega t)$$

and

$$y_2 = A_2 \sin (kx - \omega t + \phi)$$

The resultant displacement of the point where the waves meet is

$$
\begin{aligned}
y &= y_1 + y_2 \\
&= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t + \phi) \\
&= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t)\cos\phi + A_2 \cos(kx - \omega t)\sin\phi \\
&= (A_1 + A_2\cos\phi)\sin(kx - \omega t) + A_2 \sin\phi\cos(kx - \omega t) \\
&= A\cos\theta\sin(kx - \omega t) + A\sin\theta\cos(kx - \omega t)
\end{aligned}
$$

or $$y = A\sin(kx - \omega t + \theta)$$

Here, $$A_1 + A_2\cos\phi = A\cos\theta$$

and $$A_2\sin\phi = A\sin\theta$$

or $$A^2 = (A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2$$

or $$\boxed{A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}} \qquad \text{...(i)}$$

and $$\tan\theta = \frac{A\sin\theta}{A\cos\theta} = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$$

2. The above result can be obtained by graphical method as well. Assume a vector $\mathbf{A}_1$ of length A_1 to represent the amplitude of firstwave.

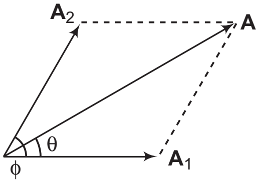


Fig. 18.2

Another vector $\mathbf{A}_2$ of length A_2, making an angle ϕ with $\mathbf{A}_1$ represent the amplitude of second wave. The resultant of $\mathbf{A}_1$ and $\mathbf{A}_2$ represent the amplitude of resulting function y. The angle θ represents the phase difference between the resulting function and the first wave.

Resultant Intensity

In the previous chapter, we have read that intensity of a wave is given by

$$I = \frac{1}{2}\rho\omega^2 A^2 v \quad \text{or} \quad I \propto A^2$$

So, if ρ, ω and v are same for the both interfering waves, then Eq. (i) can also be written as

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi} \qquad \text{...(ii)}$$

Here, proportionality constant ($I \propto A^2$) cancels out on right hand side and left hand side.

Note *(i) Eqs (i) and (ii) are two equations for finding resultant amplitude and resultant intensity at some point due to two coherent sources.*

(ii) In the above equations ϕ is the constant phase difference at that point. As the sources are coherent value of this constant phase difference will be different at different points.

(iii) The special case of above two equations is, when the individual amplitudes (or intensities) are equal.

or $A_1 = A_2 = A_0$ *(say)* $\Rightarrow$ $I_1 = I_2 = I_0$ *(say)*

In this case, Eqs. (i) and (ii) become

$$\boxed{A = 2A_0 \cos\frac{\phi}{2}} \qquad \text{...(iii)}$$

and

$$\boxed{I = 4I_0 \cos^2\frac{\phi}{2}} \qquad \text{...(iv)}$$

(iv) From Eqs. (i) to (iv) we can see that, for given values of A_1, A_2, I_1 *and* I_2 *the resultant amplitude and the resultant intensity are the functions of only* ϕ.

(v) If three or more than three waves (due to coherent sources) meet at some point then there is no direct formula for finding resultant amplitude or resultant intensity. In this case, first of all we will find resultant amplitude by vector method (either by using polygon law of vector addition or component method) and then by the relation $I \propto A^2$, *we can also determine the resultant intensity.*

For example, if resultant amplitude comes out to be $\sqrt{2}$ *times then resultant intensity will become two times.*

18.3 Interference

For interference phenomena to take place, sources must be coherent. So, phase difference at some point should remain constant. The value of this constant phase difference will be different at different points. And since the sources are coherent, therefore the following four equations can be applied for finding resultant amplitude and intensity (in case of two sources)

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} \qquad \text{...(i)}$$

$$I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi \qquad \text{...(ii)}$$

$$A = 2A_0\cos\frac{\phi}{2} \qquad (\text{if } A_1 = A_2 = A_0) \qquad \text{...(iii)}$$

$$I = 4I_0\cos^2\frac{\phi}{2} \qquad (\text{if } I_1 = I_2 = I_0) \qquad \text{...(iv)}$$

For given values of A_1, A_2, I_1 and I_2 the resultant amplitude and resultant intensity are the functions of only ϕ.

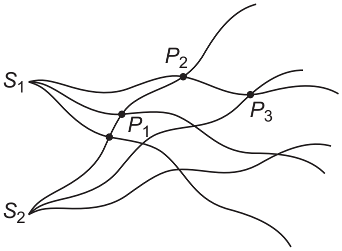


Fig. 18.3

Now, suppose S_1 and S_2 are two coherent sources, then we can see that the two waves are meeting at several points (P_1, P_2, P_3 ... etc). At different points path difference Δx will be different and therefore phase difference $\Delta\phi$ or ϕ will also be different. Because the phase difference depends on the path difference ($\Delta\phi$ or $\phi = \frac{2\pi}{\lambda}\cdot\Delta x$).

And since phase difference at different points is different, therefore from the above four equations we can see that resultants amplitude and intensity will also be different. But whatever is the intensity at some point, it will remain constant at that point because the sources are coherent and the phase difference is constant at that point.

Constructive Interference

These are the points where resultant amplitude or intensity is maximum or

$$A_{\max} = A_1 + A_2 \qquad \text{[from Eq. (i)]}$$

or $$A_{\max} = \pm 2A_0 \qquad \text{[from Eq. (iii)]}$$

and $$I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2 \qquad \text{[from Eq. (ii)]}$$

or $$I_{\max} = 4I_0 \qquad \text{[from Eq. (iv)]}$$

at those points where

$$\cos\phi = +1 \qquad \text{[from Eqs. (i) or (ii)]}$$

or $$\phi = 0, 2\pi, 4\pi, \ldots, 2n\pi \qquad (\text{where, } n = 0, 1, 2)$$

$\therefore$ $$\Delta x = 0, \lambda, 2\lambda, \ldots, n\lambda \qquad \left[\text{as } \Delta x = \phi\left(\frac{\lambda}{2\pi}\right)\right]$$

Destructive Interference

These are the points where resultant amplitude or intensity is minimum or

$$A_{\min} = A_1 \sim A_2 \qquad \text{[from Eq. (i)]}$$

or $$A_{\min} = 0 \qquad \text{[from Eq. (iii)]}$$

and $$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2 \qquad \text{[from Eq. (ii)]}$$

or $$I_{\min} = 0 \qquad \text{[from Eq. (iv)]}$$

at those points where,

$$\cos\phi = -1 \qquad \text{[from Eqs. (i) or (ii)]}$$

or $$\phi = \pi, 3\pi \ldots (2n-1)\pi \qquad (\text{where, } n = 1, 2 \ldots)$$

$\therefore$ $$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \ldots (2n-1)\frac{\lambda}{2} \qquad \left[\text{as } \Delta x = \phi\left(\frac{\lambda}{2\pi}\right)\right]$$

Extra Points to Remember

- In amplitude it hardly matters whether its $+2A_0$ or $-2A_0$. This is the reason we have taken, $A_{\max} = \pm 2A_0$
- In interference, two or more than two waves from coherent sources meet at several points. At different points Δx, $\Delta\phi$ or ϕ, resultant amplitude and therefore resultant intensity will be different (varying from $I_{\max}$ to $I_{\min}$). But whatever is the resultant intensity at some point, it remains constant at that point.
- In interference,

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{I_1/I_2} + 1}{\sqrt{I_1/I_2} - 1}\right)^2$$

$$= \left(\frac{A_1/A_2 + 1}{A_1/A_2 - 1}\right)^2 = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2 = \left(\frac{A_{\max}}{A_{\min}}\right)^2$$

Example 18.1 *In interference, two individual amplitudes are A_0 each and the intensity is I_0 each. Find resultant amplitude and intensity at a point, where*

(a) phase difference between two waves is 60°

(b) path difference between two waves is $\frac{\lambda}{3}$.

Solution (a) Substituting $\phi = 60°$ in the equations,

$$A = 2A_0 \cos\frac{\phi}{2} \qquad \dots(i)$$

and

$$I = 4I_0 \cos^2\frac{\phi}{2} \qquad \dots(ii)$$

We get, $A = \sqrt{3}A_0$ and $I = 3I_0$ **Ans.**

(b) Given, $\Delta x = \frac{\lambda}{3}$

$$\therefore \qquad \phi \text{ or } \Delta\phi = \left(\frac{2\pi}{\lambda}\right)\cdot\Delta x$$

$$= \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{3}\right) = \frac{2\pi}{3} \text{ or } 120°$$

Now, substituting $\phi = 120°$ in the above two equations we get

$A = A_0$ and $I = I_0$ **Ans.**

Example 18.2 *Three waves from three coherent sources meet at some point. Resultant amplitude of each is A_0. Intensity corresponding to A_0 is I_0. Phase difference between first wave and second wave is 60°. Path difference between first wave and third wave is $\frac{\lambda}{3}$. The first wave lags behind in phase angle from second and third wave. Find resultant intensity at this point.*

Solution Here, the sources are three. So, we don't have any direct formula for finding the resultant intensity. First we will find the resultant amplitude by vector method and then by the relation $I \propto A^2$, we can also find the resulting intensity.

Further, a path difference of $\frac{\lambda}{3}$ is equivalent to a phase difference of 120° ($\Delta\phi$ or $\phi = \frac{2\pi}{\lambda}\cdot\Delta x$).

Hence, the phase difference first and second is 60° and between first and third is 120°. So, vector diagram for amplitude is as shown below.

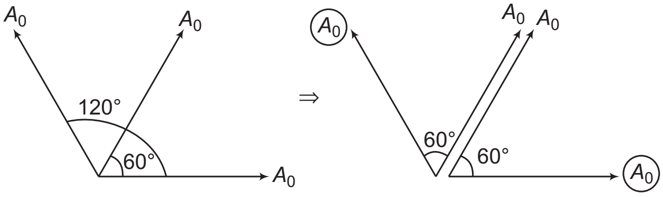


Fig. 18.4

Now, resultant of first and third acting at 120° is also A_0 (as $A = 2A_0 \cos\frac{\phi}{2}$ and $\phi = 120°$) and since the first and third are equal, so this resultant A_0 passes through the bisector line of these two or in the direction of second amplitude vector. Therefore, the resultant amplitude is

$$A = A_0 + A_0 = 2A_0$$

and the resultant intensity is

$$I = 4I_0 \qquad \text{(as } I \propto A^2\text{)}$$ **Ans.**

Example 18.3 *Two waves of equal frequencies have their amplitudes in the ratio of 3 : 5. They are superimposed on each other. Calculate the ratio of maximum and minimum intensities of the resultant wave.*

Solution Given, $\frac{A_1}{A_2} = \frac{3}{5}$

$$\therefore \qquad \sqrt{\frac{I_1}{I_2}} = \frac{3}{5} \qquad \text{(as } I \propto A^2\text{)}$$

Maximum intensity is obtained, where

$$\cos\phi = 1$$

and

$$I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

Minimum intensity is found, where

$$\cos\phi = -1$$

and

$$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

Hence,

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{\frac{I_1}{I_2}} + 1}{\sqrt{\frac{I_1}{I_2}} - 1}\right)^2$$

$$= \left(\frac{3/5 + 1}{3/5 - 1}\right)^2 = \frac{64}{4} = \frac{16}{1}$$ **Ans.**

Example 18.4 *In interference,* $\frac{I_{\max}}{I_{\min}} = \alpha$, *find*

(a $\frac{A_{\max}}{A_{\min}}$ *(b)* $\frac{A_1}{A_2}$ *(c)* $\frac{I_1}{I_2}$

Solution (a) $$\frac{A_{\max}}{A_{\min}} = \sqrt{\frac{I_{\max}}{I_{\min}}} = \sqrt{\alpha}$$ **Ans.**

(b) $$\frac{A_{\max}}{A_{\min}} = \sqrt{\alpha} = \frac{A_1 + A_2}{A_1 - A_2} = \frac{A_1/A_2 + 1}{A_1/A_2 - 1}$$

Solving this equation, we get

$$\frac{A_1}{A_2} = \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}$$ **Ans.**

(c) $$\frac{I_1}{I_2} = \left(\frac{A_1}{A_2}\right)^2 \left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}\right)^2$$

INTRODUCTORY EXERCISE 18.1

1. The ratio of intensities of two waves is 9:16. If these two waves interfere, then determine the ratio of the maximum and minimum possible intensities.
2. In Interference two individual amplitudes are 5 units and 3 units. Find
   (a) $\frac{A_{\text{max}}}{A_{\text{min}}}$ (b) $\frac{I_{\text{max}}}{I_{\text{min}}}$
3. Three waves due to three coherent sources meet at one point. Their amplitudes are $\sqrt{2}A_0$, $3A_0$ and $\sqrt{2}A_0$. Intensity corresponding to A_0 is I_0. Phase difference between first and second is 45°. Path difference between first and third $\frac{\lambda}{4}$. In phase angle, first wave lags behind from the other two waves. Find resultant intensity at this point.

18.4 Standing Wave

Standing wave is an example of interference. When two identical waves travel in opposite directions, then they superimpose almost at every point. Now, since the waves are identical (or their frequencies are same), sources are coherent. So, interference will take place. By their superposition (or interference) standing waves (also called stationary waves) are formed. By identical wave we mean that all properties (like f, ω, T, λ and k) are same. Only amplitudes may be different but still we prefer equal amplitudes ($A_1 = A_2 = A_0$). The following are listed some of the important points in standing wave.

(i) Here, the constructive interference points are called antinodes (denoted by A) and destructive interference points are called nodes (denoted by N).

(ii) Distance between two successive nodes (or two successive antinodes) is $\frac{\lambda}{2}$ or $\frac{\pi}{k}$ $\left(\text{as } k = \frac{2\pi}{\lambda}\right)$.

Similarly, distance between a node and its adjacent antinode is $\frac{\lambda}{4}$ or $\frac{\pi}{2k}$. Here, λ and k are the values corresponding to constituent waves by which stationary waves are formed.

(iii) Amplitude of oscillation in stationary wave varies from a maximum value ($A_1 + A_2$ or $2A_0$ at antinode) to a minimum value ($A_1 \sim A_2$ or zero at node). But now onwards (unless mentioned in the question) we will talk about the equal amplitudes. So, amplitude at antinode will be $2A_0$ and at node it is zero.

Thus, now we can say that, in stationary wave all particles (except nodes for the case when $A_1 = A_2$) oscillate with same frequency but different amplitudes.

(iv) All points lying between two successive nodes are in same phase. They cross their mean positions (in the same direction) simultaneously and reach their extreme positions also simultaneously. But they are out of phase with the particles lying between adjacent two nodes. This can be shown by the following figure.

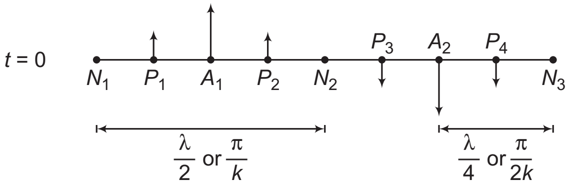

t = 0
N_1
P_1
A_1
P_2
N_2
P_3
A_2
P_4
N_3
$\frac{\lambda}{2}$ or $\frac{\pi}{k}$
$\frac{\lambda}{4}$ or $\frac{\pi}{2k}$

Fig. 18.5

In the above figure N_1, N_2 and N_3 are nodes and A_1, A_2 are antinodes. Amplitude at nodes is zero and amplitude at antinodes is $2A_0$. In between them we have taken four more points P_1, P_2, P_3 and P_4. Suppose amplitude at these points is A_0 (between $2A_0$ and 0). Now, at $t = 0$ all particles are at their mean positions. Particles P_1, A_1 and P_2 are moving upwards (as they are in same phase) and the particles P_3, A_2 and P_4 are moving downwards. At this time, they have maximum speed. For example, in case of sine wave,

$$v_{P_1} = \omega A_0 \qquad (v_{\max} = \omega A \text{ in SHM})$$

$$v_{A_1} = \omega(2A_0)$$

Now, in the chapter of SHM we have read that at time $t = \frac{T}{12}$ a particle reaches from its mean position to half its extreme position and in time $t = \frac{T}{4}$ it reaches from mean position to extreme position. So, at different times positions of different particles are as shown below.

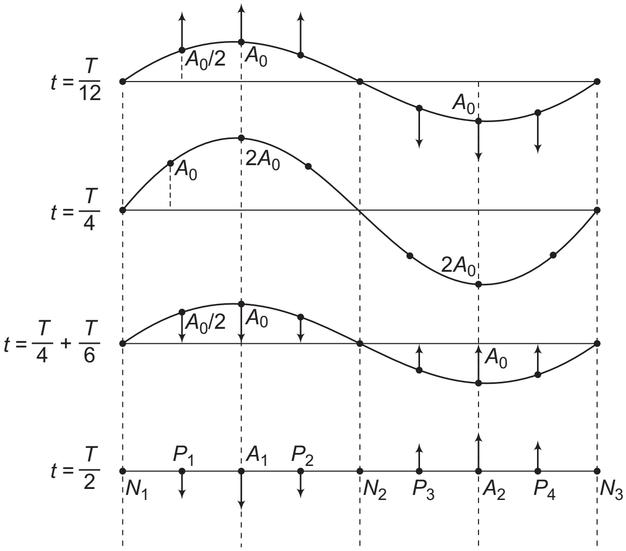

$t = \frac{T}{12}$
$A_0/2$
A_0
A_0
$t = \frac{T}{4}$
A_0
$2A_0$
$2A_0$
$t = \frac{T}{4} + \frac{T}{6}$
$A_0/2$
A_0
A_0
$t = \frac{T}{2}$
P_1
A_1
P_2
N_1
N_2
P_3
A_2
P_4
N_3

Fig. 18.6

(v) Equation of a stationary wave depends on the equations of constituent waves by which stationary wave is formed. Let us take an example

Suppose the two identical waves travelling in opposite directions are

$$y_1 = A \sin (kx - \omega t)$$

and $$y_2 = A \sin (kx + \omega t)$$

By the principle of superposition, their sum is

$$y = y_1 + y_2$$

or $$y = A [\sin (kx - \omega t) + \sin (kx + \omega t)]$$

By using the identity,

$$\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right), \quad \text{we obtain}$$

$$\boxed{y = 2A \sin kx \cos \omega t} \qquad \text{...(i)}$$

This expression is different from wave representations that we have encountered upto now. It doesn't have the form $f\ (x \pm vt)$ or $f\ (ax \pm bt)$ and therefore, does not describe a travelling wave. Instead Eq. (i) represents what is known as a **standing wave.**

Eq. (i) can also be written as

$$y = A\,(x) \cos \omega t \qquad \text{...(ii)}$$

where, $$A_x = 2A \sin kx \qquad \text{...(iii)}$$

This equation of a standing wave [Eq. (ii)] is really an equation of simple harmonic motion, whose amplitude [Eq. (iii)] is a function of x.

$$A_x = 0, \text{ where } \sin kx = 0$$

or $$kx = 0, \pi, 2\pi, \ldots, n\pi \qquad (n = 0, 1, 2, \ldots)$$

Substituting $k = \dfrac{2\pi}{\lambda}$, we have

$$A_x = 0 \qquad \text{where,} \quad \boxed{x = 0, \frac{\lambda}{2}, \lambda, \ldots, \frac{n\lambda}{2}}$$

These are the points which never displace from their mean position. These are known as the **nodes** of the standing wave. The distance between two adjacent nodes is $\dfrac{\lambda}{2}$.

Further, from Eq. (iii), we can see that maximum value of $|A_x|$ is $2A$, where

$$\sin kx = \pm 1$$

or $$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, (2n-1)\frac{\pi}{2} \qquad (n = 0, 1, 2, \ldots)$$

or $$\boxed{x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \ldots, (2n-1)\frac{\lambda}{4}} \qquad \left(k = \frac{2\pi}{\lambda} \right)$$

These are the points of maximum displacement called **antinodes**. The distance between two adjacent antinodes is also $\dfrac{\lambda}{2}$, while that between a node and an antinode is $\dfrac{\lambda}{4}$.

Extra Points to Remember

Table 18.1 comparison between travelling and stationary waves

S.No.	Travelling waves	Stationary waves
1.	In these waves, all particles of the medium oscillate with same frequency and amplitude	In these waves, all particles except nodes oscillate with same frequency but different amplitudes. Amplitudes is zero at nodes and maximum at antinodes
2.	At any instant phase difference between any two particles can have any value between 0 and 2π.	At any instant phase difference between any two particles can be either zero or π.
3.	In these waves, at no instant all the particles of the medium pass through their mean positions simultaneously.	In these waves, all particles of the medium pass through their mean positions simultaneously twice in each time period.
4.	These waves transmit energy in the medium.	These waves do not transmit energy in the medium provided $A_1 = A_2$.

Example 18.5 *The displacement of a standing wave on a string is given by*

$$y(x, t) = 0.4 \sin(0.5x) \cos(30t)$$

where x and y are in centimetres.

(a) Find the frequency, amplitude and wave speed of the component waves.

(b) What is the particle velocity at x = 2.4 cm at t = 0.8 s ?

Solution (a) The given equation can be written as the sum of two component waves as

$$y(x, t) = 0.2 \sin(0.5x - 30t) + 0.2 \sin(0.5x + 30t)$$

Hence, the two component waves are

$$y_1(x, t) = 0.2 \sin(0.5x - 30t)$$

travelling in positive x-direction and

$$y_2(x, t) = 0.2 \sin(0.5x + 30t)$$

travelling in negative x-direction.

Now, $\omega = 30 \text{ rad/s}$

and $k = 0.5 \text{ cm}^{-1}$

$$\text{Frequency, } f = \frac{\omega}{2\pi} = \frac{15}{\pi} \text{ Hz}$$ **Ans.**

$$\text{Amplitude, } A = 0.2 \text{ cm}$$ **Ans.**

and $$\text{wave speed, } v = \frac{\omega}{k} = \frac{30}{0.5} = 60 \text{ cm/s}$$ **Ans.**

(b) Particle velocity,

$$v_P(x,t) = \frac{\partial y}{\partial t} = -12\sin(0.5x)\sin(30t)$$

$$\therefore \quad v_P(x = 2.4\text{ cm}, t = 0.8\text{ s})$$

$$= -12\sin(1.2)\sin(24)$$

$$= 10.12\text{ cm/s}$$ **Ans.**

Example 18.6 *The vibrations of a string of length 60 cm fixed at both ends are represented by the equation* $y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$, *where x and y are in cm and t in seconds.*

(a) What is the maximum displacement of a point at x = 5 cm?

(b) Where are the nodes located along the string?

(c) What is the velocity of the particle at x = 7.5 cm and t = 0.25 s?

(d) Write down the equations of the component waves whose superposition gives the above wave.

Solution (a) At $x = 5$ cm, the standing wave equation gives

$$y = 4\sin\left(\frac{5\pi}{15}\right)\cos(96\pi t)$$

$$= 4\sin\frac{\pi}{3}\cos(96\pi t)$$

$$= 4 \times \frac{\sqrt{3}}{2}\cos(96\pi t)$$

$\therefore$ Maximum displacement $= 2\sqrt{3}$ cm

(b) The nodes are the points of permanent rest. Thus, they are those points for which

$$\sin\left(\frac{\pi x}{15}\right) = 0$$

i.e. $$\frac{\pi x}{15} = n\pi, \text{ where } n = 0, 1, 2, 3, 4 \ldots$$

$x = 15n$, i.e. at $x = 0$, 15, 30, 45 and 60 cm.

(c) The particle velocity is equal to

$$\left(\frac{\partial y}{\partial t}\right) = 4\sin\left(\frac{\pi x}{15}\right)(96\pi)(-\sin 96\pi t)$$

$$= -384\pi\sin\left(\frac{\pi x}{15}\right)\sin(96\pi t)$$

At $x = 7.5$ cm and $t = 0.25$ s, we get

$$\left(\frac{\partial y}{\partial t}\right) = -384\pi\sin\left(\frac{\pi}{2}\right)\sin(24\pi) = 0$$

(d) The equations of the component waves are

$$y_1 = 2\sin\left(\frac{\pi x}{15} + 96\pi t\right)$$

and $$y_2 = 2\sin\left(\frac{\pi x}{15} - 96\pi t\right)$$

as we can see that $$y = y_1 + y_2$$

INTRODUCTORY EXERCISE 18.2

1. In stationary wave, phase difference between two particles can't be $\frac{\pi}{3}$. Is this statement true or false?

2. A string vibrates according to the equation $y = 5\sin\frac{\pi x}{3}\cos 40\pi t$

where, x and y are in centimetres and t is in seconds

(a) What is the speed of the component wave?

(b) What is the distance between the adjacent nodes?

(c) What is the velocity of the particle of the string at the position $x = 1.5$ cm when $t = \frac{9}{8}$ s?

3. If two waves differ only in amplitude and are propagated in opposite directions through a medium, will they produce standing waves. Is energy transported?

4. Two sinusoidal waves travelling in opposite directions interfere to produce a standing wave described by the equation

$$y = (1.5\text{ m})\sin(0.400\,x)\cos(200\,t)$$

where, x is in metres and t is in seconds. Determine the wavelength, frequency and speed of the interfering waves.

18.5 Normal Modes of a String

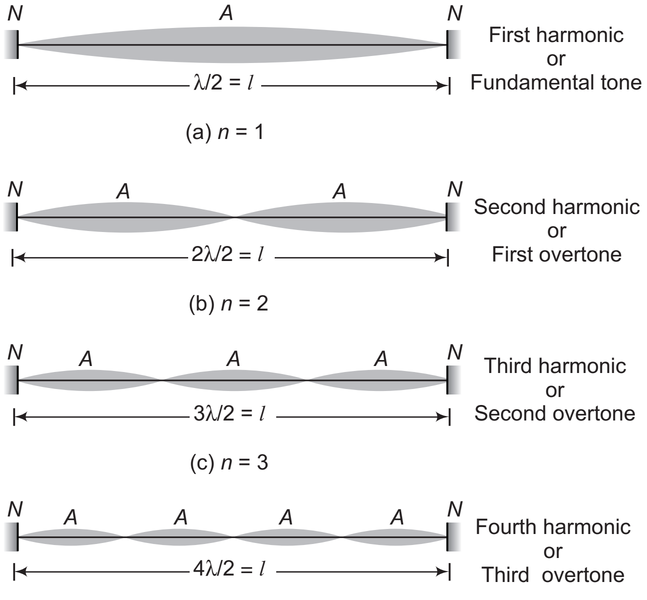
In an unbounded continuous medium, there is no restriction on the frequencies or wavelengths of the standing waves. However, if the waves are confined in space, for example, when a string is tied at both ends,standing waves can be set-up for a discrete set of frequencies or wavelengths. Consider a string of definite length l, rigidly held at both ends. When we setup a sinusoidal wave on such a string, it gets reflected from the fixed ends. By the superposition of two identical waves travelling in opposite directions standing waves are established on the string. The only requirement we have to satisfy is that the end points be nodes as these points cannot oscillate. They are permanently at rest. There may be any number of nodes in between or none at all, so that the wavelength associated with the standing waves can take many different values. The distance between adjacent nodes is $\lambda/2$, so that in a string of length l there must be exactly an integral number n of half wavelengths $\lambda/2$. That is,

$$\frac{n\lambda}{2} = l$$

or $$\lambda = \frac{2l}{n} \qquad (n = 1, 2, 3\ldots)$$

But $\lambda = \dfrac{v}{f}$ and $v = \sqrt{\dfrac{T}{\mu}}$, so that the natural frequencies of oscillation of the system are

$$\boxed{f = n\left(\frac{v}{2l}\right) = \frac{n}{2l}\sqrt{\frac{T}{\mu}}} \qquad (n = 1, 2, 3\ldots)$$


N
A
N
First harmonic
or
Fundamental tone
λ/2 = l
N
A
A
N
Second harmonic
or
First overtone
2λ/2 = l
N
A
A
A
N
Third harmonic
or
Second overtone
3λ/2 = l
N
A
A
A
A
N
Fourth harmonic
or
Third overtone
4λ/2 = l

(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

(d) $n = 4$

Fig. 18.7

The smallest frequency f_1 corresponds to the largest wavelength $(n = 1)$, $\lambda_1 = 2l$.

$$f_1 = \frac{v}{2l}$$

This is called the **fundamental frequency**. The other standing wave frequencies are

$$f_2 = \frac{2v}{2l} = 2f_1$$

$$f_3 = \frac{3v}{2l} = 3f_1 \text{ and so on}$$

These frequencies are called **harmonics**. Musicians sometimes call them **overtones**. Students are advised to remember these frequencies by name.

For example,

f_1 = fundamental tone or first harmonic

$f_2 = 2f_1$ = first overtone or second harmonic

$f_3 = 3f_1$ = second overtone or third harmonic and so on.

Extra Points to Remember

- We have seen that the natural frequencies of oscillation of a stretched wire are given by

$$f = \frac{n}{2l}\sqrt{\frac{T}{\mu}}$$

Now, tension in the wire can be produced by the following two methods.

Method 1

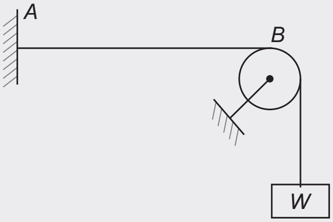


Fig. 18.8

By suspending a weight W from the wire AB as shown. In this case, transverse stationary waves are produced in wire AB.

Method 2 Wire AB is stretched at some higher temperature (than the normal room temperature). When temperature decreases, thermal stresses are produced and the wire comes under tension. Detailed discussion of thermal expansion will be covered in chapter 20.

- If by somehow, tension is decreased (for example: when weight is partially or fully immersed in a liquid), then from the above expression we can see that the set of frequencies decrease.
- As n increases, T, μ, l and therefore $v\left(=\sqrt{\frac{T}{\mu}}\right)$ remain unchanged. Frequency increases, therefore wavelength $\left(\lambda = \frac{v}{f}\right)$ and the loop size $(= \lambda/2)$ decrease.
- Even and odd both harmonics $(n = 1, 2, 3, \ldots)$ are obtained with stretched wire fixed at both ends.

Example 18.7 *In terms of T, μ and l, find frequency of*

(a) fourth overtone mode

(b) third harmonic

Solution (a) Fourth overtone mode means

$$n = 5 \qquad (n = 1, 2, 3, 4, 5)$$

Therefore, using the equation

$$f = \frac{n}{2l}\sqrt{\frac{T}{\mu}} = \frac{5}{2l}\sqrt{\frac{T}{\mu}}$$ **Ans.**

(b) Third harmonic means

$$n = 3$$

$$\therefore \qquad f = \frac{3}{2l}\sqrt{\frac{T}{\mu}}$$ **Ans.**

Example 18.8 *A massless rod BD is suspended by two identical massless strings AB and CD of equal lengths. A block of mass m is suspended from point P such that BP is equal to x. If the fundamental frequency of the left wire is twice the fundamental frequency of right wire, then the value of x is* **(JEE 2006)**

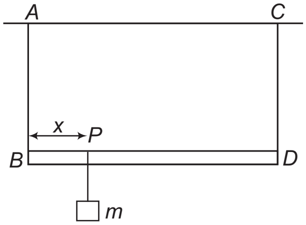


Fig. 18.9

(a) $l/5$ (b) $l/4$ (c) $4l/5$ (d) $3l/4$

Solution $f \propto v \propto \sqrt{T} \quad \Rightarrow \quad f_{AB} = 2f_{CD}$

$\therefore$ $$T_{AB} = 4T_{CD} \qquad \text{...(i)}$$

Further $$\Sigma\tau_P = 0$$

$\therefore$ $$T_{AB}(x) = T_{CD}(l - x)$$

or $$4x = l - x \qquad (\text{as } T_{AB} = 4T_{CD})$$

or $$x = l/5$$

$\therefore$ The correct option is (a).

Example 18.9 *An object of specific gravity ρ is hung from a thin steel wire. The fundamental frequency for transverse standing waves in the wire is 300 Hz. The object is immersed in water, so that one half of its volume is submerged. The new fundamental frequency (in Hz) is* **(JEE 1995)**

(a) $300\left(\dfrac{2\rho - 1}{2\rho}\right)^{1/2}$ (b) $300\left(\dfrac{2\rho}{2\rho - 1}\right)^{1/2}$ (c) $300\left(\dfrac{2\rho}{2\rho - 1}\right)$ (d) $300\left(\dfrac{2\rho - 1}{2\rho}\right)$

Solution The diagrammatic representation of the given problem is shown in figure. The expression of fundamental frequency is

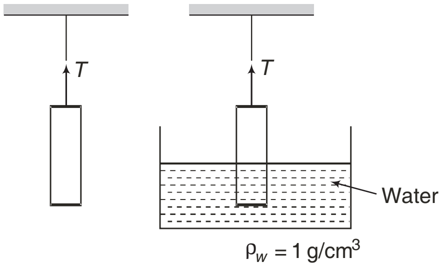


Fig. 18.10

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

In air, $T = mg = (V\rho)g$

$$\therefore \quad \nu = \frac{1}{2l}\sqrt{\frac{V\rho g}{\mu}} \qquad \text{...(i)}$$

When the object is half immersed in water, then

$$T' = mg - \text{upthrust} = V\rho g - \left(\frac{V}{2}\right)\rho_w g = \left(\frac{V}{2}\right) g\,(2\rho - \rho_w)$$

The new fundamental frequency is

$$\nu' = \frac{1}{2l} \times \sqrt{\frac{T'}{\mu}} = \frac{1}{2l}\sqrt{\frac{(Vg/2)(2\rho - \rho_w)}{\mu}} \qquad \text{...(ii)}$$

$$\therefore \quad \frac{\nu'}{\nu} = \left(\sqrt{\frac{2\rho - \rho_w}{2\rho}}\right) \quad \text{or} \quad \nu' = \nu\left(\frac{2\rho - \rho_w}{2\rho}\right)^{1/2}$$

$$= 300\left(\frac{2\rho - 1}{2\rho}\right)^{1/2} \text{Hz} \qquad \textbf{Ans.}$$

∴ The correct option is (a).

INTRODUCTORY EXERCISE 18.3

1. Sound waves of frequency 660 Hz fall normally on a perfectly reflecting wall. The shortest distance from the wall at which the air particles have maximum amplitude of vibration is …… m. Speed of sound = 330 m/s. **(JEE 1984)**
2. If the frequencies of the second and fifth harmonics of a string differ by 54 Hz. What is the fundamental frequency of the string?
3. A wire is attached to a pan of mass 200 g that contains a 2.0 kg mass as shown in the figure. When plucked, the wire vibrates at a fundamental frequency of 220 Hz. An additional unknown mass *M* is then added to the pan and a fundamental frequency of 260 Hz is detected. What is the value of *M*?

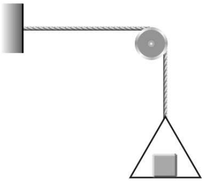


Fig. 18.11

4. A wire fixed at both ends is 1.0 m long and has a mass of 36 g. One of its oscillation frequencies is 250 Hz and the next higher one is 300 Hz.
   (a) Which harmonics do these frequencies represent?
   (b) What is the tension in the wire?
5. Two different stretched wires have same tension and mass per unit length. Fifth overtone frequency of the first wire is equal to second harmonic frequency of the second wire. Find the ratio of their lengths.

18.6 Reflection and Transmission of a Wave

Following points are important in reflection and transmission of a wave :

(i) Speed From wave point of view that medium is called a denser medium in which speed of wave is less.

For example, for a sound wave (or longitudinal wave) air is denser medium compared to water because speed of sound in air is less. On the other hand, air is rarer medium for an electromagnetic wave as speed of electromagnetic wave in air is more. When a wave strikes the boundary separating two different media, part of it is reflected and part is transmitted.

Since, wave speed depends upon the medium, in reflection wave speed does not change because medium does not change. In transmission, medium changes, therefore wave speed also changes. In denser medium it decreases.

(ii) Frequency Frequency (ω and T also) depends on source. In reflection as well as transmission source does not change. Therefore, frequency does not change.

(iii) Wavelength Wavelength (and therefore wave number $k = \dfrac{2\pi}{\lambda}$ also) is self adjusted in a value, $\lambda = \dfrac{v}{f}$. In reflection v and f do not change, therefore λ and k do not change. In transmission, f remains unchanged but v changes. Therefore, λ and k both change. In a denser medium, v decreases, therefore λ also decreases but k increases.

(iv) Phase change A phase change of π or 180° occurs only during the reflection from a denser medium. In any other case (in transmission or reflection from a rarer medium) phase change is zero.

(v) Amplitude In reflection as well as transmission, amplitude changes.

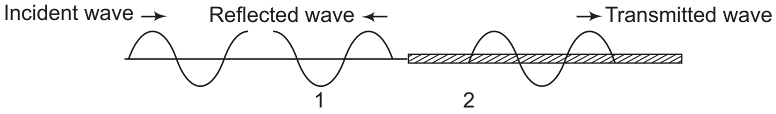


Fig. 18.12

If amplitude of incident wave in medium-1 is A_i, it is partly reflected and partly transmitted at the boundary of two media-1 and 2. Wave speeds in two media are v_1 and v_2. If amplitudes of reflected and transmitted waves are A_r and A_t, then

$$\boxed{A_r = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_i} \quad \text{and} \quad \boxed{A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i}$$

From the above two expressions, we can make the following conclusions :

Conclusion 1 If $v_1 = v_2$, then $A_r = 0$ and $A_t = A_i$

Basically $v_1 = v_2$ means both media are same from wave point of view. So, in this case there is no reflection ($A_r = 0$), only transmission ($A_t = A_i$) is there.

Conclusion 2 If $v_2 < v_1$, then A_r comes out to be negative. Now, $v_2 < v_1$ means the second medium is denser. A_r in this case is negative means, there is a phase change of π.

Conclusion 3 If $v_2 > v_1$, then $A_t > A_i$. This implies that amplitude always increases as the wave travels from a denser medium to rarer medium (as $v_2 > v_1$, so second medium is rarer).

(vi) Power At the boundary of two media,
energy incident per second = energy reflected per second + energy transmitted per second.

or

Power incident = power reflected + power transmitted

or

$$\boxed{P_i = P_r + P_t}$$

Let us summarise the above discussion in tabular form as below.

Table 18.2

Wave property	Reflection	Transmission (Refraction)
v	does not change	changes
f, T, ω	do not change	do not change
λ, k	do not change	change
A, I	change	change
ϕ	$\Delta\phi = 0$, from a rarer medium $\Delta\phi = \pi$, from a denser medium	$\Delta\phi = 0$

Example 18.10 *Two strings 1 and 2 are taut between two fixed supports (as shown in figure) such that the tension in both strings is same. Mass per unit length of 2 is more than that of 1. Explain which string is denser for a transverse travelling wave.*

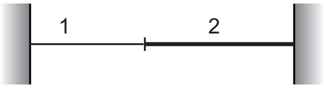


Fig. 18.13

Solution Speed of a transverse wave on a string,

$$v = \sqrt{\frac{T}{\mu}} \quad \text{or} \quad v \propto \frac{1}{\sqrt{\mu}}$$

Now, $\mu_2 > \mu_1$ (given)

$\therefore$ $v_2 < v_1$

i.e. medium 2 is denser and medium 1 is rarer.

Example 18.11 *A sound wave and a light wave are reflected and refracted (or transmitted) from water surface. Show the changes in different physical quantities associated with a wave.*

Solution **Sound wave** We have seen in article 18.6, that water is a rarer medium for sound. When a wave travels from a denser to rarer medium, it bends away from normal. Its speed, wavelength and amplitude increase but angular wave number decreases. Further, in reflection from a rarer medium and in transmission there is no change in phase angle.

In the figure shown, 1 is incident wave 2 is reflected wave and 3 is transmitted wave.

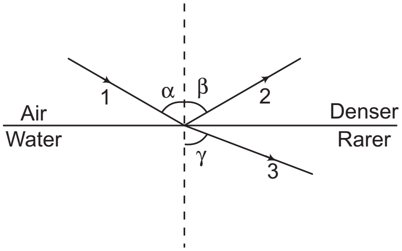


Fig. 18.14

$\alpha = \beta, \gamma > \alpha, f_1 = f_2 = f_3, v_1 = v_2, \lambda_1 = \lambda_2, k_1 = k_2, v_1 < v_3, \lambda_3 > \lambda_1 = \lambda_2, k_3 < k_1$

$\Delta\phi_{12} = \Delta\phi_{13} = 0°$.

Electromagnetic Wave For electromagnetic wave, water is denser medium. So, in water, ray of light will bend towards normal.

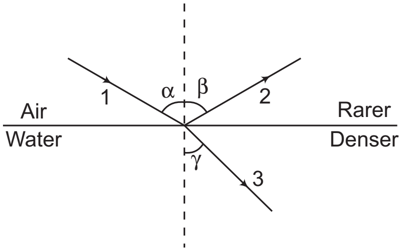


Fig. 18.15

Further, speed and wavelength in this medium will decrease. But value of k will increase.

$\alpha = \beta, \gamma < \alpha, f_1 = f_2 = f_3, v_1 = v_2, \lambda_1 = \lambda_2, k_1 = k_2, v_3 < v_1, \lambda_3 < \lambda_1, k_3 > k_1$

$\Delta\phi_{12} = \pi$ and $\Delta\phi_{13} = 0°$

Example 18.12 *A harmonic wave is travelling on string 1. At a junction with string 2 it is partly reflected and partly transmitted. The linear mass density of the second string is four times that of the first string, and that the boundary between the two strings is at $x = 0$. If the expression for the incident wave is*

$$y_i = A_i \cos(k_1 x - \omega_1 t)$$

(a) What are the expressions for the transmitted and the reflected waves in terms of A_i, k_1 and ω_1?

(b) Show that the average power carried by the incident wave is equal to the sum of the average power carried by the transmitted and reflected waves.

Solution (a) Since, $v = \sqrt{T/\mu}$, $T_2 = T_1$ and $\mu_2 = 4\mu_1$

We have,
$$v_2 = \frac{v_1}{2} \qquad \text{...(i)}$$

From Table 18.2, we can see that the frequency does not change, that is

$$\omega_1 = \omega_2 \qquad \text{...(ii)}$$

Also, because $k = \dfrac{\omega}{v}$, the wave numbers of the harmonic waves in the two strings are related by

$$k_2 = \frac{\omega_2}{v_2} = \frac{\omega_1}{v_1/2} = 2\frac{\omega_1}{v_1} = 2k_1 \qquad \text{...(iii)}$$

The amplitudes are $$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i = \left[\frac{2(v_1/2)}{v_1 + (v_1/2)}\right] = \frac{2}{3} A_i \qquad \text{...(iv)}$$

and $$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_i = \left[\frac{(v_1/2) - v_1}{v_1 + (v_1/2)}\right] A_i = -\frac{A_i}{3} \qquad \text{...(v)}$$

Now with Eqs. (ii), (iii) and (iv), the transmitted wave can be written as

$$y_t = \frac{2}{3} A_i \cos(2k_1 x - \omega_1 t) \qquad \textbf{Ans.}$$

Similarly the reflected wave can be expressed as

$$y_r = -\frac{A_i}{3} \cos(k_1 x + \omega_1 t)$$

$$= \frac{A_i}{3} \cos(k_1 x + \omega_1 t + \pi) \qquad \textbf{Ans.}$$

(b) The average power of a harmonic wave on a string is given by

$$P = \frac{1}{2} \rho A^2 \omega^2 S v = \frac{1}{2} A^2 \omega^2 \mu v \qquad (\text{as } \rho S = \mu)$$

Now, $$P_i = \frac{1}{2} \omega_1^2 A_i^2 \mu_1 v_1 \qquad \text{...(vi)}$$

$$P_t = \frac{1}{2} \omega_1^2 \left(\frac{2}{3} A_i\right)^2 (4\mu_1) \left(\frac{v_1}{2}\right) = \frac{4}{9} \omega_1^2 A_i^2 \mu_1 v_1 \qquad \text{...(vii)}$$

and $$P_r = \frac{1}{2} \omega_2^2 \left(-\frac{A_i}{3}\right)^2 (\mu_1)(v_1) = \frac{1}{18} \omega_1^2 A_i^2 \mu_1 v_1 \qquad \text{...(viii)}$$

From Eqs. (vi), (vii) and (viii), we can show that

$$P_i = P_t + P_r \qquad \textbf{Hence proved.}$$

Example 18.13 *From energy conservation principle prove the relations,*

$$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_i \quad \text{and} \quad A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i$$

Here, symbols have their usual meanings.

Solution By conservation of energy, the average incident power equals the average reflected power plus the average transmitted power

or $$P_i = P_r + P_t$$

∴ $$\frac{1}{2} \mu_1 \omega^2 A_i^2 v_1 = \frac{1}{2} \mu_1 \omega^2 A_r^2 v_1 + \frac{1}{2} \mu_2 \omega^2 A_t^2 v_2$$

or $$\left(\frac{T}{v_1^2}\right)\omega^2 A_i^2 v_1 = \left(\frac{T}{v_1^2}\right)\omega^2 A_r^2 v_1 + \left(\frac{T}{v_2^2}\right)\omega^2 A_t^2 v_2$$

or $$\frac{A_i^2}{v_1} = \frac{A_r^2}{v_1} + \frac{A_t^2}{v_2} \qquad \text{...(i)}$$

Further $$A_i + A_r = A_t \qquad \text{...(ii)}$$

Solving these two equations for A_r and A_t, we get

$$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_i$$

and $$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i$$

Hence proved.

INTRODUCTORY EXERCISE 18.4

1. Two pulses of identical shape overlap such that the displacement of the rope is momentarily zero at all points, what happens to the energy at this time?
2. The pulse shown in figure has a speed of 10 cm/s.

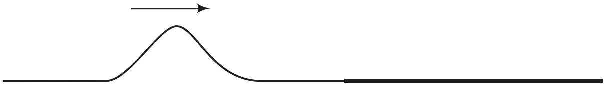


Fig. 18.16

   (a) If the linear mass density of the right string is 0.25 that of the left string, at what speed does the transmitted pulse travel?

   (b) Compare the heights of the transmitted pulse and the reflected pulse to that of the incident pulse.

3. The harmonic wave $y_i = (2.0 \times 10^{-3}) \cos \pi (2.0x - 50t)$ travels along a string toward a boundary at $x = 0$ with a second string. The wave speed on the second string is 50 m/s. Write expressions for reflected and transmitted waves. Assume SI units.

Final Touch Points

1. Until now, we have come across the following three sets of equations

$$\left.\begin{array}{l} y = A\sin(\omega t \pm \phi) \\ y = A\cos(\omega t \pm \phi) \end{array}\right\} \text{SHM}$$

$$\left.\begin{array}{l} y = A\sin(kx \pm \omega t \pm \phi) \\ y = A\cos(kx \pm \omega t \pm \phi) \end{array}\right\} \text{Travelling wave}$$

$$\left.\begin{array}{lcl} y = A\sin kx\cos\omega t & \text{or} & 2A\sin kx\cos\omega t \\ y = 2A\sin\omega t\cos kx & \text{or} & A\sin\omega t\cos kx \\ y = A\sin kx\sin\omega t & \text{or} & 2A\sin kx\sin\omega t \\ y = 2A\cos kx\cos\omega t & \text{or} & A\cos kx\cos\omega t \end{array}\right\} \text{Standing wave}$$

Note *In standing waves, we have given four sets of equations. The equation of standing wave basically depends on the component waves. Further, if the maximum amplitude is 2A, it means amplitude of travelling waves is A and if it is A, then amplitude of travelling waves is* $\frac{A}{2}$.

2. **Resonance** If the string (discussed in Art 18.5) is set vibrating in anyone of the normal modes and left to itself, the oscillations gradually die out. The motion is damped by dissipation of energy through the elastic supports at the ends and by the resistance of the air to the motion. We can pump energy into the system by applying a driving force. If the driving frequency is equal to any natural frequency of the string, the string will vibrate at that frequency with a larger amplitude. This phenomenon is called **resonance**. Because the string has a large number of natural frequencies, resonance can occur at many different frequencies.

3. **Reason of phase change of π in reflection from a denser medium** When a transverse wave is reflected at the rigid support, a phase change of π takes place in the displacement. Suppose a transverse pulse is produced along a string PQ whose end Q is attached to a rigid support. If the pulse travels in the form of a crest, then on reaching the rigid support, the pulse would exert an upward force. Since the support is rigid, it remains unaffected but a reaction force act in the downward direction on the string which reverses the sign of the displacement of the particle at Q. As a result, an inverted pulse in the form of a trough travels back as shown.

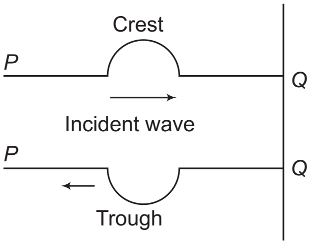


Fig. 18.17

This concept can also be understood by a second thought process. Incident wave and reflected wave are two identical waves travelling in opposite directions. So, they should produce a stationary wave. Point Q is a fixed point, which cannot oscillate at all. So, Q should become a node. This is possible only when the two displacements (one due to incident wave y_1 and the other due to reflected wave y_2) are always equal and opposite ($y = y_1 + y_2 = 0$ or $y_1 = -y_2$). Or, they have a phase difference of π.

Solved Examples

TYPED PROBLEMS

Type 1. *To find length of oscillating wire from the equation of stationary wave*

How to Solve?

- From the given equation of stationary wave find the value of wave number k and then λ. Now, from the given mode of oscillations we can also find number of loops. Suppose number of loops are n and one loop size is $\frac{\lambda}{2}$. Then, total length of wire will be

$$l = n\left(\frac{\lambda}{2}\right)$$

Example 1 *A stretched wire is oscillating in third overtone mode. Equation of transverse stationary wave produced in this wire is*

$$y = A\sin(6\pi x)\sin(20\pi t)$$

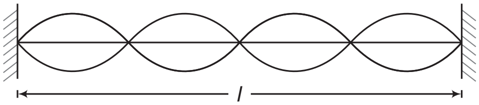
Here, x is in metres. Find the length of the wire.

Solution From the given equation we can see that

$$k = 6\pi \text{ m}^{-1}$$

$$\therefore \qquad \lambda = \frac{2\pi}{k} = \frac{2\pi}{6\pi} = \frac{1}{3}\text{ m}$$

Now, third overtone mode means four loops as shown below



One loop size is $\frac{\lambda}{2}$, therefore total length of wire,

$$l = 4\left(\frac{\lambda}{2}\right) = 2\lambda = 2\left(\frac{1}{3}\right)\text{m} = \frac{2}{3}\text{ m}$$

Ans.

Type 2. *To find amplitude at some given point in stationary wave*

Concept

In a stationary wave amplitude at different points varies from 0 at node (when $A_1 = A_2 = A_0$) to maximum ($= 2A_0$) at antinode. If by some how, A_{max}, k and position of node or position of antinode is known then we can find amplitude at any general point. For example, if position of node is known, then take this point as $x = 0$. Now, amplitude at any general point x is

$$A_x = A_{\text{max}} \sin kx$$

In this expression we can see that $A_x = 0$ at $x = 0$.

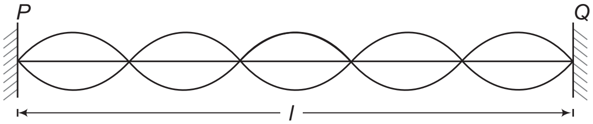
Similarly, if position of antinode is known then amplitude at general point x is

$$A_x = A_{\text{max}} \cos kx$$

Again we can see that, $A_x = A_{\text{max}}$ at $x = 0$.

Example 2 *Length of a stretched wire is 2m. It is oscillating in its fourth overtone mode. Maximum amplitude of oscillations is 2mm. Find amplitude of oscillation at a distance of 0.2 m from one fixed end.*

Solution Fourth overtone mode means five loops.



$$\therefore \quad l = 5\left(\frac{\lambda}{2}\right)$$

$$\text{or} \quad \lambda = \frac{2l}{5}$$

$$= \frac{2 \times 2}{5} = 0.8 \text{ m}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.8}$$

$$= (2.5\,\pi)\text{ m}^{-1}$$

Now, P is a node. So, take $x = 0$ at P.

Then, at a distance x from P,

$$A_x = A_{\text{max}} \sin kx$$

$$= (2\text{ mm}) \sin (2.5\,\pi)(0.2)$$

$$= (2\text{ mm}) \sin\left(\frac{\pi}{2}\right)$$

$$= 2\text{ mm}$$ **Ans.**

Note *At $x = 0.2$ m, we are getting $A_x = A_{\text{max}}$.*

Hence, $x = 0.2$ m is an antinode.

Type 3. *To find energy of oscillation in a given portion in stationary wave*

Concept

In SHM, energy of oscillation of a single particle is given by

$$E = \frac{1}{2} m\omega^2 A^2$$

In a travelling sine wave, all particles oscillate with same amplitude and same frequency. Energy density (or energy per unit volume) is given by

$$u = \frac{1}{2} \rho\omega^2 A^2$$

Since ρ, ω and A is same at all points. Therefore, energy density will be same at all points. So, in each unit volume equal amount of energy will be stored. Therefore, to find total energy of oscillation in a given volume, we just multiply the given volume by this constant energy density.

In stationary wave, amplitude is different at different positions. So, energy density will be non-uniform. Hence, total energy of oscillation in a given volume can be obtained by integration.

There is an alternate method also. By adding two energy densities of the travelling waves (from which standing wave is formed) first of all we find the total energy density. Then, we multiply the given volume with this energy density.

Example 3 *A standing wave is formed by two harmonic waves, $y_1 = A \sin (kx - \omega t)$ and $y_2 = A \sin (kx + \omega t)$ travelling on a string in opposite directions. Mass density of the string is ρ and area of cross-section is s. Find the total mechanical energy between two adjacent nodes on the string.*

Solution The distance between two adjacent nodes is $\frac{\lambda}{2}$ or $\frac{\pi}{k}$.

$\therefore$ Volume of string between two nodes will be

$$V = (\text{area of cross-section}) \times (\text{distance between two nodes})$$
$$= (S)\left(\frac{\pi}{k}\right)$$

Energy density (energy per unit volume) of a travelling wave is given by

$$u = \frac{1}{2} \rho A^2 \omega^2$$

A standing wave is formed by two identical waves travelling in opposite directions. Therefore, the energy stored between two nodes in a standing wave.

$$E = 2\left[\text{energy stored in a distance of } \frac{\pi}{k} \text{ of a travelling wave}\right]$$
$$= 2\ (\text{energy density})\ (\text{volume})$$
$$= 2\left(\frac{1}{2}\rho A^2\omega^2\right)\left(\frac{\pi S}{k}\right)$$

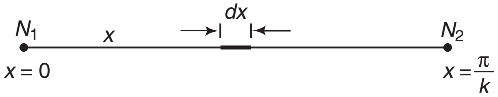
or $$E = \frac{\rho A^2 \omega^2 \pi S}{k}$$ **Ans.**

Alternate Method The equation of the standing wave

$$y = y_1 + y_2 = 2A \sin kx \cos \omega t = A_x \cos \omega t$$

Here, $A_x = 2A \sin kx$

i.e. first node is at $x = 0$ and the next node is at $x = \frac{\pi}{k}$. Let us take an element of length dx at a distance x from N_1. Mass of this element is dm $(= \rho S dx)$. This can be treated as point mass. This element oscillates simple harmonically with angular frequency ω and amplitude $2A \sin kx$.



Hence, energy of this element

$$dE = \frac{1}{2}(dm)\,(2A \sin kx)^2(\omega^2)$$

$$dE = \frac{1}{2}(\rho S dx)\,(2A \sin kx)^2 \omega^2$$

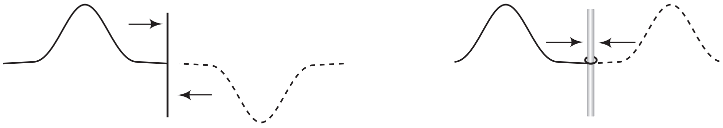
Integrating this with the limits from $x = 0$ to $x = \frac{\pi}{k}$, we get the same result.

Type 4. *Based on the formation of the reflected pulse, either from a rigid boundary or from a free boundary*

Concept

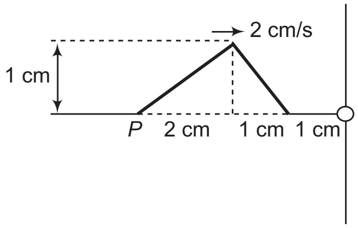
The formation of the reflected pulse can be obtained by overlapping two pulses travelling in opposite directions.

The net displacement at any point is given by the principle of superposition.



(a) Rigid boundary (b) Free boundary

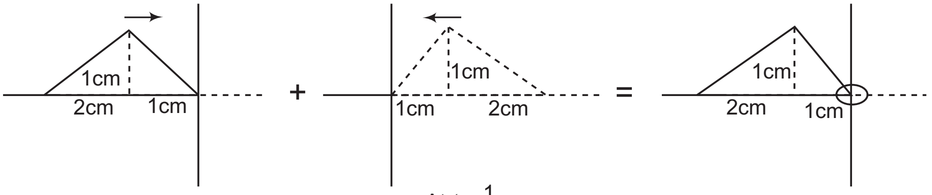
Example 4 *A triangular wave pulse moving at 2 cm/s on a rope approached an end at which it is free to slide on a vertical pole.*



(a) Draw the pulse at $\frac{1}{2}$ s interval until it is completely reflected.

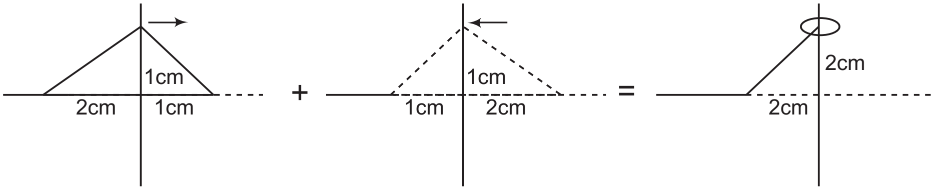
(b) What is the particle speed on the trailing edge P at the instant depicted?

Solution (a) Reflection of pulse from a free boundary is really the superposition of two identical waves travelling in opposite directions. This can be shown as under.


1cm
2cm
1cm
+
1cm
1cm
2cm
=
1cm
2cm
1cm

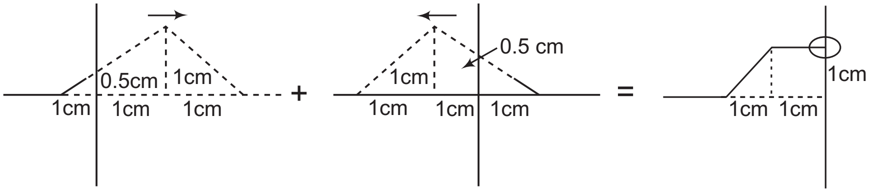
At $t = \frac{1}{2}$ s

(a) (b) (c)


1cm
2cm
1cm
+
1cm
1cm
2cm
=
2cm
2cm

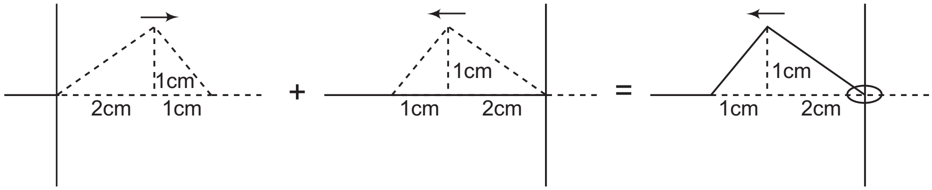
At t = 1 s

(d) (e) (f)


0.5cm
1cm
1cm
1cm
1cm
+
0.5 cm
1cm
1cm
1cm
1cm
=
1cm
1cm
1cm

At $t = 1\frac{1}{2}$ s

(g) (h) (i)


1cm
2cm
1cm
+
1cm
1cm
2cm
=
1cm
1cm
2cm

At t = 2 s

(j) (k) (l)

In every $\frac{1}{2}$ s, each pulse (one real moving towards right and one imaginary moving towards left) travels a distance of 1 cm as the wave speed is 2 cm/s.

(b) Particle speed, $v_P = |-v(\text{slope})|$

Here, v = wave speed = 2 cm/s

and $\text{slope} = \frac{1}{2}$

$\therefore$ Particle speed = 1 cm/s

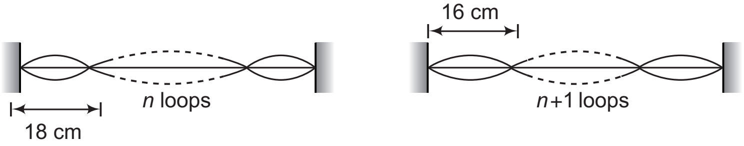
Miscellaneous Examples

Example 5 *A string fixed at both ends has consecutive standing wave modes for which the distances between adjacent nodes are* 18 cm *and* 16 cm, *respectively.*

(a) What is the minimum possible length of the string?

(b) If the tension is 10 N *and the linear mass density is* 4 g/m, *what is the fundamental frequency?*

Solution (a)



Let l be the length of the string. Then,

$$18n = l \qquad \ldots(\text{i})$$

$$16(n+1) = l \qquad \ldots(\text{ii})$$

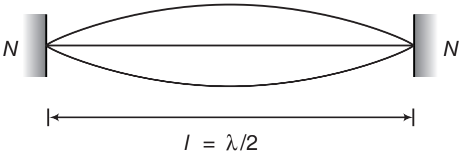
From Eqs. (i) and (ii), wet get

$$n = 8$$

and $$l = 144 \text{ cm}$$

Therefore, the minimum possible length of the string can be 144 cm. **Ans.**

(b) For fundamental frequency, $l = \dfrac{\lambda}{2}$



or $$\lambda = 2l = 288 \text{ cm}$$
$$= 2.88 \text{ m}$$

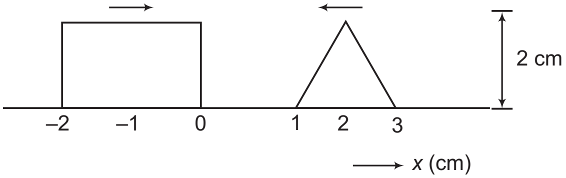
Speed of wave on the string,

$$v = \sqrt{\frac{T}{\mu}}$$
$$= \sqrt{\frac{10}{4 \times 10^{-3}}} = 50 \text{ m/s}$$

∴ Fundamental frequency,

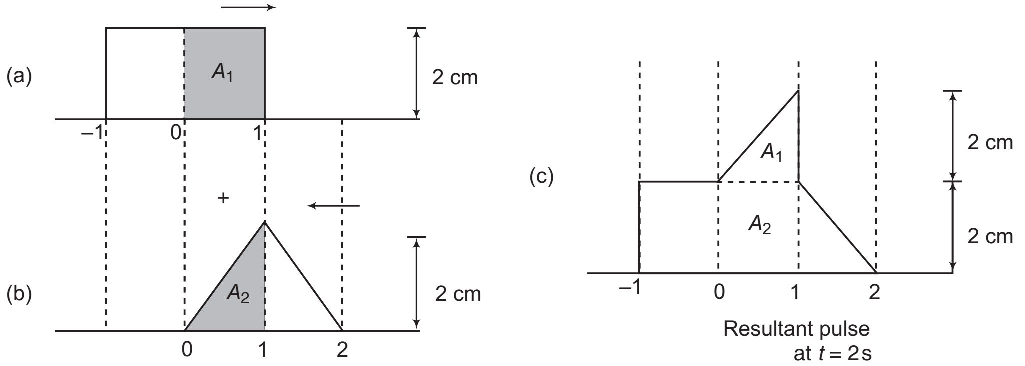
$$f = \frac{v}{\lambda} = \frac{50}{2.88}$$
$$= 17.36 \text{ Hz}$$ **Ans.**

Example 6 *Figure shows a rectangular pulse and triangular pulse approaching towards each other. The pulse speed is* 0.5 cm/s. *Sketch the resultant pulse at* $t = 2$ s.



Solution In 2 s each pulse will travel a distance of 1 cm.

The two pulses overlap between 0 and 1 cm as shown in figure. So, A_1 and A_2 can be added as shown in Fig. (c).



Resultant pulse at $t = 2$s

Example 7 *Two wires are fixed on a sonometer. Their tensions are in the ratio* $8:1$, *their lengths are in the ratio* $36:35$, *the diameters are in the ratio* $4:1$ *and densities are in the ratio* $1:2$. *Find the value of lower frequency if higher frequency is* 360 *Hz.*

Solution Given,

$$\frac{T_1}{T_2} = \frac{8}{1}, \frac{L_1}{L_2} = \frac{36}{35}$$

$$\frac{D_1}{D_2} = \frac{4}{1}, \frac{\rho_1}{\rho_2} = \frac{1}{2}$$

Let μ_1 and μ_2 be the linear mass densities, then

$$\therefore \quad \mu_1 = \pi \times \frac{D_1^2}{4} \times \rho_1 \text{ and } \mu_2 = \pi \times \frac{D_2^2}{4} \times \rho_2 \qquad (\mu = \rho S)$$

$$\therefore \quad \frac{\mu_1}{\mu_2} = \left(\frac{D_1}{D_2}\right)^2 \times \frac{\rho_1}{\rho_2} = \left(\frac{4}{1}\right)^2 \times \frac{1}{2} = \frac{8}{1}$$

$$\therefore \quad \frac{f_1}{f_2} = \frac{L_2}{L_1} \times \sqrt{\frac{T_1}{T_2} \times \frac{\mu_2}{\mu_1}} = \frac{35}{36}\sqrt{\frac{8}{1} \times \frac{1}{8}} = \frac{35}{36} \qquad \left(f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}\right)$$

$$f_2 > f_1$$

We have $f_2 = 360$

$\therefore$ $f_1 = 350$ **Ans.**

Example 8 *In a stationary wave pattern that forms as a result of reflection of waves from an obstacle the ratio of the amplitude at an antinode and a node is* $\beta = 1.5$. *What percentage of the energy passes across the obstacle?*

Solution
$$\frac{A_{\max}}{A_{\min}} = \frac{A_i + A_r}{A_i - A_r}$$

This ratio is given as 1.5 or $\frac{3}{2}$.

$\therefore$
$$\frac{A_i + A_r}{A_i - A_r} = \frac{3}{2} \quad \text{or} \quad \frac{1 + \frac{A_r}{A_i}}{1 - \frac{A_r}{A_i}} = \frac{3}{2}$$

Solving this equation, we get
$$\frac{A_r}{A_i} = \frac{1}{5}$$

$\therefore$
$$\frac{I_r}{I_i} = \left(\frac{A_r}{A_i}\right)^2 = \frac{1}{25}$$

or
$$I_r = 0.04\, I_i$$

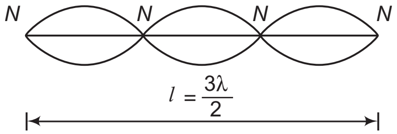
i.e. 4% of the incident energy is reflected or 96% energy passes across the obstacle. **Ans.**

Example 9 *A string of linear mass density* 5.0×10^{-3} *kg/m is stretched under a tension of* 65 *N between two rigid supports* 60 *cm apart.*

(a) If the string is vibrating in its second overtone so that the amplitude at one of its antinodes is 0.25 *cm, what are the maximum transverse speed and acceleration of the string at antinodes?*

(b) What are these quantities at a distance 5.0 *cm from an node?*

Solution (a) In second overtone,



$$l = \frac{3\lambda}{2}$$

or
$$\lambda = \frac{2l}{3} = \frac{2 \times 60}{3}$$
$$= 40 \text{ cm} = 0.4 \text{ m}$$
$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4} = 5\pi \text{ m}^{-1}$$
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{65}{5.0 \times 10^{-3}}}$$
$$= 114 \text{ m/s}$$

$\therefore$
$$\omega = kv = 570\pi \text{ rad/s}$$

Maximum transverse speed at antinode $= A_0\omega$

Here, A_0 = amplitude of antinode = 0.25 cm = 2.5×10^{-3} m

$$\therefore \qquad \text{Maximum speed} = (2.5 \times 10^{-3})(570\,\pi) \text{ m/s}$$

$$= 4.48 \text{ m/s}$$ **Ans.**

$$\text{Maximum acceleration} = \omega^2 A_0$$

$$= (570\pi)^2 \, (2.5 \times 10^{-3}) \text{ m/s}^2$$

$$= 8.0 \times 10^3 \text{ m/s}^2$$ **Ans.**

(b) At a distance x from the node, the amplitude can be written as

$$A = A_0 \sin kx = (2.5 \times 10^{-3}) \sin(5\pi x) \text{ metre}$$

Here, x is in metres.

Therefore, at $x = 5.0 \text{ cm} = 5.0 \times 10^{-2}$ m

$$A = (2.5 \times 10^{-3}) \sin\,(5\pi \times 5.0 \times 10^{-2})$$

$$= 1.8 \times 10^{-3} \text{ m}$$

Maximum speed = $A\omega$

$$= (1.8 \times 10^{-3})\,(570\pi) \text{ m/s}$$

$$= 3.22 \text{ m/s}$$ **Ans.**

and maximum acceleration = $\omega^2 A$

$$= (570\pi)^2 \, (1.8 \times 10^{-3}) \text{ m/s}^2$$

$$= 5.8 \times 10^3 \text{ m/s}^2$$ **Ans.**

Example 10 *An aluminium wire of cross-sectional area* 10^{-6} m^2 *is joined to a steel wire of the same cross-sectional area. This compound wire is stretched on a sonometer pulled by a weight of* 10 *kg. The total length of the compound wire between the bridges is* 1.5 *m of which the aluminium wire is* 0.6 *m and the rest is steel wire. Transverse vibrations are setup in the wire by using an external source of variable frequency. Find the lowest frequency of excitation for which the standing waves are formed such that the joint in the wire is a node. What is the total number of nodes at this frequency? The density of aluminium is* 2.6×10^3 kg/m^3 *and that of steel is* 1.04×10^4 $kg/m^3 (g = 10 \; m/s^2)$.

Solution Let n_a loops are formed in aluminium wire and n_s in steel. Then,

$$f_a = f_s$$

or $$n_a\left(\frac{v_a}{2l_a}\right) = n_s\left(\frac{v_s}{2l_s}\right)$$

or $$\frac{n_a}{n_s} = \left(\frac{v_s}{v_a}\right)\left(\frac{l_a}{l_s}\right)$$

But, $$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho s}} \propto \frac{1}{\sqrt{\rho}}$$

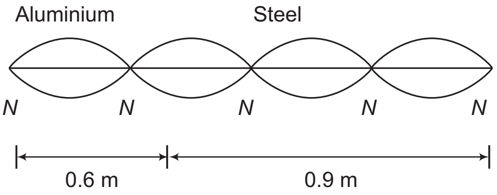
Therefore, $$\frac{v_s}{v_a} = \sqrt{\frac{\rho_a}{\rho_s}}$$

$$\therefore \qquad \frac{n_a}{n_s} = \sqrt{\frac{\rho_a}{\rho_s}} \cdot \frac{l_a}{l_s}$$

Substituting the values, we have

$$\frac{n_a}{n_s} = \sqrt{\frac{2.6 \times 10^3}{1.04 \times 10^4}} \cdot \frac{0.6}{0.9} = \frac{1}{3}$$

i.e. at lowest frequency, one loop is formed in aluminium wire and three loops are formed in steel wire as shown in figure.



$$\therefore \qquad f_{\min} = n_a\left(\frac{v_a}{2l_a}\right) = \frac{1}{2l_a}\sqrt{\frac{T}{\rho_a S}} \qquad (T = mg = 100\text{ N})$$

$$= \frac{1}{2 \times 0.6}\sqrt{\frac{10 \times 10}{2.6 \times 10^3 \times 10^{-6}}}$$

or $\qquad f_{\min} = 163.4$ Hz **Ans.**

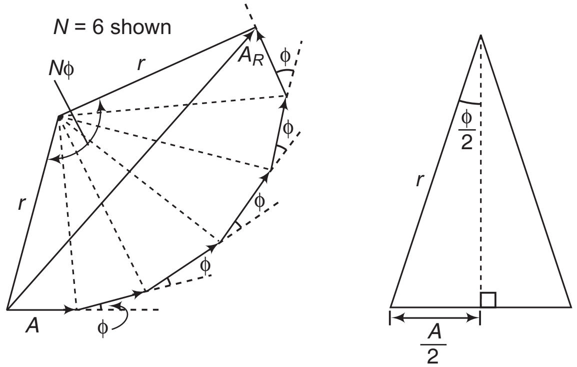
Total number of nodes are five as shown is figure. **Ans.**

Note *In such type of problems nature of junction will be known to us. Then, we have to equate frequencies on the two sides. By equating the frequencies, we find* $\frac{n_1}{n_2}$*. Suppose this comes out to be* 0.4*. Write it,* $\frac{n_1}{n_2} = \frac{2}{5}$*.*

At lowest oscillation frequency 2 *loops are formed on side* 1 *and* 5 *on side* 2*. At next higher frequency* 4 *loops will be formed on side* 1 *and* 10 *on side* 2 *and so on.*

Example 11 *Find the resultant amplitude and phase of a point at which N sinusoidal waves interfere. All the waves have same amplitude A and their phases increase in arithmetic progression of common difference ϕ.*

Solution The diagram for their sum is shown in figure for $N = 6$. The resultant amplitude is A_R. The apex angle of every isosceles triangle is ϕ. So, the angle subtended by the resultant is $N\phi$. Since, the heads of the vectors are all at the same distance r from the apex of the diagram.



(a) (b)

From Fig. (a),

$$\frac{A_R}{2} = r \sin \frac{N\phi}{2} \qquad \text{...(i)}$$

and from Fig. (b),

$$\frac{A}{2} = r \sin \frac{\phi}{2} \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

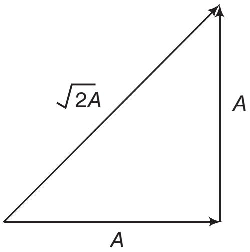
$$\frac{A_R}{A} = \frac{\sin \frac{N\phi}{2}}{\sin \frac{\phi}{2}}$$

or

$$\boxed{A_R = A\left(\frac{\sin \frac{N\phi}{2}}{\sin \frac{\phi}{2}}\right)} \qquad \textbf{Ans.}$$

Note *Suppose $N = 2$ and $\phi = 90°$, then*

$$A_R = \sqrt{2}A$$



Similarly, we can also check the above result for other special cases.

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Two waves $y_1 = A \sin(\omega t + kx)$ and $y_2 = A \cos(\omega t - kx)$ are superimposed, then $x = 0$ becomes a node.

Reason : At node net displacement due to two waves should be zero.

2. Assertion : Stationary waves are so called because particles are at rest in stationary waves.

Reason : They are formed by the superposition of two identical waves travelling in opposite directions.

3. Assertion : When a wave travels from a denser medium to rarer medium, then amplitude of oscillation increases.

Reason : In denser medium, speed of wave is less compared to a rarer medium.

4. Assertion : A wire is stretched and then fixed at two ends. It oscillates in its second overtone mode. There are total four nodes and three antinodes.

Reason : In second overtone mode, length of wire should be $l = \frac{3\lambda}{2}$, where λ is wavelength.

5. Assertion : If we see the oscillations of a stretched wire at higher overtone mode, frequency of oscillation increases but wavelength decreases.

Reason : From $v = f\lambda$, $\lambda \propto \frac{1}{f}$ as $v =$ constant.

6. Assertion : Standing waves are formed when amplitudes of two constituent waves are equal.

Reason : At any point net displacement at a given time is resultant of displacement of constituent waves.

7. Assertion : In a standing wave $x = 0$ is a node. Then, total mechanical energy lying between $x = 0$ and $x = \frac{\lambda}{8}$ is not equal to the energy lying between $x = \frac{\lambda}{8}$ and $x = \frac{\lambda}{4}$.

Reason : In standing waves different particles oscillate with different amplitudes.

8. Assertion : Ratio of maximum intensity and minimum intensity in interference is 25 : 1.

The amplitude ratio of two waves should be 3 : 2.

Reason : $\frac{I_{\max}}{I_{\min}} = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2$

9. **Assertion :** Three waves of equal amplitudes interfere at a point. Phase difference between two successive waves is $\frac{\pi}{2}$. Then, resultant intensity is same as the intensity due to individual wave.

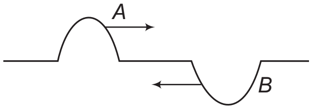
Reason : For interference to take place sources must be coherent.

10. **Assertion :** For two sources to be coherent phase difference between two waves at all points should be same.

Reason : Two different light sources are never coherent.

Objective Questions

1. Two identical harmonic pulses travelling in opposite directions in a taut string approach each other. At the instant when they completely overlap, the total energy of the string will be



(a) zero
(b) partly kinetic and partly potential
(c) purely kinetic
(d) purely potential

2. Three coherent waves having amplitudes 12 mm, 6 mm and 4 mm arrive at a given point with successive phase difference of $\pi/2$. Then, the amplitude of the resultant wave is

(a) 7 mm
(b) 10 mm
(c) 5 mm
(d) 4.8 mm

3. Two transverse waves A and B superimpose to produce a node at $x = 0$. If the equation of wave A is $y = a \cos (kx + \omega t)$, then the equation of wave B is

(a) $+a \cos (kx - \omega t)$
(b) $-a \cos (kx + \omega t)$
(c) $-a \cos (kx - \omega t)$
(d) $+a \cos (\omega t - kx)$

4. In a stationary wave system, all the particles of the medium

(a) have zero displacement simultaneously at some instant
(b) have maximum displacement simultaneously at some instant
(c) are at rest simultaneously at some instant
(d) all of the above

5. In a standing wave on a string

(a) In one time period all the particles are simultaneously at rest twice
(b) All the particles must be at their positive extremes simultaneously once in a time period
(c) All the particles may be at their positive extremes simultaneously twice in a time period
(d) All the particles are never at rest simultaneously

6. If λ_1, λ_2 and λ_3 are the wavelength of the waves giving resonance to the fundamental, first and second overtone modes respectively in a string fixed at both ends. The ratio of the wavelengths $\lambda_1 : \lambda_2 : \lambda_3$ is

(a) $1 : 2 : 3$
(b) $1 : 3 : 5$
(c) $1 : \frac{1}{2} : \frac{1}{3}$
(d) $1 : \frac{1}{3} : \frac{1}{5}$

7. In a standing wave, node is a point of

(a) maximum strain
(b) maximum pressure
(c) maximum density
(d) All of these

8. For a certain stretched string, three consecutive resonance frequencies are observed as 105, 175 and 245 Hz respectively. Then, the fundamental frequency is

(a) 30 Hz
(b) 45 Hz
(c) 35 Hz
(d) None of these

9. A string 1 m long is drawn by a 300 Hz vibrator attached to its end. The string vibrates in three segments. The speed of transverse waves in the string is equal to

(a) 100 m/s
(b) 200 m/s
(c) 300 m/s
(d) 400 m/s

10. Three resonant frequencies of string with both rigid ends are 90, 150 and 210 Hz. If the length of the string is 80 cm, what is the speed of the transverse wave in the string?

(a) 45 m/s
(b) 75 m/s
(c) 48 m/s
(d) 80 m/s

11. The period of oscillations of a point is 0.04 s and the velocity of propagation of oscillation is 300 m/s. The difference of phases between the oscillations of two points at distance 10 m and 16 m respectively from the source of oscillations is

(a) 2π
(b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$
(d) π

12. A transverse wave described by an equation $y = 0.02 \sin(x + 30t)$, where x and t are in metre and second, is travelling along a wire of area of cross-section 1 mm^2 and density 8000 kgm^{-3}. What is the tension in the string?

(a) 20 N
(b) 7.2 N
(c) 30 N
(d) 14.4 N

Subjective Questions

1. Two waves are travelling in the same direction along a stretched string. The waves are 90° out of phase. Each wave has an amplitude of 4.0 cm. Find the amplitude of the resultant wave.

2. Two wires of different densities are soldered together end to end then stretched under tension T. The wave speed in the first wire is twice that in the second wire.

(a) If the amplitude of incident wave is A, what are amplitudes of reflected and transmitted waves?
(b) Assuming no energy loss in the wire, find the fraction of the incident power that is reflected at the junction and fraction of the same that is transmitted.

3. A wave is represented by

$$y_1 = 10 \cos(5x + 25\,t)$$

where, x is measured in metres and t in seconds. A second wave for which

$$y_2 = 20 \cos\left(5x + 25t + \frac{\pi}{3}\right)$$

interferes with the first wave. Deduce the amplitude and phase of the resultant wave.

4. Two waves passing through a region are represented by

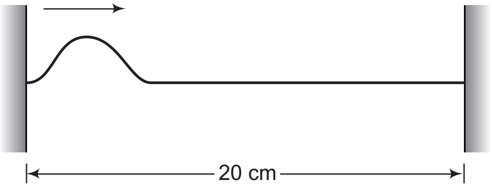
$$y_1 = (1.0\text{ cm}) \sin[(3.14\text{ cm}^{-1})\,x - (157\text{ s}^{-1})\,t]$$

and

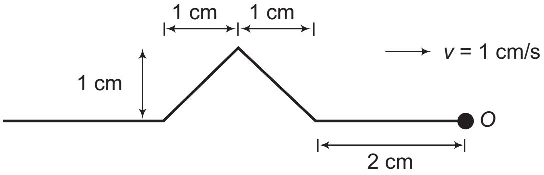
$$y_2 = (1.5\text{ cm}) \sin[(1.57\text{ cm}^{-1})\,x - (314\text{ s}^{-1})\,t]$$

Find the displacement of the particle at $x = 4.5$ cm at time $t = 5.0$ ms.

5. A string of length 20 cm and linear mass density 0.4 g/cm is fixed at both ends and is kept under a tension of 16 N. A wave pulse is produced at $t = 0$ near an end as shown in figure which travels towards the other end. (a) When will the string have the shape shown in the figure again? (b) Sketch the shape of the string at a time half of that found in part (a).



6. A wave pulse on a string has the dimensions shown in figure. The wave speed is $v = 1$ cm/s.



(a) If point O is a fixed end, draw the resultant wave on the string at $t = 3$ s and $t = 4$ s.

(b) Repeat part (a) for the case in which O is a free end.

7. Two sinusoidal waves combining in a medium are described by the equations

$$y_1 = (3.0 \text{ cm}) \sin \pi (x + 0.60\, t)$$

and

$$y_2 = (3.0 \text{ cm}) \sin \pi (x - 0.60\, t)$$

where, x is in centimetres and t is in seconds. Determine the maximum displacement of the medium at

(a) $x = 0.250$ cm,

(b) $x = 0.500$ cm and

(c) $x = 1.50$ cm.

(d) Find the three smallest values of x corresponding to antinodes.

8. A standing wave is formed by the interference of two travelling waves, each of which has an amplitude $A = \pi$ cm, angular wave number $k = (\pi/2)$ per centimetre.

(a) Calculate the distance between two successive antinodes.

(b) What is the amplitude of the standing wave at $x = 0.50$ cm from a node?

9. Find the fundamental frequency and the next three frequencies that could cause a standing wave pattern on a string that is 30.0 m long, has a mass per unit length of 9.00×10^{-3} kg/m and is stretched to a tension of 20.0 N.

10. A string vibrates in its first normal mode with a frequency of 220 vibrations/s. The vibrating segment is 70.0 cm long and has a mass of 1.20 g.

(a) Find the tension in the string.

(b) Determine the frequency of vibration when the string vibrates in three segments.

11. A 60.0 cm guitar string under a tension of 50.0 N has a mass per unit length of 0.100 g/cm. What is the highest resonance frequency of the string that can be heard by a person able to hear frequencies upto 20000 Hz?

12. A wire having a linear density of 0.05 g/cm is stretched between two rigid supports with a tension of 450 N. It is observed that the wire resonates at a frequency of 420 Hz. The next higher frequency at which the same wire resonates is 490 Hz. Find the length of the wire.

13. The vibrations from an 800 Hz tuning fork set up standing waves in a string clamped at both ends. The wave speed in the string is known to be 400 m/s for the tension used. The standing wave is observed to have four antinodes. How long is the string?

14. A string vibrates in 4 segments to a frequency of 400 Hz.

(a) What is its fundamental frequency?

(b) What frequency will cause it to vibrate into 7 segments?

15. A sonometer wire has a total length of 1 m between the fixed ends. Where should the two bridges be placed below the wire so that the three segments of the wire have their fundamental frequencies in the ratio 1 : 2 : 3 ?

16. A guitar string is 90 cm long and has a fundamental frequency of 124 Hz. Where should it be pressed to produce a fundamental frequency of 186 Hz?

17. Adjacent antinodes of a standing wave on a string are 15.0 cm apart. A particle at an antinode oscillates in simple harmonic motion with amplitude 0.850 cm and period 0.0750 s. The string lies along the $+x$-axis and is fixed at $x = 0$.

(a) Find the displacement of a point on the string as a function of position and time.

(b) Find the speed of propagation of a transverse wave in the string.

(c) Find the amplitude at a point 3.0 cm to the right of an antinode.

18. A 1.50 m long rope is stretched between two supports with a tension that makes the speed of transverse waves 48.0 m/s. What are the wavelength and frequency of

(a) the fundamental?

(b) the second overtone?

(c) the fourth harmonic?

19. A thin taut string tied at both ends and oscillating in its third harmonic has its shape described by the equation $y(x,t) = (5.60 \text{ cm}) \sin[(0.0340 \text{ rad/cm})x] \sin[(50.0 \text{ rad/s})t]$, where the origin is at the left end of the string, the x-axis is along the string and the y-axis is perpendicular to the string.

(a) Draw a sketch that shows the standing wave pattern.

(b) Find the amplitude of the two travelling waves that make up this standing wave.

(c) What is the length of the string?

(d) Find the wavelength, frequency, period and speed of the travelling wave.

(e) Find the maximum transverse speed of a point on the string.

(f) What would be the equation $y(x, t)$ for this string if it were vibrating in its eighth harmonic?

20. A wire with mass 40.0 g is stretched so that its ends are tied down at points 80.0 cm apart. The wire vibrates in its fundamental mode with frequency 60.0 Hz and with an amplitude at the antinodes of 0.300 cm.

(a) What is the speed of propagation of transverse wave in the wire?

(b) Compute the tension in the wire.

(c) Find the maximum transverse velocity and acceleration of particles in the wire.

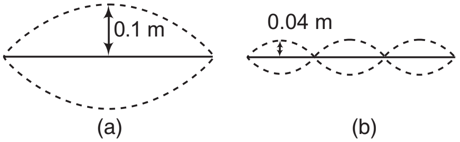
21. Two harmonic waves are represented in SI units by

$$y_1(x, t) = 0.2 \sin(x - 3.0\,t) \quad \text{and} \quad y_2(x, t) = 0.2 \sin(x - 3.0\,t + \phi)$$

(a) Write the expression for the sum $y = y_1 + y_2$ for $\phi = \dfrac{\pi}{2}$ rad.

(b) Suppose the phase difference ϕ between the waves is unknown and the amplitude of their sum is 0.32 m, what is ϕ?

22. Figure shows different standing wave patterns on a string of linear mass density 4.0×10^{-2} kg/m under a tension of 100 N. The amplitude of antinodes is indicated in each figure. The length of the string is 2.0 m.



(i) Obtain the frequencies of the modes shown in figures (a) and (b).

(ii) Write down the transverse displacement y as a function of x and t for each mode. (Take the initial configuration of the wire in each mode to be as shown by the dark lines in the figure).

23. A 160 g rope 4 m long is fixed at one end and tied to a light string of the same length at the other end. Its tension is 400 N.

(a) What are the wavelengths of the fundamental and the first two overtones?

(b) What are the frequencies of these standing waves?

[**Hint :** In this case, fixed end is a node and the end tied with the light string is antinode.]

24. A string fastened at both ends has successive resonances with wavelengths of 0.54 m for the nth harmonic and 0.48 m for the $(n+1)$ th harmonic.

(a) Which harmonics are these?

(b) What is the length of the string?

(c) What is the wavelength of the fundamental frequency?

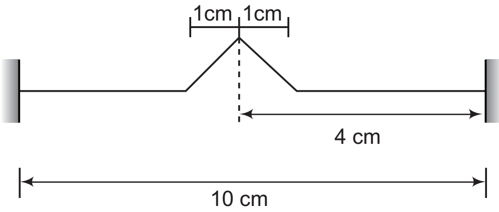
25. A wave $y_i = 0.3 \cos(2.0x - 40t)$ is travelling along a string toward a boundary at $x = 0$. Write expressions for the reflected waves if

(a) the string has a fixed end at $x = 0$ and

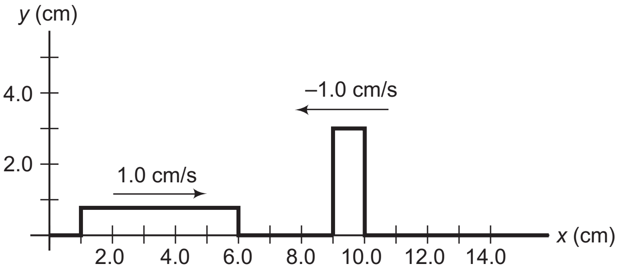
(b) the string has a free end at $x = 0$.

Assume SI units.

26. A string that is 10 cm long is fixed at both ends. At $t = 0$, a pulse travelling from left to right at 1 cm/s is 4.0 cm from the right end as shown in figure. Determine the next two times when the pulse will be at that point again. State in each case whether the pulse is upright or inverted.



27. Two pulses travelling in opposite directions along a string are shown for $t = 0$ in the figure. Plot the shape of the string at $t = 1.0$, 2.0, 3.0, 4.0 and 5.0 s respectively.



LEVEL 2

Single Correct Option

1. When tension of a string is increased by 2.5 N, the initial frequency is altered in the ratio of $3:2$. The initial tension in the string is

(a) 6 N (b) 5 N (c) 4 N (d) 2 N

2. The lengths of two wires of same material are in the ratio 1 : 2, their tensions are in the ratio 1 : 2 and their diameters are in the ratio 1 : 3. The ratio of the notes they emit when sounded together by the same source is

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $2\sqrt{3}$ (d) $3\sqrt{2}$

3. If f_1, f_2 and f_3 are the fundamental frequencies of three segments into which a string is divided, then the original fundamental frequency f_0 of the whole string is

(a) $f_0 = f_1 + f_2 + f_3$

(b) $\dfrac{1}{f_0} = \dfrac{1}{f_1} + \dfrac{1}{f_2} + \dfrac{1}{f_3}$

(c) $\dfrac{1}{\sqrt{f_0}} = \dfrac{1}{\sqrt{f_1}} + \dfrac{1}{\sqrt{f_2}} + \dfrac{1}{\sqrt{f_3}}$

(d) None of these

4. Three one-dimensional mechanical waves in an elastic medium is given as

$$y_1 = 3A\sin(\omega t - kx), \quad y_2 = A\sin(\omega t - kx + \pi) \text{ and } \quad y_3 = 2A\sin(\omega t + kx)$$

are superimposed with each other. The maximum displacement amplitude of the medium particle would be

(a) $4A$

(b) $3A$

(c) $2A$

(d) A

5. A string is stretched so that its length is increased by $\dfrac{1}{\eta}$ of its original length. The ratio of fundamental frequency of transverse vibration to that of fundamental frequency of longitudinal vibration will be

(a) $\eta : 1$

(b) $1 : \eta$

(c) $\sqrt{\eta} : 1$

(d) $1 : \sqrt{\eta}$

6. A string of length 1 m and linear mass density 0.01 kg/m is stretched to a tension of 100 N. When both ends of the string are fixed, the three lowest frequencies for standing wave are f_1, f_2 and f_3. When only one end of the string is fixed, the three lowest frequencies for standing wave are n_1, n_2 and n_3. Then,

(a) $n_3 = 5n_1 = f_3 = 125$ Hz

(b) $f_3 = 5f_1 = n_2 = 125$ Hz

(c) $f_3 = n_2 = 3f_1 = 150$ Hz

(d) $n_2 = \dfrac{f_1 + f_2}{2} = 75$ Hz

7. A sonometer wire has a length 114 cm between two fixed ends. Where should two bridges be placed so as to divide the wire into three segments whose fundamental frequencies are in the ratio 1 : 3 : 4

(a) $l_1 = 72$ cm, $l_2 = 24$ cm, $l_3 = 18$ cm

(b) $l_1 = 60$ cm, $l_2 = 40$ cm, $l_3 = 14$ cm

(c) $l_1 = 52$ cm, $l_2 = 30$ cm, $l_3 = 32$ cm

(d) $l_1 = 65$ cm, $l_2 = 30$ cm, $l_3 = 19$ cm

8. The frequency of a sonometer wire is f. The frequency becomes $f/2$ when the mass producing the tension is completely immersed in water and on immersing the mass in a certain liquid, frequency becomes $f/3$. The relative density of the liquid is

(a) 1.32 (b) 1.67 (c) 1.41 (d) 1.18

9. A string of length 1.5 m with its two ends clamped is vibrating in fundamental mode. Amplitude at the centre of the string is 4 mm. Distance between the two points having amplitude 2 mm is

(a) 1 m (b) 75 cm
(c) 60 cm (d) 50 cm

10. A man generates a symmetrical pulse in a string by moving his hand up and down. At $t = 0$, the point in his hand moves downwards from mean position. The pulse travels with speed 3 m/s on the string and his hand passes 6 times in each second from the mean position. Then, the point on the string at a distance 3 m will reach its upper extreme first time at $t =$

(a) 1.25 s (b) 1 s (c) 11/12 s (d) $\frac{23}{24}$ s

11. Among two interfering sources, let S_1 be ahead of the phase by 90° relative to S_2. If an observation point P is such that $PS_1 - PS_2 = 1.5\,\lambda$, the phase difference between the waves from S_1 and S_2 reaching P is

(a) 3π (b) $\frac{5\pi}{2}$ (c) $\frac{7\pi}{2}$ (d) 4π

12. A wire having a linear density 0.1 kg/m is kept under a tension of 490 N. It is observed that it resonates at a frequency of 400 Hz. The next higher frequency is 450 Hz. Find the length of the wire.

(a) 0.4 m (b) 0.7 m (c) 0.6 m (d) 0.49 m

13. A string vibrates in 5 segments to a frequency of 480 Hz. The frequency that will cause it to vibrate in 2 segments will be

(a) 96 Hz (b) 192 Hz (c) 1200 Hz (d) 2400 Hz

14. A wave travels on a light string. The equation of the waves is $y = A\sin(kx - \omega t + 30°)$. It is reflected from a heavy string tied to an end of the light string at $x = 0$. If 64% of the incident energy is reflected, then the equation of the reflected wave is

(a) $y = 0.8\,A\sin(kx - \omega t + 30° + 180°)$ (b) $y = 0.8\,A\sin(kx + \omega t + 30° + 180°)$
(c) $y = 0.8\,A\sin(kx - \omega t - 30°)$ (d) $y = 0.8\,A\sin(kx - \omega t + 30°)$

15. The tension, length, diameter and density of a string B are double than that of another string A. Which of the following overtones of B is same as the fundamental frequency of A?

(a) 1st (b) 2nd (c) 3rd (d) 4th

More than One Correct Options

1. If the tension in a stretched string fixed at both ends is increased by 21%, the fundamental frequency is found to be changed by 15 Hz. Then, the

(a) original frequency is 150 Hz
(b) velocity of propagation of the transverse wave along the string increases by 5%
(c) velocity of propagation of the transverse wave along the string increases by 10%
(d) fundamental wavelength on the string does not change

2. For interference to take place

(a) sources must be coherent (b) sources must have same amplitude
(c) waves should travel in opposite directions (d) sources must have same frequency

3. Regarding stationary waves, choose the correct options.

(a) This is an example of interference (b) Amplitudes of waves may be different
(c) Particles at nodes are always at rest (d) Energy is conserved

4. When a wave travels from a denser to rarer medium, then

(a) speed of wave increases
(b) wavelength of wave decreases
(c) amplitude of wave increases
(d) there is no change in phase angle

5. A wire is stretched and fixed at two ends. Transverse stationary waves are formed in it. It oscillates in its third overtone mode. The equation of stationary wave is

$$y = A \sin kx \cos \omega t$$

Choose the correct options.

(a) Amplitude of constituent waves is $\frac{A}{2}$
(b) The wire oscillates in three loops
(c) The length of the wire is $\frac{4\pi}{k}$
(d) Speed of stationary wave is $\frac{\omega}{k}$

6. Which of the following equations can form stationary waves?

(i) $y = A \sin (\omega t - kx)$
(ii) $y = A \cos (\omega t - kx)$
(iii) $y = A \sin (\omega t + kx)$
(iv) $y = A \cos (\omega t + kx)$

(a) (i) and (ii)
(b) (i) and (iii)
(c) (iii) and (iv)
(d) (ii) and (iv)

7. Two waves

$$y_1 = A \sin (\omega t - kx)$$

and

$$y_2 = A \sin (\omega t + kx)$$

superimpose to produce a stationary wave, then

(a) $x = 0$ is a node
(b) $x = 0$ is an antinode
(c) $x = \frac{\pi}{k}$ is a node
(d) $\pi = \frac{2\pi}{k}$ is an antinode

Comprehension Based Questions

Passage

Incident wave $y = A \sin\left(ax + bt + \frac{\pi}{2}\right)$ *is reflected by an obstacle at* $x = 0$ *which reduces intensity of reflected wave by 36%. Due to superposition, the resulting wave consists of a standing wave and a travelling wave given by*

$$y = -1.6 \sin ax \sin bt + cA \cos (bt + ax)$$

where A, a, b *and* c *are positive constants.*

1. Amplitude of reflected wave is

(a) $0.6\,A$
(b) $0.8\,A$
(c) $0.4\,A$
(d) $0.2\,A$

2. Value of c is

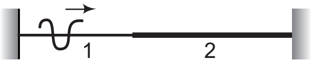
(a) 0.2
(b) 0.4
(c) 0.6
(d) 0.3

3. Position of second antinode is

(a) $x = \frac{\pi}{3a}$
(b) $x = \frac{3\pi}{a}$
(c) $x = \frac{3\pi}{2a}$
(d) $x = \frac{2\pi}{3a}$

Match the Columns

1. In the figure shown, mass per unit length of string-2 is nine times that of string-1. Tension in both the strings is same. A transverse wave is incident at the boundary as shown. Part of wave is reflected in medium-1 and part is transmitted to medium-2. Match the following two columns.



Column I	Column II
(a) $\lvert A_1/A_2 \rvert$	(p) 9
(b) v_1/v_2	(q) 1
(c) I_1/I_2	(r) 3
(d) P_2/P_1	(s) data insufficient

Here A is amplitude, v the speed of wave, I the intensity and P the power. Abbreviation 1 is used for reflected wave and 2 for transmitted wave.

2. Transverse waves are produced in a stretched wire. Both ends of the string are fixed. Let us compare between second overtone mode (in numerator) and fifth harmonic mode (in denominator). Match the following two columns.

Column I	Column II
(a) Frequency ratio	(p) 2/3
(b) Number of nodes ratio	(q) 4/5
(c) Number of antinodes ratio	(r) 3/5
(d) Wavelength ratio	(s) 5/3

3. A wave travels from a denser medium to a rarer medium, then match the following two columns.

Column I	Column II
(a) speed of wave	(p) will increase
(b) wavelength of wave	(q) will decrease
(c) amplitude of wave	(r) will remain unchanged
(d) frequency of wave	(s) may increase or decrease

4. Two waves of same amplitude and same intensity interfere at one point. Phase difference between them is θ. Match the following two columns.

Column I	Column II
(a) Resultant amplitude for $\theta = 60^\circ$	(p) 2 times
(b) Resultant amplitude for $\theta = 120^\circ$	(q) 3 times
(c) Resultant intensity for $\theta = 90^\circ$	(r) 4 times
(d) Resultant intensity for $\theta = 0^\circ$	(s) None of these

5. A wire is stretched and fixed at its two ends. Its second overtone frequency is 210 Hz. Then, match the following two columns.

Column I	Column II
(a) Fundamental frequency	(p) 210 Hz
(b) Third harmonic frequency	(q) 350 Hz
(c) Third overtone frequency	(r) 280 Hz
(d) Second harmonic frequency	(s) None of these

Subjective Questions

1. Three pieces of string, each of length L, are joined together end-to-end, to make a combined string of length $3L$. The first piece of string has mass per unit length μ_1, the second piece has mass per unit length $\mu_2 = 4\mu_1$ and the third piece has mass per unit length $\mu_3 = \mu_1/4$.

(a) If the combined string is under tension F, how much time does it take a transverse wave to travel the entire length $3L$? Give your answer in terms of L, F and μ_1.

(b) Does your answer to part (a) depend on the order in which the three piece are joined together? Explain.

2. In a stationary wave that forms as a result of reflection of waves from an obstacle, the ratio of the amplitude at an antinode to the amplitude at node is 6. What percentage of energy is transmitted?

3. A standing wave $y = a \sin kx \cos \omega t$ is maintained in a homogeneous rod with cross-sectional area S and density ρ. Find the total mechanical energy confined between the sections corresponding to the adjacent nodes.

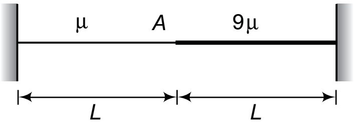
4. A string of mass per unit length μ is clamped at both ends such that one end of the string is at $x = 0$ and the other is at $x = l$. When string vibrates in fundamental mode, amplitude of the mid-point of the string is a and tension in the string is T. Find the total oscillation energy stored in the string.

5. A long wire PQR is made by joining two wires PQ and QR of equal radii. PQ has a length 4.8 m and mass 0.06 kg. QR has length 2.56 m and mass 0.2 kg. The wire PQR is under a tension of 80 N. A sinusoidal wave pulse of amplitude 3.5 cm is sent along the wire PQ from the end P. No power is dissipated during the propagation of the wave pulse. Calculate :

(a) The time taken by the wave pulse to reach the other end R.

(b) The amplitude of the reflected and transmitted wave pulse after the incident wave pulse crosses the joint Q.

6. A light string is tied at one end to fixed support and to a heavy string of equal length L at the other end as shown in figure. Mass per unit length of the strings are μ and 9μ and the tension is T. Find the possible values of frequencies such that point A is a node/antinode.

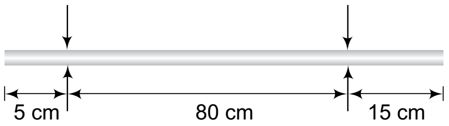


7. A string fixed at both ends is vibrating in the lowest possible mode of vibration for which a point at quarter of its length from one end is a point of maximum displacement. The frequency of vibration in this mode is 100 Hz. What will be the frequency emitted when it vibrates in the next mode such that this point is again a point of maximum displacement ?

8. Sources separated by 20 m vibrate according to the equation $y_1 = 0.06 \sin \pi t$ and $y_2 = 0.02 \sin \pi t$. They send out waves along a rod with speed 3 m/s. What is the equation of motion of a particle 12 m from the first source and 8 m from the second, y_1, y_2 are in m?

9. Three component sinusoidal waves progressing in the same direction along the same path have the same period but their amplitudes are $A, \frac{A}{2}$ and $\frac{A}{3}$. The phases of the variation at any position x on their path at time $t = 0$ are $0, -\frac{\pi}{2}$ and $-\pi$ respectively. Find the amplitude and phase of the resultant wave.

10. A metal rod of length 1 m is clamped at two points as shown in the figure. Distance of the clamp from the two ends are 5 cm and 15 cm, respectively. Find the minimum and next higher frequency of natural longitudinal oscillation of the rod. Given that Young's modulus of elasticity and density of aluminium are $Y = 1.6 \times 10^{11}\ \text{Nm}^{-2}$ and $\rho = 2500\ \text{kgm}^{-3}$, respectively.



11. $y_1 = 8 \sin(\omega t - kx)$ and $y_2 = 6 \sin(\omega t + kx)$ are two waves travelling in a string of area of cross-section s and density ρ. These two waves are superimposed to produce a standing wave.

(a) Find the energy of the standing wave between two consecutive nodes.

(b) Find the total amount of energy crossing through a node per second.

Answers

Introductory Exercise 18.1

1. 49 : 1 **2.** (a) 4 : 1 (b) 16 : 1 **3.** 25 I_0

Introductory Exercise 18.2

1. True **2.** (a) 120 cm/s (b) 3 cm (c) zero

3. Yes, Yes **4.** 15.7 m, 31.8 Hz, 500 m/s

Introductory Exercise 18.3

1. 0.125 **2.** 18 Hz **3.** 0.873 kg

4. (a) 5th and 6th (b) 360 N **5.** $\frac{l_1}{l_2} = 3$

Introductory Exercise 18.4

1. Energy is in the form of kinetic energy **2.** (a) 20 cms^{-1} (b) $\frac{A_r}{A_i} = \frac{1}{3}, \frac{A_t}{A_i} = \frac{4}{3}$

3. $y_r = \frac{2}{3} \times 10^{-3} \cos \pi(2.0x + 50t)$, $y_t = \frac{8}{3} \times 10^{-3} \cos \pi(x - 50t)$

Exercises

LEVEL 1

Assertion and Reason

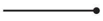
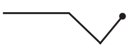
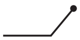
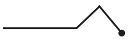
1. (d) **2.** (d) **3.** (b) **4.** (b) **5.** (a) **6.** (d) **7.** (a) **8.** (a) **9.** (b) **10.** (d)

Objective Questions

1. (c) **2.** (b) **3.** (c) **4.** (d) **5.** (a) **6.** (c) **7.** (d) **8.** (c) **9.** (b) **10.** (c)
11. (d) **12.** (b)

Subjective Questions

1. 5.66 cm **2.** (a) $-\frac{A}{3}, \frac{2}{3}A$ (b) $\frac{1}{9}, \frac{8}{9}$ **3.** 26.46 cm, $(5x + 25t + 0.714)$ rad **4.** $\frac{-1}{2\sqrt{2}}$ cm

5. (a) 0.02 s **6.** (a) (i)  (ii)  (b) (i)  (ii) 

7. (a) 4.24 cm (b) 6.00 cm (c) 6.00 cm (d) 0.500 cm, 1.50 cm, 2.50 cm

8. (a) 2 cm (b) $\sqrt{2}\pi$ cm **9.** 0.786 Hz, 1.57 Hz, 2.36 Hz, 3.14 Hz

10. (a) 163 N (b) 660 Hz **11.** 19977 Hz **12.** 2.142 m **13.** 1.0 m

14. (a) 100 Hz (b) 700 Hz

15. One bridge at 6/11 m from one end and the other at 2/11 m from the other end.

16. The string should be pressed at 60 cm from one end.

17. (a) $y(x, t) = (0.85 \text{ cm}) \sin\left[\frac{2\pi x}{0.3 \text{ cm}}\right] \sin\left[\frac{2\pi t}{0.075 \text{ s}}\right]$ (b) 4.00 m/s (c) 0.688 cm

18. (a) 3.0 m, 16.0 Hz (b) 1.0 m, 48.0 Hz (c) 0.75 m, 64.0 Hz

19. (b) 2.80 cm (c) 277 cm (d) 185 cm, 7.9 Hz, 0.126 s, 1470 cm/s (e) 280 cm/s
(f) $y(x, t) = (5.6\text{ cm}) \sin[(0.0907\text{ rad/cm})x] \sin[(133\text{ rad/s})t]$

20. (a) 96.0 m/s (b) 461 N (c) 1.13 m/s, 426.4 m/s^2

21. (a) $y = 0.28 \sin\left(x - 3.0t + \frac{\pi}{4}\right)$ (b) $\phi = \pm 1.29$ rad

22. (i) Fundamental 12.5 Hz, third harmonic 37.5 Hz
(ii) (a) $y = 0.1 \sin \frac{\pi x}{2} \sin 25\pi t$ (b) $y = 0.04 \sin \frac{3\pi x}{2} \sin 75\pi t$

23. (a) 16 m, 5.33 m, 3.2 m (b) 6.25 Hz, 18.75 Hz, 31.25 Hz

24. (a) 8th and 9th (b) 2.16 m (c) 4.32 m

25. (a) $y(x, t) = 0.3 \cos(2.0x + 40t + \pi)$
(b) $y(x, t) = 0.3 \cos(2.0x + 40t)$ SI units

26. 8 s, inverted, 20 s upright

27. See the hints

LEVEL 2

Single Correct Option

1.(d) **2.**(d) **3.**(b) **4.**(a) **5.**(d) **6.**(d) **7.**(a) **8.**(d) **9.**(a) **10.**(c)
11.(b) **12.**(b) **13.**(b) **14.**(b) **15.**(c)

More than One Correct Options

1.(a,c,d) **2.**(a,d) **3.**(a,b,d) **4.**(a,c,d) **5.**(a,c) **6.**(b,d) **7.**(b,d)

Comprehension Based Questions

1.(b) **2.**(a) **3.**(c)

Match the Columns

1. (a) → q (b) → r (c) → s (d) → r **2.** (a) → r (b) → p (c) → r (d) → s
3. (a) → p (b) → p (c) → p (d) → r **4.** (a) → s (b) → s (c) → p (d) → r
5. (a) → s (b) → p (c) → r (d) → s

Subjective Questions

1. (a) $\frac{7L}{2}\sqrt{\frac{\mu_1}{F}}$ (b) No **2.** 49% **3.** $\frac{\pi S\rho\omega^2 a^2}{4k}$ **4.** $\frac{\pi^2 a^2 T}{4l}$

5. (a) 0.14 s (b) –1.5 cm, 2.0 cm

6. $\frac{f}{2}, f, \frac{3f}{2}$... etc. when A is a node, $\frac{3}{4}f, \frac{5}{4}f, \frac{7}{4}f$... etc. when A is an antinode. Here, $f = \frac{1}{L}\sqrt{\frac{T}{\mu}}$

7. 300 Hz **8.** $0.05 \sin \pi t - 0.0173 \cos \pi t$ **9.** $\frac{5}{6}A, -\tan^{-1}\left(\frac{3}{4}\right)$

10. 40 kHz, 120 kHz **11.** (a) $\frac{50\pi}{k}\rho\omega^2 S$ (b) $\frac{2\rho\omega^3 S}{k}$

19

Sound Waves

Chapter Contents

19.1 Introduction

19.2 Displacement Wave, Pressure Wave and Density Wave

19.3 Speed of a Longitudinal Wave

19.4 Sound Waves in Gases

19.5 Sound Intensity

19.6 Interference in Sound Wave and Stationary Wave

19.7 Standing Longitudinal Waves in Organ Pipes

19.8 Beats

19.9 The Doppler's Effect

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19.1 Introduction

Of all the mechanical waves that occur in nature, the most important in our everyday life are longitudinal waves in a medium, usually air, called **sound waves.** The reason is that the human ear is tremendously sensitive and can detect sound waves even of very low intensity. The human ear is sensitive to waves in the frequency range from about 20 to 20000 Hz called the **audible range**, but we also use the term sound for similar waves with frequencies above **(ultrasonic)** and below **(infrasonic)** the range of human hearing. Our main concern in this chapter is with sound waves in air, but sound can travel through any gas, liquid or solid.

In this chapter, we will discuss several important properties of sound waves, including **pressure wave** and **density wave.** We will find that superposition of two sound waves differing slightly in frequency causes a phenomenon called **beats.** When a source of sound or a listener moves through the air, the listener may hear a different frequency than the one emitted by the source. This is the **Doppler's effect.**

Note *Sound wave (rather we can call audible sound wave) is a part of longitudinal wave with frequency varying from 20 Hz to 20,000 Hz.*

19.2 Displacement Wave, Pressure Wave and Density Wave

Upto this point we have described mechanical waves primarily in terms of displacement. As we have discussed in chapter-17, a sinusoidal wave equation $y(x, t)$ travelling in positive x-direction can be written as

$$\boxed{y(x, t) = A \sin(kx - \omega t)}$$

which gives the instantaneous displacement y of a particle in the medium at position x at time t.

Note that in a longitudinal wave the displacements are parallel to the direction of motion of the wave. So, x and y are measured parallel to each other, not perpendicular as in a transverse wave. The amplitude A is the maximum displacement of a particle in a medium from its equilibrium position.

These displacements are along the direction of the motion of the wave lead to **variations in the density and pressure** of air. Hence, a sound wave can also be described in terms of variations of pressure or density at various points. The pressure fluctuations are of the order of $1\text{ Pa}\ (= 1\text{ N/m}^2)$, whereas atmospheric pressure is about 10^5 Pa. In a sinusoidal sound wave in air, the pressure fluctuates above and below atmospheric pressure p_0 with the same frequency as the motions of the air particles. Similarly, the density of air also vibrates sinusoidally above and below its normal level. So, we can express a sound wave either in terms of $y(x, t)$ or $\Delta p(x, t)$ or $\Delta\rho(x, t)$. All the three equations are related to one another. For example, amplitude of pressure variation $(\Delta p)_m$ is related to amplitude of displacement A by the equation,

$$\Delta p_m = BAk$$

where B is the bulk modulus of the medium. Moreover, pressure or density wave is 90° out of phase with the displacement wave, i.e. when the displacement is zero, the pressure and density changes are either maximum or minimum and when the displacement is a maximum or minimum, the pressure and density changes are zero. Now, let us find the relation between them.

Relation between Displacement Wave and Pressure Wave

Figure shows a harmonic displacement wave moving through air contained in a long tube of cross-sectional area S.

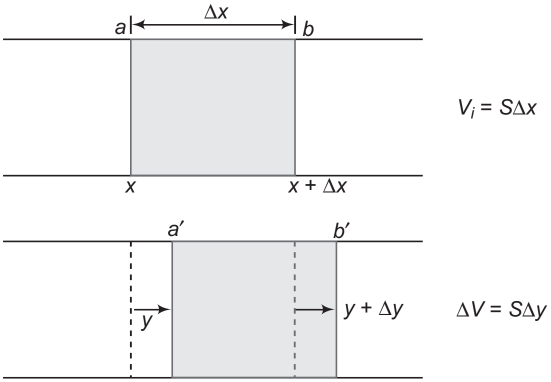


Fig. 19.1

The volume of gas that has a thickness Δx in the horizontal direction is $V_i = S\Delta x$. The change in volume ΔV is $S\Delta y$, where Δy is the difference between the value of y at $x + \Delta x$ and the value of y at x. From the definition of bulk modulus, the pressure variation in the gas is

$$\Delta p = -B\frac{\Delta V}{V_i} = -B\left(\frac{S\Delta y}{S\Delta x}\right) = -B\left(\frac{\Delta y}{\Delta x}\right)$$

As Δx approaches zero, the ratio $\frac{\Delta y}{\Delta x}$ becomes $\frac{\partial y}{\partial x}$. (The partial derivative indicates that we are interested in the variation of y with position at a fixed time).

Therefore, $$\boxed{\Delta p = -B\frac{\partial y}{\partial x}} \qquad \text{...(i)}$$

So, this is the equation which relates the displacement equation with the pressure equation. Suppose the displacement equation is

$$\boxed{y = A\cos(kx - \omega t)} \qquad \text{...(ii)}$$

Then, $$\frac{\partial y}{\partial x} = -kA\sin(kx - \omega t) \qquad \text{...(iii)}$$

and from Eqs. (i) and (iii), we find that

$$\boxed{\begin{gathered}\Delta p = BAk\sin(kx - \omega t) = (\Delta p)_m\sin(kx - \omega t)\\ (\Delta p)_m = BAk\end{gathered}} \qquad \text{...(iv)}$$

Here,

where Δp_m is the amplitude of pressure variation. From Eqs. (ii) and (iv), we see that pressure equation is 90° out of phase with displacement equation. When the displacement is zero, the pressure variation is either maximum or minimum and *vice-versa.* Fig. 19.2 (a) shows displacement from equilibrium of air molecules in a harmonic sound wave *versus* position at some instants. Points x_1 and x_3 are points of zero displacement.

Now, refer Fig. 19.2 (b) : Just to the left of x_1, the displacement

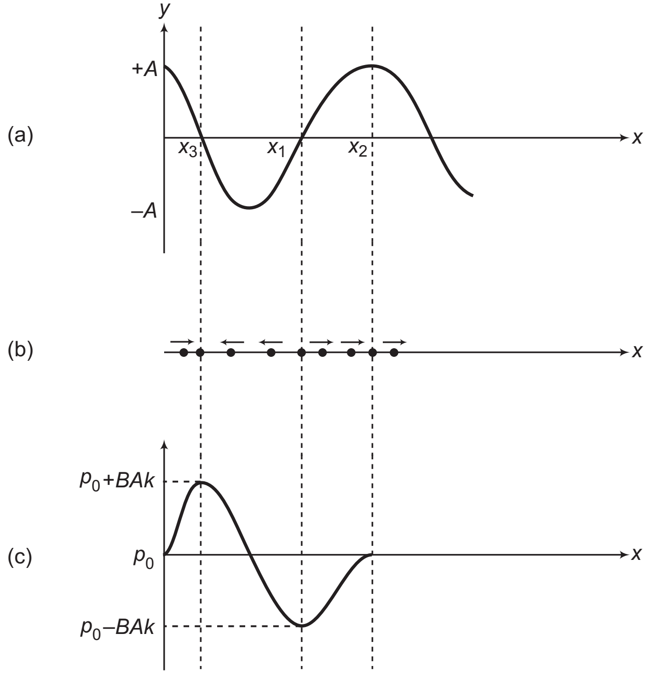


Fig. 19.2

is negative indicating that the gas molecules are displaced to left away from point x_1 at this instant. Just to the right of x_1, the displacement is positive indicating that the molecules are displaced to the right, which is again away from point x_1. So, at point x_1 the pressure of the gas is minimum. So, if p_0 is the atmospheric pressure (normal pressure), the pressure at x_1 will be

$$p(x_1) = p_0 - (\Delta p)_m = p_0 - BAk$$

At point x_3, the pressure (and hence the density also) is maximum because the molecules on both sides of that point are displaced toward point x_3. Hence,

$$p(x_3) = p_0 + (\Delta p)_m = p_0 + BAk$$

At point x_2, the pressure (and hence the density) does not change because the gas molecules on both sides of that point have equal displacements in the same direction or

$$p(x_2) = p_0$$

From Fig. 19.2 (a) and (c), we see that pressure change and displacement are 90° out of phase.

Relation between Pressure Wave and Density Wave

In this section, we will find the relation between pressure wave and density wave.

According to definition of bulk modulus (B),

$$B = \left(-\frac{dp}{dV/V}\right) \qquad \ldots\text{(i)}$$

Further, $$\text{volume} = \frac{\text{mass}}{\text{density}} \quad \text{or} \quad V = \frac{m}{\rho}$$

or $$dV = -\frac{m}{\rho^2} \cdot d\rho = -\frac{V}{\rho} \cdot d\rho \quad \text{or} \quad \frac{dV}{V} = -\frac{d\rho}{\rho}$$

Substituting in Eq. (i), we get

$$d\rho = \frac{\rho\,(dp)}{B} = \frac{dp}{v^2} \qquad \left(v = \sqrt{\frac{B}{\rho}} \Rightarrow \frac{\rho}{B} = \frac{1}{v^2} \right)$$

Or, this can be written as

$$\boxed{\Delta\rho = \frac{\rho}{B} \cdot \Delta p = \frac{1}{v^2} \cdot \Delta p}$$

So, this relation relates the pressure equation with the density equation. For example, if

$$\Delta p = (\Delta p)_m \sin\,(kx - \omega t)$$

then $$\Delta\rho = (\Delta\rho)_m \sin\,(kx - \omega t)$$

where, $$\boxed{(\Delta\rho)_m = \frac{\rho}{B}\,(\Delta p)_m = \frac{(\Delta p)_m}{v^2}}$$

Thus, density equation is in phase with the pressure equation and this is 90° out of phase with the displacement equation.

Example 19.1 *Corresponding to displacement equation,*

$$y = A \sin(kx + \omega t)$$

of a longitudinal wave make its pressure and density wave also. Bulk modulus of the medium is B and density is ρ.

Solution Given, $y(x,\,t) = A \sin\,(kx + \omega t)$

$$\frac{\partial y}{\partial x} = kA \cos\,(kx + \omega t) \qquad \ldots\text{(i)}$$

Now, pressure equation is given by

$$\Delta p = -B \left(\frac{\partial y}{\partial x} \right)$$

Substituting the value of $\frac{\partial y}{\partial x}$ from Eq. (i), we have

$$\Delta p = -BAk \cos\,(kx + \omega t) \qquad \textbf{Ans.}$$

Similarly, density equation is given by

$$(\Delta p) = \frac{\rho}{B}\,(\Delta p) = \frac{\rho}{B} \left(-B \frac{\partial y}{\partial x} \right) = -\rho \left(\frac{\partial y}{\partial x} \right)$$

Substituting the values of $\frac{\partial y}{\partial x}$ from Eq. (i), we have,

$$\Delta\rho = -\rho Ak \cos\,(kx + \omega t) \qquad \textbf{Ans.}$$

Example 19.2 *(a) What is the displacement amplitude for a sound wave having a frequency of* 100 *Hz and a pressure amplitude of* 10 *Pa?*

(b) The displacement amplitude of a sound wave of frequency 300 *Hz is* 10^{-7} *m. What is the pressure amplitude of this wave? Speed of sound in air is* 340 *m/s and density of air is* 1.29 *kg/m*^{3}.

Solution (a) $(\Delta p)_m = BAk$

Here, $$k = \frac{\omega}{v} = \frac{2\pi f}{v}$$

and $$B = \rho v^2 \qquad \left(\text{as } v = \sqrt{\frac{B}{\rho}}\right)$$

$$\therefore \quad (\Delta p)_m = (\rho v^2)(A)\left(\frac{2\pi f}{v}\right)$$

$$\therefore \quad A = \frac{(\Delta p)_m}{2\pi v \rho f} \qquad \text{...(i)}$$

Substituting the values, we have

$$A = \frac{(10)}{2 \times 3.14 \times 340 \times 1.29 \times 100}$$

$$= 3.63 \times 10^{-5} \text{ m}$$ **Ans.**

(b) From Eq. (i) $(\Delta p)_m = 2\pi f \rho v A$

Substituting the values, we have

$$(\Delta p)_m = 2 \times 3.14 \times 300 \times 1.29 \times 340 \times 10^{-7}$$

$$= 8.26 \times 10^{-2} \text{ N/m}^2$$ **Ans.**

INTRODUCTORY EXERCISE 19.1

1. Calculate the bulk modulus of air from the following data for a sound wave of wavelength 35 cm travelling in air. The pressure at a point varies between $(10^5 \pm 14)$ Pa and the particles of the air vibrate in SHM of amplitude 5.5×10^{-6} m.
2. Find the minimum and maximum wavelengths of sound in water that is in the audible range for an average human ear. Speed of sound in water is 1450 m/s.
3. A typical loud sound wave with a frequency of 1 kHz has a pressure amplitude of about 10 Pa
   (a) At $t = 0$, the pressure is a maximum at some point x_1. What is the displacement at that point at $t = 0$?
   (b) What is the maximum value of the displacement at any time and place? Take the density of air to be 1.29 kg/m^3 and speed of sound in air is 340 m/s.
4. The pressure variation in a sound wave in air is given by
$$\Delta p = 12 \sin(8.18x - 2700t + \pi/4) \text{ N/m}^2$$
Find the displacement amplitude. Density of air = 1.29 kg/m^3.

19.3 Speed of a Longitudinal Wave

First we calculate the speed at which a longitudinal pulse propagates through a fluid. We will apply Newton's second law to the motion of an element of the fluid and from this we derive the wave equation.

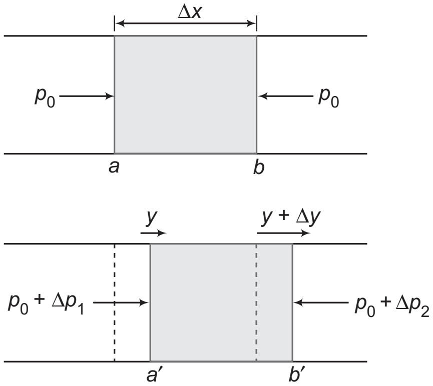


Fig. 19.3

Consider a fluid element *ab* confined to a tube of cross-sectional area S as shown in figure. The element has a thickness Δx. We assume that the equilibrium pressure of the fluid is p_0. Because of the disturbance, the section '*a*' of the element moves a distance y from its mean position and section '*b*' moves a distance $y + \Delta y$ to a new position b'. The pressure on the left side of the element becomes $p_0 + \Delta p_1$ and on the right side it becomes $p_0 + \Delta p_2$. If ρ is the equilibrium density, the mass of the element is $\rho S \Delta x$. (When the element moves its mass does not change, even though its volume and density do change).

The net force acting on the element is

$$F = (\Delta p_1 - \Delta p_2)\, S$$

and its acceleration is

$$a = \frac{\partial^2 y}{\partial t^2}$$

Thus, Newton's second law applied to the motion of the element is

$$(\Delta p_1 - \Delta p_2)\, S = ma = \rho S \Delta x \frac{\partial^2 y}{\partial t^2} \qquad \text{...(i)}$$

Next we divide both sides by Δx and note that in the limit as $\Delta x \to 0$, we have $(\Delta p_1 - \Delta p_2)/\Delta x \to \partial p/\partial x$, Eq. (i) then take the form

$$-\frac{\partial p}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2} \qquad \text{...(ii)}$$

The excess pressure Δp may be written as

$$\Delta p = -B \frac{\partial y}{\partial x} \quad \Rightarrow \quad \frac{\partial p}{\partial x} = -B \frac{\partial^2 y}{\partial x^2}$$

When this is used in Eq. (ii), we obtain the wave equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{B} \cdot \frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = \frac{B}{\rho}\frac{\partial^2 y}{\partial x^2}$$

Comparing this equation with the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We have $$\boxed{v = \sqrt{\frac{B}{\rho}}}$$ (speed of longitudinal wave in a fluid)

This is the speed of longitudinal waves within a **gas** or a **liquid.**

When a longitudinal wave propagates in a solid rod or bar, the rod expands sideways slightly when it is compressed longitudinally and the speed of a longitudinal wave in a rod is given by

$$\boxed{v = \sqrt{\frac{Y}{\rho}}}$$ (speed of a longitudinal wave in a solid rod)

19.4 Sound Waves in Gases

In the preceding section, we derived equation

$$v = \sqrt{B/\rho}$$

for the speed of longitudinal waves in a fluid of bulk modulus B and density ρ. We use this to find the speed of sound in an ideal gas. The bulk modulus of the gas, however depends on the process. When a wave travels through a gas, are the compressions and expansions adiabatic, or is there enough heat conduction between adjacent layers of gas to maintain a nearly constant temperature throughout?

Because thermal conductivities of gases are very small, it turns out that for ordinary sound frequencies (20 Hz to 20,000 Hz) propagation of sound is very nearly adiabatic. Thus, in the above equation, we use the **adiabatic bulk modulus (B_s)**, which is given by

$$B_s = \gamma p$$

Here, γ is the ratio of molar heat capacity C_p/C_V. Thus,

$$\boxed{v = \sqrt{\frac{\gamma p}{\rho}}}$$

We can get a useful alternative form of the above equation by substituting density ρ of an ideal gas

$$\rho = \frac{pM}{RT}$$

where, R is the gas constant, M the molecular mass and T is the absolute temperature combining all these equations, we can write

$$\boxed{v = \sqrt{\frac{\gamma RT}{M}}}$$ (speed of sound in an ideal gas)

Effect of Temperature, Pressure and Humidity on the Speed of Sound in Air

(i) **Effect of temperature** From the equation,

$$v = \sqrt{\frac{\gamma RT}{M}}$$

We can see that,

$$\boxed{v \propto \sqrt{T}} \quad \text{or} \quad \frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}}$$

At STP, the temperature is 0°C or 273 K. If the speed of sound at 0°C is v_0, its value at t°C will satisfy

$$\frac{v_t}{v_0} = \sqrt{\frac{273+t}{273}} = \left(1+\frac{t}{273}\right)^{\frac{1}{2}} = 1+\frac{t}{546}$$

$$\therefore \qquad v_t = v_0\left(1+\frac{t}{546}\right)$$

If the speed of sound in air at 0°C (v_0) be taken 332 m/s, then

$$v_t = 332\left(1+\frac{t}{546}\right)$$

or

$$\boxed{v_t = 332 + 0.61t}$$

Thus, the velocity of sound in air increases roughly by 0.61 m/s per degree centigrade rise in temperature.

(ii) **Effect of pressure** From the formula for the speed of sound in a gas $v = \sqrt{\frac{\gamma p}{\rho}}$, it appears that $v \propto \sqrt{p}$. But actually it is not so. Because

$$\frac{p}{\rho} = \frac{RT}{M} = \text{constant at constant temperature.}$$

That is, at constant temperature if p changes then ρ also changes in such a way that p/ρ remains constant. Hence, in the formula $v = \sqrt{\frac{\gamma p}{\rho}}$, the value of p/ρ does not change when p changes. From this it is clear that if the temperature of the gas remains constant, then there is no effect of the pressure change on the speed of sound.

(iii) **Effect of humidity** The density of moist air (i.e. air mixed with water-vapour) is less than the density of dry air. This is because in moist air heavy dust particles settle down due to condensation. Hence, density of air gets decreased thus increasing the speed of sound. Therefore, assuming the value of γ for moist air same as for dry air (which is actually slightly less than that for dry air) it is clear from the formula $v = \sqrt{\frac{\gamma p}{\rho}}$ that the speed of sound in moist air is slightly greater than in dry air.

Example 19.3 *Calculate the speed of longitudinal waves in the following gases at $0°C$ and $1\ atm\ (= 10^5\ Pa)$:*

(a) oxygen for which the bulk modulus is 1.41×10^5 Pa and density is $1.43\ kg/m^3$.
(b) helium for which the bulk modulus is 1.7×10^5 Pa and the density is $0.18\ kg/m^3$.

Solution (a)

$$v_{O_2} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.41 \times 10^5}{1.43}} = 314 \text{ m/s}$$ **Ans.**

(b)

$$v_{He} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.7 \times 10^5}{0.18}} = 972 \text{ m/s}$$ **Ans.**

Example 19.4 *Find speed of sound in hydrogen gas at $27°C$. Ratio C_p/C_V for H_2 is 1.4. Gas constant $R = 8.31\ J/mol\text{-}K$.*

Solution $\because \quad v = \sqrt{\dfrac{\gamma RT}{M}}$

Here, $T = 27 + 273 = 300$ K, $\gamma = 1.4$, $R = 8.31$ J/mol-K, $M = 2 \times 10^{-3}$ kg/mol

$$\therefore \quad v = \sqrt{\frac{1.4 \times 8.31 \times 300}{2 \times 10^{-3}}} = 1321 \text{ m/s}$$ **Ans.**

Note *In the above formula, put $M = 2 \times 10^{-3}$ kg/mol. Don't put it 2 kg.*

Example 19.5 *At what temperature will the speed of sound in hydrogen be the same as in oxygen at $100°C$? Molar masses of oxygen and hydrogen are in the ratio 16 : 1.*

Solution $v = \sqrt{\dfrac{\gamma RT}{M}}$

$$v_{H_2} = v_{O_2}$$

$$\therefore \quad \sqrt{\frac{\gamma_{H_2} R T_{H_2}}{M_{H_2}}} = \sqrt{\frac{\gamma_{O_2} R T_{O_2}}{M_{O_2}}}$$

$$\gamma_{H_2} = \gamma_{O_2}$$ (as both are diatomic)

$$\therefore \qquad T_{H_2} = \left(\frac{M_{H_2}}{M_{O_2}}\right)(T_{O_2}) = \left(\frac{1}{16}\right)(100 + 273)$$

$$= 23.31\,\text{K}$$

$$\approx -249.7^\circ\,\text{C} \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 19.2

1. Calculate the temperature at which the velocity of sound in air is double its velocity at 0°C.
2. Calculate the difference in the speeds of sound in air at –3°C, 60 cm pressure of mercury and 30°C, 75 cm pressure of mercury. The speed of sound in air at 0°C is 332 m/s.
3. In a liquid with density 900 kg/m^3, longitudinal waves with frequency 250 Hz are found to have wavelength 8.0 m. Calculate the bulk modulus of the liquid.
4. Calculate the speed of sound in oxygen at 273 K.

19.5 Sound Intensity

Travelling sound waves, like all other travelling waves, transfer energy from one region of space to another. We define the **intensity** of a wave (denoted by I) to be the time average rate at which energy is transported by the wave, per unit area across a surface perpendicular to the direction of propagation. We have already derived an expression for the intensity of a mechanical wave

$$\boxed{I = \frac{1}{2}\rho A^2 \omega^2 v} \qquad \ldots\text{(i)}$$

For a sound wave,

$$(\Delta p)_m = BAk = BA\left(\frac{\omega}{v}\right) \quad \text{or} \quad \omega = \frac{(\Delta p)_m v}{BA}$$

Substituting this value in Eq. (i), we have

$$I = \frac{1}{2}\rho A^2 \frac{(\Delta p)_m^2 v^2}{B^2 A^2} v \qquad (\rho v^2 = B)$$

or

$$\boxed{I = \frac{v\,(\Delta p)_m^2}{2B}} \qquad \ldots\text{(ii)}$$

Thus, intensity of a sound wave can be calculated by either of the Eqs. (i) or (ii).

Sound Intensity in Decibels

The physiological sensation of loudness is closely related to the intensity of wave producing the sound. At a frequency of 1 kHz people are able to detect sounds with intensities as low as 10^{-12} W/m^2. On the other hand, an intensity of 1 W/m^2 can cause pain and prolonged exposure to sound at this level will damage a person's ears. Because the range in intensities over which people hear is so large, it is convenient to use a logarithmic scale to specify intensities. This scale is defined as follows.

If the intensity of sound in watts per square metre is I, then the intensity level β in decibels (dB) is given by

$$\boxed{\beta = 10 \log \frac{I}{I_0}} \qquad \text{...(iii)}$$

where the base of the logarithm is 10, and $I_0 = 10^{-12}\ \text{W/m}^2$ (roughly the minimum intensity that can be heard).

On the decibel scale, the pain threshold of $1\ \text{W/m}^2$ is then

$$\beta = 10 \log \frac{1}{10^{-12}} = 120\ \text{dB}$$

Note *For the comparison of two different sounds in dB ; we can modify Eq. (iii) as*

$$\boxed{\beta_2 - \beta_1 = 10 \log_{10} \frac{I_2}{I_1}} \qquad \text{...(iv)}$$

Table 19.1 gives typical values for the intensity levels of some of the common sounds.

Table 19.1 Sound intensity levels in decibels
(Threshold of hearing = 0 dB; threshold of pain = 120 dB)

Source of sound	dB
Rusting leaves	10
Whisper	20
Quiet room	30
Normal level of speech (inside)	65
Street traffic (inside car)	80
Riveting tool	100
Thunder	110
Indoor rock concert	120

Example 19.6 *For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about* 6.0×10^{-5} *Pa. Calculate the corresponding intensity in* W/m^2*. Take speed of sound in air as 344 m/s and density of air 1.2* kg/m^3.

Solution $\because \quad I = \dfrac{v(\Delta p)_m^2}{2B}$

Substituting $B = \rho v^2$, the above equation reduces to

$$I = \frac{(\Delta p)_m^2}{2\rho v} = \frac{(6.0 \times 10^{-5})^2}{2 \times 1.2 \times 344}$$

$$= 4.4 \times 10^{-12}\ \text{W/m}^2 \qquad \textbf{Ans.}$$

Example 19.7 *Find intensity of sound in dB if its intensity in Watt/m^2 is 10^{-10}.*

Solution Using the equation,

$$\beta = 10\log_{10}\frac{I}{I_0}, \text{ we have}$$

$$\beta = 10\log_{10}\left(\frac{10^{-10}}{10^{-12}}\right) = 10\log_{10}(10^2)$$

$$= 20\,\text{dB}$$ **Ans.**

Example 19.8 *A point source of sound emits a constant power with intensity inversely proportional to the square of the distance from the source. By how many decibels does the sound intensity level drops when you move from point P_1 to P_2? Distance of P_2 from the source is two times the distance of source from P_1.*

Solution We label the two points 1 and 2, and we use the equation $\beta = 10\log\frac{I}{I_0}$ (dB) twice.

The difference in sound intensity level $\beta_2 - \beta_1$ is given by

$$\beta_2 - \beta_1 = (10\,\text{dB})\left(\log\frac{I_2}{I_0} - \log\frac{I_1}{I_0}\right)$$

$$= (10\,\text{dB})[(\log I_2 - \log I_0) - (\log I_1 - \log I_0)]$$

$$= (10\,\text{dB})\log\frac{I_2}{I_1}$$

Now, $$I \propto \frac{1}{r^2} \Rightarrow \therefore \frac{I_2}{I_1} = \left(\frac{r_1}{r_2}\right)^2 = \frac{1}{4} \quad \text{as} \quad r_2 = 2r_1$$

$\therefore$ $$\beta_2 - \beta_1 = (10\,\text{dB})\log\left(\frac{1}{4}\right) = -6.0\,\text{dB}$$ **Ans.**

INTRODUCTORY EXERCISE 19.3

1. A sound wave in air has a frequency of 300 Hz and a displacement amplitude of 6.0×10^{-3} mm. For this sound wave calculate the
   (a) pressure amplitude (b) intensity (c) sound intensity level (in dB)
   Speed of sound = 344 m/s and density of air = 1.2 kg/m^3.
2. Most people interpret a 9.0 dB increase in sound intensity level as a doubling in loudness. By what factor must the sound intensity be increased to double the loudness?
3. A baby's mouth is 30 cm from her father's ear and 3.0 m from her mother's ear. What is the difference between the sound intensity levels heard by the father and by the mother.
4. The faintest sound that can be heard has a pressure amplitude of about 2×10^{-5} N/m^2 and the loudest that can be heard without pain has a pressure amplitude of about 28 N/m^2. Determine in each case (a) the intensity of the sound both in W/m^2 and in dB and (b) the amplitude of the oscillations if the frequency is 500 Hz. Assume an air density of 1.29 kg/m^3 and a velocity of sound is 345 m/s.

19.6 Interference in Sound Wave and Stationary Wave

In chapter 18, we have already discussed about the principle of superposition, interference and stationary wave. In the similar manner, we can make stationary waves with longitudinal (or sound) waves also.

When two identical sound waves travel in opposite directions, then by their superposition (and hence interference) longitudinal stationary waves are formed.

But, as we know that, with every longitudinal travelling wave three equations are associated $y(x, t)$, $\Delta P\,(x, t)$ and $\Delta\rho\,(x, t)$. So, with every longitudinal stationary wave three equations are associated, $y(x, t)$, $\Delta P\,(x, t)$ and $\Delta\rho\,(x, t)$.

Following three points are important with longitudinal stationary wave :

(i) If individual displacement amplitudes of constituent travelling waves are A_0 each, then displacement amplitude in stationary wave varies from 0 to $2A_0$.

(ii) If two identical (same frequency) longitudinal waves travel in opposite directions, standing waves are produced by their superposition. If the equations of the two waves are written as

$$\Delta p_1 = (\Delta p)_m \sin\,(kx - \omega t)$$

and

$$\Delta p_2 = (\Delta p)_m \sin\,(kx + \omega t)$$

then from the principle of superposition, the resultant wave is

$$\Delta p = \Delta p_1 + \Delta p_2 = 2\,(\Delta p)_m \sin kx \cos \omega t \qquad \ldots\text{(i)}$$

This equation is similar to the equation obtained in chapter 18 for standing waves on a string.

From Eq. (i), we can see that

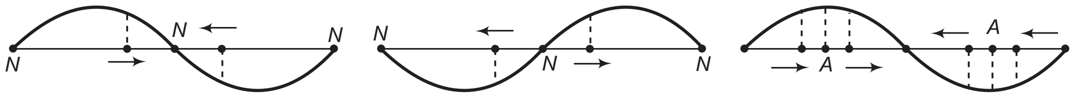
$$\Delta p = 0 \quad \text{at } x = 0, \lambda/2, \lambda, \ldots, \text{etc.} \qquad \text{(pressure nodes)}$$

and

$$\Delta p = \text{maximum at } x = \lambda/4, 3\lambda/4, \ldots, \text{etc.} \qquad \text{(pressure antinodes)}$$

The distance between two adjacent nodes or between two adjacent antinodes is $\lambda/2$. Longitudinal standing waves can be produced in air columns trapped in tubes of cylindrical shape. Organ pipes are such vibrating air columns.

(iii) Because the pressure wave is 90° out of phase with the displacement wave. Consequently, the displacement node behaves as a pressure antinode and *vice-versa*.



(a) (b) (c)

Fig. 19.4

This can be explained by realizing that two small volume elements of fluid on opposite sides of a displacement node are vibrating in opposite phase. Hence, when they approach each other [see Fig. (a)] the pressure at this node is a maximum, and when they recede from each other, [see Fig. (b)] the pressure at this node is a minimum. Similarly, two small elements of fluid which are on opposite sides of a displacement antinode vibrate in phase and therefore give rise to no pressure variations at the antinode [see Fig. (c)].

Extra Points to Remember

- Most of the problems of interference can be solved by calculating the path difference Δx and then by putting

$$\Delta x = 0, \lambda, 2\lambda, \ldots \qquad \text{(constructive interference)}$$

$$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \ldots \qquad \text{(destructive interference)}$$

provided the waves emitted from S_1 and S_2 are in phase.

- If the two waves emitted from S_1 and S_2 have already a phase difference of π, the conditions of maximas and minimas are interchanged, i.e. path difference

$$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \ldots \qquad \text{(for constructive interference)}$$

and $$\Delta x = \lambda, 2\lambda, \ldots \qquad \text{(for destructive interference)}$$

Example 19.9 *Two sound sources S_1 and S_2 emit pure sinusoidal waves in phase. If the speed of sound is 350 m/s, then*

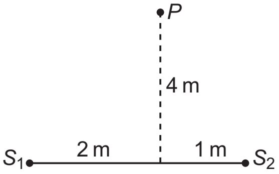


Fig. 19.5

(a) for what frequencies does constructive interference occur at P?

(b) for what frequencies does destructive interference occur at point P?

Solution Path difference, $\Delta x = S_1P - S_2P$

$$= \sqrt{(2)^2 + (4)^2} - \sqrt{(1)^2 + (4)^2}$$

$$= 4.47 - 4.12 = 0.35 \text{ m}$$

(a) Constructive interference occurs when the path difference is an integer multiple of wavelength.

or $$\Delta x = n\lambda = \frac{nv}{f} \quad \text{or} \quad f = \frac{nv}{\Delta x} \qquad \text{where, } n = 1, 2, 3, \ldots$$

$\therefore$ $$f = \frac{350}{0.35}, \frac{2 \times 350}{0.35}, \frac{3 \times 350}{0.35}, \ldots$$

$$f = 1000 \text{ Hz}, 2000 \text{ Hz}, 3000 \text{ Hz}, \ldots, \text{etc.}$$ **Ans.**

(b) Destructive interference occurs when the path difference is a half-integer multiple of wavelengths

or $$\Delta x = (2n+1)\frac{\lambda}{2} \qquad n = 0, 1, 2, \ldots$$

or $$\Delta x = (2n+1)\frac{v}{2f}$$

$\therefore$ $$f = \frac{(2n+1)\,v}{2\,\Delta x}$$

$$= \frac{350}{2\times 0.35}, \frac{3\times 350}{2\times 0.35}, \frac{5\times 350}{2\times 0.35}, \ldots$$

$$= 500\text{ Hz}, 1500\text{ Hz}, 2500\text{ Hz}, \ldots$$ **Ans.**

INTRODUCTORY EXERCISE 19.4

1. Two sound waves emerging from a source reach a point simultaneously along two paths. When the path difference is 12 cm or 36 cm, then there is a silence at that point. If the speed of sound in air be 330 m/s, then calculate maximum possible frequency of the source.
2. A wave of frequency 500 cycle/s has a phase velocity of 350 m/s.
   (a) How far apart are two points 60° out of phase?
   (b) What is the phase difference between two displacements at a certain point at time 10^{-3} s apart?

19.7 Standing Longitudinal Waves in Organ Pipes

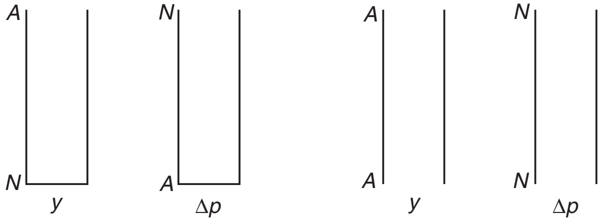
When longitudinal waves propagate in a fluid in a pipe with finite length, the waves are reflected from the ends in a same way the transverse waves on a string are reflected at its ends. The superposition of the waves travelling in opposite directions forms a longitudinal standing wave.

Transverse waves on a string, including standing waves, are usually described only in terms of the displacement of the string. But longitudinal standing waves in a fluid may be described either in terms of the displacement of the fluid or in terms of the pressure variation in the fluid. To avoid confusion we will use the terms **displacement node** and **displacement antinode** to refer to points where particles of the fluid have zero displacement and maximum displacement respectively. Similarly, **pressure node** and **pressure antinode** refer to the points where pressure and density variation in the fluid is zero or maximum, respectively.

Conditions at the Boundary of an Organ Pipe

Let us take an example of a closed organ pipe. In a closed pipe one end is closed and the other is open. When a longitudinal wave encounters the closed end of the pipe it gets reflected from this end. But the reflected wave is 180° out of phase with the incident wave, i.e. a compression is reflected as a compression and a rarefaction is reflected as a rarefaction. This is a necessary condition because the displacement of the small volume elements at the closed end must always be zero. Hence, a closed end is a displacement node.

A sound wave is also reflected from an open end. You may wonder how a sound wave can reflect from an open end, since there may not appear to be a change in the medium at this point. At the open end pressure is the same as atmospheric pressure and does not vary. Thus, there is a pressure node (or displacement antinode) at this end. A compression is therefore reflected as a rarefaction and rarefaction as a compression. Now let us see how this reflection takes place. When a rarefaction reaches an open end, the surrounding air rushes towards this region and creates a compression that travels back along the pipe. Similarly, when a compression reaches an open end, the air expands to form a rarefaction. This can be said in a different way as : at the open end of the tube, fluid elements are free to move, so there is a displacement antinode. Thus, in a nutshell we can say that closed end of an organ pipe is a displacement node or pressure antinode and open end of the pipe is displacement antinode or pressure node.



A
N
y
N
A
Δp
A
A
y
N
N
Δp

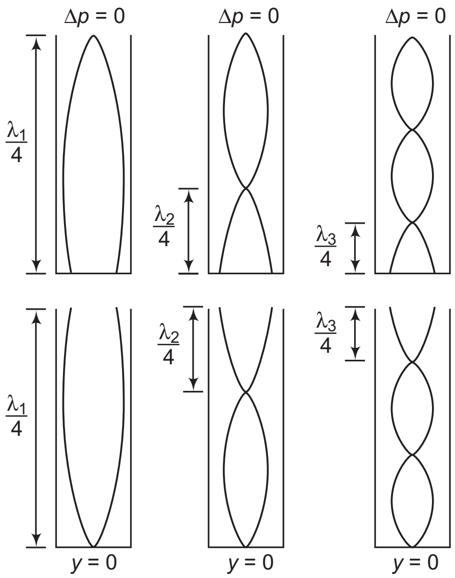
(a) Closed organ pipe (b) Open organ pipe

Fig. 19.6

Similarly, both ends of an open organ pipe (open at both ends) are displacement antinodes or pressure nodes.

Standing Waves in a Closed Organ Pipe

To get resonance in a closed organ pipe sound waves are sent in by a source (normally a tuning fork) near the open end. Resonance corresponds to a pressure antinode at the closed end and a pressure node at the open end. The standing wave patterns for the three lowest harmonics in this situation are shown in figure. Since the node-antinode separation is $\frac{\lambda}{4}$, the resonance condition for the first harmonic is, $l = \frac{\lambda_1}{4}$.



Δp = 0
Δp = 0
Δp = 0
$\frac{\lambda_1}{4}$
$\frac{\lambda_2}{4}$
$\frac{\lambda_3}{4}$
$\frac{\lambda_1}{4}$
$\frac{\lambda_2}{4}$
$\frac{\lambda_3}{4}$
y = 0
y = 0
y = 0

Fig. 19.7

Similarly, the resonance conditions for the higher harmonics are $l = \frac{3\lambda_2}{4}, \frac{5\lambda_3}{4}, \ldots$, etc. or, $\lambda_n = \frac{4l}{n}$ (where $n = 1, 3, 5\ldots$). The natural frequencies of oscillation of the air in the tube closed at one end and open at the other are, therefore,

$$\boxed{f_n = \frac{v}{\lambda_n} = n\frac{v}{4l} = nf_1 \qquad (n = 1, 3, 5, \dots)}$$

Here, v = speed of sound in the tube

$$f_1 = \frac{v}{4l} \qquad \text{(fundamental frequency or the first harmonic)}$$

$$f_3 = 3\frac{v}{4l} = 3f_1 \qquad \text{(first overtone or the third harmonic)}$$

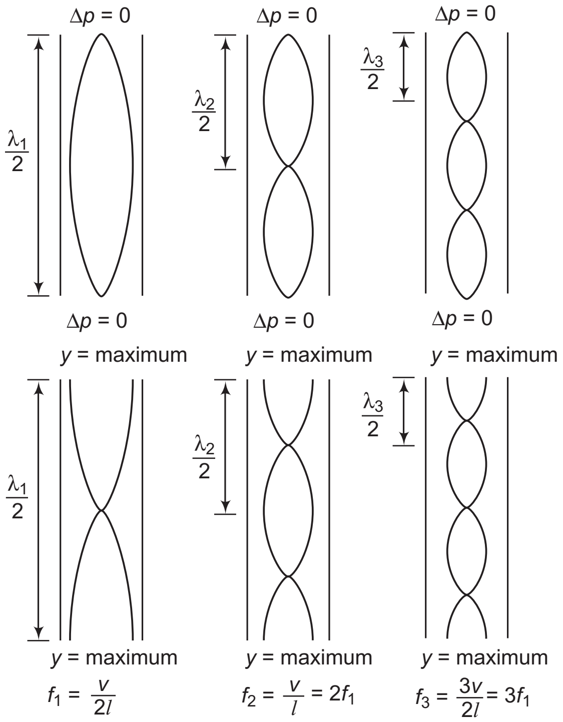
and $$f_5 = 5\frac{v}{4l} = 5f_1 \qquad \text{(second overtone or the fifth harmonic)}$$

and so on.

Thus, in a pipe closed at one end and open at the other, the natural frequencies of oscillation form a harmonic series that includes only odd integer multiples of the fundamental frequency.

Standing Waves in an Open Organ Pipe

Since both ends of the tube are open, there are pressure nodes (or displacement antinodes) at both ends. Figure shows the resulting standing waves for the three lowest resonant frequencies since the distance between pressure nodes is $\lambda/2$, the resonance condition is $l = n\left(\frac{\lambda_n}{2}\right)$ where $n = 1, 2, 3, \dots$ and l is the length of the tube. The resonant frequencies for a tube open at both ends are then :



Δp = 0
Δp = 0
Δp = 0
$\frac{\lambda_1}{2}$
$\frac{\lambda_2}{2}$
$\frac{\lambda_3}{2}$
Δp = 0
Δp = 0
Δp = 0
y = maximum
y = maximum
y = maximum
$\frac{\lambda_1}{2}$
$\frac{\lambda_2}{2}$
$\frac{\lambda_3}{2}$
y = maximum
y = maximum
y = maximum
$f_1 = \frac{v}{2l}$
$f_2 = \frac{v}{l} = 2f_1$
$f_3 = \frac{3v}{2l} = 3f_1$

Fig. 19.8

$$\boxed{f_n = \frac{v}{\lambda_n} = n\frac{v}{2l} = nf_1 \qquad (n = 1, 2, 3, \dots)}$$

Here, $$f_1 = \frac{v}{2l}$$

(fundamental frequency or first harmonic)

$$f_2 = \frac{v}{l} = 2f_1$$

(first overtone or second harmonic)

$$f_3 = 3\left(\frac{v}{2l}\right) = 3f_1$$

(second overtone or third harmonic)

and so on.

Thus, in a pipe open at both ends, the natural frequencies of oscillation form a harmonic series that includes all integral multiples of the fundamental frequency.

Extra Points to Remember

- **Laplace end correction** In practice, the pressure nodes lie slightly beyond the ends of the tube. A compression reaching an open end does not reflect until it passes beyond the end. For a thin walled tube of circular cross-section, this **end correction** is approximately 0.6 r where r is the tube's radius. Hence, the effective length of the tube is longer than the true length l.

  Therefore, Laplace correction $e = 0.6\,r$ (in closed pipe) and $2e = 1.2\,r$ (in open pipe)

  Hence,
  $$f = n\left[\frac{v}{2(l+1.2r)}\right] \quad \text{(in open pipe)}$$
  and
  $$f = n\left[\frac{v}{4(l+0.6\,r)}\right] \quad \text{(in closed pipe)}$$

  However, we neglect this small correction if the length of the tube is much larger than its diameter.

- If an open pipe and a closed pipe are of same lengths, then fundamental frequency of open pipe $\left(=\frac{v}{2l}\right)$ is two times the fundamental frequency of closed pipe $\left(=\frac{v}{4l}\right)$.

- In the above equations v is the speed of constituent longitudinal waves by which stationary wave is formed. This speed in air is given by
  $$v = \sqrt{\frac{\gamma RT}{M}}$$
  If temperature is increased or some light gas is filled in the pipe, then v will increase.So, this set of frequencies will also increase.

- As n increases, v remains unchanged, f increases. Therefore, $\lambda (= v/f)$ and the loop size $\left(=\frac{\lambda}{2}\right)$ decrease.

Example 19.10 *Third overtone of a closed organ pipe is in unison with fourth harmonic of an open organ pipe. Find the ratio of the lengths of the pipes.*

Solution Third overtone of closed organ pipe means seventh harmonic. Given

$$(f_7)_{\text{closed}} = (f_4)_{\text{open}}$$
$$7\left(\frac{v}{4l_c}\right) = 4\left(\frac{v}{2l_o}\right)$$
$$\therefore \quad \frac{l_c}{l_o} = \frac{7}{8}$$
Ans.

Example 19.11 *An open organ pipe has a fundamental frequency of 300 Hz. The first overtone of a closed organ pipe has the same frequency as the first overtone of this open pipe. How long is each pipe?*
(Speed of sound in air = 330 m/s)

Solution Fundamental frequency of an open organ pipe,

$$f_1 = \frac{v}{2l_o} \Rightarrow l_o = \frac{v}{2f_1} = \frac{330}{2\times 300}$$
$$= 0.55\text{ m}$$
$$= 55\text{ cm}$$
Ans.

Given, first overtone of closed organ pipe = first overtone of open organ pipe

Hence, $$3\left(\frac{v}{4l_c}\right) = 2\left(\frac{v}{2l_o}\right)$$

$\therefore$ $$l_c = \frac{3}{4} l_o = \left(\frac{3}{4}\right)(0.55)$$

$$= 0.4125 \text{ m}$$

$$= 41.25 \text{ cm}$$ **Ans.**

Example 19.12 *A cylindrical tube, open at both ends, has a fundamental frequency f in air. The tube is dipped vertically in water so that half of its length is in water. The fundamental frequency of the air column is now* **(JEE 1981)**

(a) $f/2$ *(b)* $3f/4$ *(c)* f *(d)* $2f$

Solution Initially, the tube was open at both ends and then it is closed.

$$f_o = \frac{v}{2l_o} \quad \text{and} \quad f_c = \frac{v}{4l_c}$$

Since, tube is half dipped in water, $l_c = \dfrac{l_o}{2}$

$\therefore$ $$f_c = \frac{v}{4\left(\dfrac{l_o}{2}\right)} = \frac{v}{2l_o} = f_o = f$$

Hence, the correct option is (c).

Example 19.13 *A closed organ pipe of length L and an open organ pipe contain gases of densities* ρ_1 *and* ρ_2*, respectively. The compressibility of gases are equal in both the pipes. Both the pipes are vibrating in their first overtone with same frequency. The length of the open organ pipe is* **(JEE 2004)**

(a) $\dfrac{L}{3}$ *(b)* $\dfrac{4L}{3}$ *(c)* $\dfrac{4L}{3}\sqrt{\dfrac{\rho_1}{\rho_2}}$ *(d)* $\dfrac{4L}{3}\sqrt{\dfrac{\rho_2}{\rho_1}}$

Solution. $f_c = f_o$ (both first overtone)

or $$3\left(\frac{v_c}{4L}\right) = 2\left(\frac{v_o}{2l_o}\right)$$

$\therefore$ $$l_o = \frac{4}{3}\left(\frac{v_o}{v_c}\right)L = \frac{4}{3}\frac{\sqrt{B/\rho_2}}{\sqrt{B/\rho_1}}L$$

$$= \frac{4}{3}\sqrt{\frac{\rho_1}{\rho_2}}L$$

Hence, the correct option is (c).

Note *Compressibility* $= \dfrac{1}{\text{Bulk modulus}}$*. Compressibility is same. Therefore, bulk modulus is also same.*

INTRODUCTORY EXERCISE 19.5

1. An open pipe is suddenly closed at one end with the result that the frequency of third harmonic of the closed pipe is found to be higher by 100 Hz than the fundamental frequency of the open pipe. The fundamental frequency of the open pipe is **(JEE 1996)**

   (a) 200 Hz (b) 300 Hz (c) 240 Hz (d) 480 Hz

2. An organ pipe P_1 closed at one end vibrating in its first harmonic and another pipe P_2 open at both ends vibrating in its third harmonic are in resonance with a given tuning fork. The ratio of the length of P_1 and P_2 is **(JEE 1988)**

   (a) 8/3 (b) 3/8 (c) 1/6 (d) 1/3

3. A tube closed at one end and containing air produces, when excited the fundamental note of frequency 512 Hz. If the tube is opened at both ends, the fundamental frequency that can be excited is (in Hz) **(JEE 1986)**

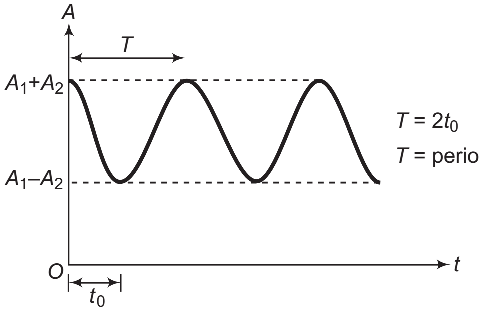
   (a) 1024 (b) 512 (c) 256 (d) 128

4. The fundamental frequency of a closed organ pipe is 220 Hz.
   (a) Find the length of this pipe.
   (b) The second overtone of this pipe has the same frequency as the third harmonic of an open pipe. Find the length of this open pipe. Take speed of sound in air 345 m/s.

5. Standing sound waves are produced in a pipe that is 0.8 m long, open at one end, and closed at the other. For the fundamental and first two overtones, where along the pipe (measured from the closed end) are
   (a) the displacement antinodes
   (b) the pressure antinodes.

6. An organ pipe has two successive harmonics with frequencies 400 and 560 Hz. The speed of sound in air is 344 m/s.
   (a) Is this an open or a closed pipe?
   (b) What two harmonics are there?
   (c) What is the length of the pipe?

19.8 Beats



$T = 2t_0$

T = period of the beat = $\dfrac{1}{f_1 - f_2}$

Fig. 19.9

When two wavetrains of the same frequency travel along the same line in opposite directions, standing waves are formed in accordance with the principle of superposition. In standing waves, amplitude is a function of distance. This illustrates a type of interference that we can call **interference in space.** The same principle of superposition leads us to another type of interference, which we can call **interference in time.** It occurs when two wavetrains of slightly different frequency travel through the same region.

If the waves are in phase at some time (say $t = 0$) the interference will be constructive and the resultant amplitude at this moment will be $A_1 + A_2$, where A_1 and A_2 are the amplitudes of individual wavetrains. But at some later time (say $t = t_0$), because the frequencies are different, the waves will be out of phase or the interference will be destructive and the resultant amplitude will be $A_1 - A_2$ (if $A_1 > A_2$). Later, we will see that the time t_0 is $\dfrac{1}{2(f_1 - f_2)}$.

Thus, the resultant amplitude oscillates between $A_1 + A_2$ and $A_1 - A_2$ with a time period $T = 2t_0 = \dfrac{1}{f_1 - f_2}$ or with a frequency $f = f_1 - f_2$ known as **beat frequency.** Thus,

$$\boxed{\text{Beat frequency, } f = f_1 - f_2}$$

Calculation of Beat Frequency

Suppose two waves of frequencies f_1 and $f_2\ (< f_1)$ are meeting at some point in space. The corresponding periods are T_1 and $T_2\ (> T_1)$. If the two waves are in phase at $t = 0$, they will again be in phase when the first wave has gone through exactly one more cycle than the second. This will happen at a time $t = T$, the period of the beat. Let n be the number of cycles of the first wave in time T, then the number of cycles of the second wave in the same time is $(n - 1)$. Hence,

$$T = nT_1 \qquad \text{...(i)}$$

and

$$T = (n - 1)T_2 \qquad \text{...(ii)}$$

Eliminating n from these two equations, we have

$$T = \frac{T_1 T_2}{T_2 - T_1} = \frac{1}{\dfrac{1}{T_1} - \dfrac{1}{T_2}} = \frac{1}{f_1 - f_2}$$

The reciprocal of the beat period is the beat frequency

$$f = \frac{1}{T} = f_1 - f_2.$$

Alternate Method

Let the oscillations at some point in space (say $x = 0$) due to two waves be

$$y_1 = A_1 \sin 2\pi f_1 t \quad \text{and} \quad y_2 = A_2 \sin 2\pi f_2 t \qquad (\omega = 2\pi f)$$

If they are in phase at some time t, then

$$2\pi f_1 t = 2\pi f_2 t \quad \text{or} \quad f_1 t = f_2 t \qquad \text{...(i)}$$

They will be again in phase at time $(t + T)$ if,

$$2\pi f_1 (t + T) = 2\pi f_2 (t + T) + 2\pi$$

or

$$f_1 (t + T) = f_2 (t + T) + 1 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $\quad T = \dfrac{1}{f_1 - f_2}$

Note *If a tuning fork is loaded with wax, its frequency decreases. On the other hand, when tuning fork is filed, its frequency increases.*

Example 19.14 *Two tuning forks A and B produce 6 beats per second. Frequency of A is 300 Hz. When B is slightly loaded with wax, beat frequency decreases. Find original frequency of B.*

Solution Since, A and B produce 6 beats per second. Therefore, original frequency of B may be 306 Hz or 294 Hz.

When B is loaded with wax, its frequency will decrease (suppose it decreases by 1Hz). So, if it is 306 Hz, it will become 305 Hz. If it is 294 Hz, it will become 293 Hz.

Frequency of A is unchanged $(= 300\,\text{Hz})$. If f_B becomes 305 Hz, then beat frequency will become 5 Hz. If f_B becomes 293 Hz, then it will become 7 Hz. But in the question it is given that beat frequency has decreased. So, the correct answer is 306 Hz.

Example 19.15 *The string of a violin emits a note of 400 Hz at its correct tension. The string is bit taut and produces 5 beats per second with a tuning fork of frequency 400 Hz. Find frequency of the note emitted by this taut string.*

Solution The frequency of vibration of a string increases with increase in the tension. Thus, the note emitted by the string will be a little more than 400 Hz. As it produces 5 beats per second with the 440 Hz tuning fork, the frequency will be 405 Hz.

Example 19.16 *Two tuning forks P and Q when set vibrating, give 4 beats per second. If a prong of the fork P is filed, the beats are reduced to 2 per second. Determine the original frequency of P, if that of Q is 250 Hz.*

Solution There are four beats between P and Q, therefore the possible frequencies of P are 246 or 254 (that is 250 ± 4) Hz.

When the prong of P is filed, its frequency becomes greater than the original frequency.

If we assume that the original frequency of P is 254, then on filing its frequency will be greater than 254. The beats between P and Q will be more than 4. But it is given that the beats are reduced to 2, therefore, 254 is not possible.

Therefore, the required frequency must be 246 Hz.

(This is true, because on filing the frequency may increase to 248, giving 2 beats with Q of frequency 250 Hz) **Ans.**

INTRODUCTORY EXERCISE 19.6

1. A tuning fork produces 4 beats per second with another tuning fork of frequency 256 Hz. The first one is now loaded with a little wax and the beat frequency is found to increase to 6 per second. What was the original frequency of the first tuning fork?
2. A tuning fork of unknown frequency makes three beats per second with a standard fork of frequency 384 Hz. The beat frequency decreases when a small piece of wax is put on a prong of the first fork. What is the frequency of this fork?

19.9 The Doppler's Effect

If a wave source and a receiver are moving relative to each other, the frequency observed by the receiver (f') is different from the actual source frequency (f). This phenomenon is called the **Doppler's effect**, named after the Austrian physicist Christian Johann Doppler (1803–1853), who discovered it in light waves.

Perhaps you might have noticed how the sound of a vehicle's horn changes as the vehicle moves past you. The frequency (pitch) of the sound you hear as the vehicle approaches you is higher than the frequency you hear as it moves away from you. The Doppler's effect applies to waves in general. Let us apply it to sound waves. We consider the special case in which the source and observer move along the line joining them. We will use the following symbols,

$v =$ speed of sound, $v_s =$ speed of source and $v_o =$ speed of observer

and $v_m =$ velocity of medium in which sound travels.

For example, if Doppler's effect is observed in air, it is wind velocity. If Doppler's effect is observed inside a river, then it is river velocity.

The general formula of the changed frequency is

$$\boxed{f' = \left(\frac{v \pm v_m \pm v_o}{v \pm v_m \pm v_s}\right) f}$$

Sign Convention

v_m We are talking about that sound which is travelling from source to observer (S to O). If the medium is also travelling in the same direction, then it means medium is supporting the sound. So, take positive sign both in numerator and denominator. If the medium travels in opposite direction, then take negative sign. If nothing is given in the question, then take it zero.

v_o and v_s The concept is approaching nature always increases the frequency. So, take positive sign with v_o (because it is in numerator) and negative sign with v_s (as it is in denominator). On the other hand, receding nature decreases the frequency. So, take negative sign with v_o and positive sign with v_s. Now, let us make some different cases.

Case 1

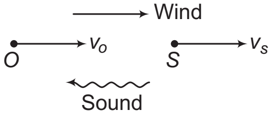


Fig. 19.10

In the given figure, sound is travelling from right to left but wind is blowing from left to right, so we will have to take negative sign with wind velocity. Observer is approaching towards source, so take positive sign with v_o. Source is receding from the observer, so take positive sign with v_s. The correct formula is given below

$$f' = \left(\frac{v - v_{\text{wind}} + v_o}{v - v_{\text{wind}} + v_s}\right) f$$

Case 2

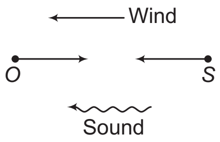


Fig. 19.11

Sound and wind both are travelling in the same direction (from S to O), so take positive sign with wind velocity. Source and observer both are approaching towards each other. So, take positive sign with v_o and negative sign with v_s. Therefore, the correct formula is

$$f' = \left(\frac{v + v_{\text{wind}} + v_o}{v + v_{\text{wind}} - v_s} \right) f$$

Now, let us derive two special cases when medium velocity is zero.

Source at Rest, Observer Moves

Suppose that the observer O moves towards the source S at speed v_o. The speed of the sound waves relative to O is $v_r = v + v_o$, but wavelength has its normal value $\lambda = \frac{v}{f}$. Thus, the frequency heard by O is

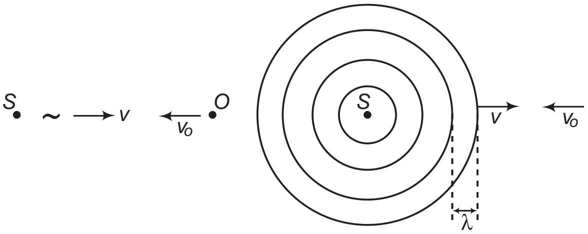


Fig. 19.12

$$f' = \frac{v_r}{\lambda} = \left(\frac{v + v_o}{v} \right) f$$

If O were moving away from S, the frequency heard by O would be

$$f' = \left(\frac{v - v_o}{v} \right) f$$

Combining these two expressions, we find

$$\boxed{f' = \left(\frac{v \pm v_o}{v} \right) f} \quad \text{...(i)}$$

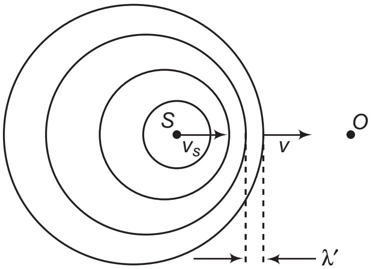
Source Moves, Observer at Rest

Fig. 19.13

Suppose that the source S moves towards O as shown in figure.

If S were at rest, the distance between two consecutive wave pulses emitted by sound would be $\lambda = \frac{v}{f} = vT$. However, in one time period S moves a distance v_sT before it emits the next pulse. As a result the wavelength is modified. Directly ahead of S the effective wavelength (for both S and O) is

$$\lambda' = vT - v_sT = (v - v_s)T = \left(\frac{v - v_s}{f}\right)$$



The wavelength in front of the source is less than the normal whereas in the rear it is larger than normal.

Fig. 19.14

The speed of sound waves relative to O is simply v. Thus, the frequency observed by O is

$$f' = \frac{v}{\lambda'} = \left(\frac{v}{v - v_s}\right)f$$

If S were moving away from O, the effective wavelength would be $\lambda' = \left(\frac{v + v_s}{f}\right)$ and the apparent frequency would be

$$f' = \left(\frac{v}{v + v_s}\right)f$$

Combining these two results, we have

$$\boxed{f' = \left(\frac{v}{v \pm v_s}\right)f} \qquad \text{...(ii)}$$

All four possibilities can be combined into one equation

$$\boxed{f' = \left(\frac{v \pm v_o}{v \mp v_s}\right)f} \qquad \text{...(iii)}$$

where the upper signs (+ numerator, – denominator) correspond to the source and observer along the line joining the two in the direction toward the other, and the lower signs in the direction away from the other.

Alternate Method

The above formulae can be derived alternately as discussed below.

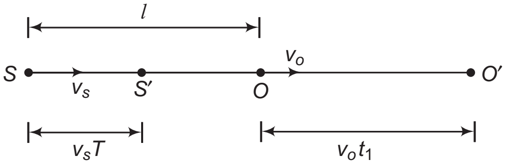


Fig. 19.15

Assume that the source and observer are moving along the same line and that the observer O is to the right of the source S. Suppose that at time $t = 0$, when the source and the observer are separated by a distance $SO = l$, the source emits a wave pulse (say p_1) that reaches the observer at a later time t_1. In that time the observer has moved a distance $v_o t_1$ and the total distance travelled by p_1 in the time t_1 has been $l + v_o t_1$. If v is the speed of sound, this distance is also vt_1. Then,

$$vt_1 = l + v_0 t_1$$

or
$$t_1 = \frac{l}{v - v_0} \qquad \text{...(iv)}$$

At time $t = T$, the source is at S' and the wave pulse (say p_2) emitted at this time will reach the observer at a time say t_2, measured from the same time $t = 0$, as before. The total distance travelled by p_2 till it is received by the observer (measured from S') is $(l - v_s T) + v_o t_2$. The actual travel time for p_2 is $(t_2 - T)$ and the distance travelled is $v\,(t_2 - T)$. Therefore,

$$v\,(t_2 - T) = (l - v_s T) + v_0 t_2$$

or
$$t_2 = \frac{l + (v - v_s)\,T}{v - v_o} \qquad \text{...(v)}$$

The time interval reckoned by the observer between the two pulses emitted by the source at S and at S' is

$$T' = t_2 - t_1 = \left(\frac{v - v_s}{v - v_o}\right) T$$

This is really the changed time period as observed by the observer. Hence, the new frequency is

$$f' = \frac{1}{T'} = \left(\frac{v - v_o}{v - v_s}\right)\frac{1}{T}$$

or
$$f' = \left(\frac{v - v_o}{v - v_s}\right) f$$

This is a result which we can expect by using equation $f' = \left(\frac{v \pm v_o}{v \mp v_s}\right) f$ in the case when source is moving towards observer and observer is moving away from the source.

Extra Points to Remember

- We have derived equation number (iii) by assuming that v_o and v_s are along the line joining source and observer. If the motion is along some other direction, the components of velocities along the line joining source and observer are considered.

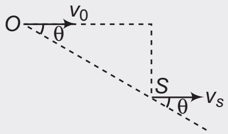


Fig. 19.16

For example in the figure shown,
$$f' = \left(\frac{v + v_o \cos\theta}{v + v_s \cos\theta}\right) f$$

- Change in frequency depends on the fact that whether the source is moving towards the observer or the observer is moving towards the source. But when the speed of source and observer are much lesser than that of sound, the change in frequency becomes independent of the fact whether the source is moving or the observer. This can be shown as under.

  Suppose a source is moving towards a stationary observer, with speed u and the speed of sound is v, then
$$f' = \left(\frac{v}{v-u}\right) f = \left(\frac{1}{1-\dfrac{u}{v}}\right) f = \left(1 - \frac{u}{v}\right)^{-1} f$$
  Using the binomial expansion, we have
$$\left(1 - \frac{u}{v}\right)^{-1} \approx 1 + \frac{u}{v} \quad \text{if} \quad u << v$$
$$\therefore \qquad f' \approx \left(1 + \frac{u}{v}\right) f \quad \text{if} \quad u << v$$
  On the other hand, if an observer moves towards a stationary source with same speed u, then
$$f' = \left(\frac{v+u}{v}\right) f = \left(1 + \frac{u}{v}\right) f$$
  which is same as above.

- As long as v_s and v_o are along the line joining S and O, Doppler's effect (or change in frequency) does not depend on the distance between S and O. For example in the given figure,

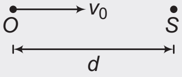


Fig. 19.17

$$f' = f\left(\frac{v + v_o}{v}\right)$$

  $f' > f$ but it is constant and independent of 'd'.

- Frequency is given by $f = \dfrac{v}{\lambda}$

  By the motion of source, λ changes, therefore frequency changes. But, from the motion of observer, relative velocity v between sound and observer changes, therefore frequency changes.

- Despite the motion of source or observer (or both), Doppler's effect is not observed (or $f' = f$) under the following four conditions.

***Condition* 1** $\mathbf{v}_s$ or $\mathbf{v}_o$ is making an angle of 90° with the line joining S and O. This is illustrated in the following figure.

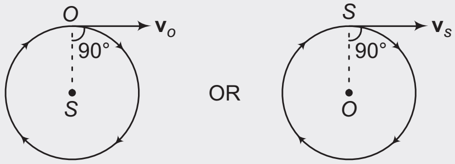


Fig. 19.18

***Condition* 2** Source and observer both are in motion but their velocities are equal or relative motion between them is zero.

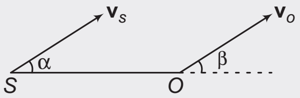


Fig. 19.19

In the figure shown, $\mathbf{v}_s = \mathbf{v}_o$ if $v_s = v_o$ and $\alpha = \beta$

Taking the components along SO, we have $f' = f\left(\dfrac{v - v_o \cos\beta}{v - v_s \cos\alpha}\right) \Rightarrow f' = f$ because $v_s = v_o$ and $\alpha = \beta$

***Condition* 3** Source and observer both are at rest. Only medium is in motion.

In the figure shown, source and observer are at rest. Only wind is blowing in a direction from source to observer. The changed frequency is

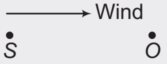


Fig. 19.20

$$f' = f\left(\frac{v + v_{\text{wind}}}{v + v_{\text{wind}}}\right) = f$$

***Condition* 4**

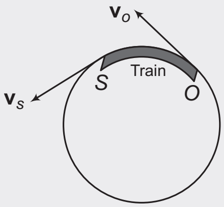


Fig. 19.21

A train is travelling on a circular track. Engine is the source of sound and guard is the observer. Although,

$$\mathbf{v}_s \neq \mathbf{v}_o$$

yet f' comes out to be f.

Exercise Derive the above result.

- **Doppler's Effect in Light**

  Light waves also show Doppler's effect. If a light source is moving away from a stationary observer, then the frequency of light waves appear to be decreased and wavelength appear to be increased and *vice-versa.*

  If the light source or the observer is moving with a velocity v such that the distance between them is decreasing, then the apparent frequency of the source will be given by

$$\boxed{f' = f\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}}$$

If the distance between the light source and the observer is increasing, then the apparent frequency of the source is given by

$$\boxed{f' = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}}$$

The change in wavelength can be determined by

$$\boxed{\Delta\lambda = \frac{v}{c}\cdot\lambda}$$

If the light source is moving away from the observer, the shift in the spectrum is towards red and if it is moving towards the observer the shift is towards the violet.

Note *The Doppler's effect in light depends only on the relative motion between the source and the observer while the Doppler's effect in sound also depends upon whether the source is moving or the observer is moving.*

Example 19.17 *A car approaching a crossing C at a speed of 20 m/s sounds a horn of frequency 500 Hz when 80 m from the crossing. Speed of sound in air is 330 m/s. What frequency is heard by an observer (at rest) 60 m from the crossing on the straight road which crosses car road at right angles?*

Solution The situation is as shown in figure.

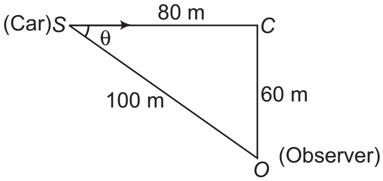


Fig. 19.22

$$\cos\theta = \frac{80}{100} = \frac{4}{5}$$

∴ Apparent frequency,

$$f_{\text{app}} = \left(\frac{v}{v - v_s\cos\theta}\right)f$$

$$= \left(\frac{330}{330 - 20\times\frac{4}{5}}\right)(500)$$

$$= 525.5\text{ Hz}$$

Ans.

Example 19.18 *A siren emitting a sound of frequency 1000 Hz moves away from you towards a cliff at a speed of 10 m/s.*

(a) What is the frequency of the sound, you hear coming directly from the siren?

(b) What is the frequency of sound you hear reflected off the cliff. Speed of sound in air is 330 m/s?

Note *For reflected sound cliff can be assumed as a plane mirror.*

Solution (a) Sound heard directly

$$f_1 = f_0\left(\frac{v}{v+v_s}\right) \quad\Rightarrow\quad v_s = 10\text{ m/s}$$

∴

$$f_1 = \left(\frac{330}{330+10}\right)\times 1000$$

$$= 970.6\text{ Hz}$$

Ans.

(b) The frequency of the reflected sound is given by

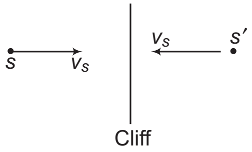


Fig. 19.23

$$f_2 = f_0\left(\frac{v}{v - v_s}\right)$$

$$f_2 = \left(\frac{330}{330 - 10}\right) \times 1000$$

$$= 1031.25\,\text{Hz}$$ **Ans.**

Example 19.19 *A whistle of frequency* 540 *Hz rotates in a circle of radius* 2 *m at a linear speed of* 30 *m/s. What is the lowest and highest frequency heard by an observer a long distance away at rest with respect to the centre of circle? Take speed of sound in air as* 330 *m/s. Can the apparent frequency be ever equal to actual?*

Solution Apparent frequency will be minimum when the source is at N and moving away from the observer.

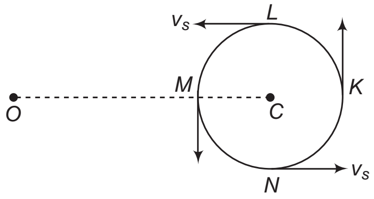


Fig. 19.24

$$f_{\min} = \left(\frac{v}{v + v_s}\right) f$$

$$= \left(\frac{330}{330 + 30}\right)(540) = 495\,\text{Hz}$$ **Ans.**

Frequency will be maximum when source is at L and approaching the observer.

$$f_{\max} = \left(\frac{v}{v - v_s}\right) f = \left(\frac{330}{330 - 30}\right)(540)$$

$$= 594\,\text{Hz}$$ **Ans.**

Further, when the source is at M and K, angle between velocity of source and line joining source and observer is 90° or $v_s \cos\theta = v_s \cos 90° = 0$. So, there will be no change in the apparent frequency.

Note *Although the source velocity v_s is not along the line joining S and O but at a long distance we can assume that it is along SO.*

Example 19.20 *Two tuning forks with natural frequencies* 340 *Hz each move relative to a stationary observer. One fork moves away from the observer while the other moves towards him at the same speed. The observer hears beats of frequency* 3 *Hz. Find the speed of the tuning fork (velocity of sound in air is* 340 *m/s).*

Note *The difference in apparent frequencies is very small (3 Hz). So, we may conclude that the speed of source (v_s) << speed of sound (v). Therefore, we can neglect the higher terms of $\frac{v_s}{v}$.*

Solution Given,

$$f_1 - f_2 = 3$$

or

$$\left(\frac{v}{v - v_s}\right) f - \left(\frac{v}{v + v_s}\right) f = 3$$

or
$$\left[\frac{1}{\left(1-\frac{v_s}{v}\right)}-\frac{1}{\left(1+\frac{v_s}{v}\right)}\right]f=3$$

or
$$\left[\left(1-\frac{v_s}{v}\right)^{-1}-\left(1+\frac{v_s}{v}\right)^{-1}\right]f=3$$

or
$$\left[\left(1+\frac{v_s}{v}\right)-\left(1-\frac{v_s}{v}\right)\right]f=3$$

or
$$\frac{2v_s f}{v}=3$$

or Speed of tuning fork, $v_s=\frac{3v}{2f}$

Substituting the values, we get

$$v_s=\frac{(3)(340)}{(2)(340)}=1.5\text{ m/s}$$ **Ans.**

INTRODUCTORY EXERCISE 19.7

1. A whistle giving out 450 Hz approaches a stationary observer at a speed of 33 m/s. The frequency heard by the observer (in Hz) is (Speed of sound = 330 m/s) **(JEE 2000)**

(a) 409 (b) 429

(c) 517 (d) 500

2. A train moves towards a stationary observer with speed 34 m/s. The train sounds a whistle and its frequency registered by the observer is f_1. If the train's speed is reduced to 17 m/s, the frequency registered is f_2. If the speed of sound is 340 m/s, then the ratio f_1/f_2 is **(JEE 1997)**

(a) 18/19 (b) 1/2

(c) 2 (d) 19/18

3. A siren placed at a railway platform is emitting sound of frequency 5 kHz. A passenger sitting in a moving train *A* records a frequency of 5.5 kHz, while the train approaches the siren. During his return journey in a different train *B* he records a frequency of 6.0 kHz while approaching the same siren. The ratio of the velocity of train *B* to that of train *A* is **(JEE 2002)**

(a) 242/252 (b) 2

(c) 5/6 (d) 11/6

4. A train is moving on a straight track with speed 20 ms^{-1}. It is blowing its whistle at the frequency of 1000 Hz. The percentage change in the frequency heard by a person standing near the track as the train passes him is close to (speed of sound = 320 ms^{-1}) **(JEE 2015 Main)**

(a) 12% (b) 6%

(c) 18% (d) 24%

Solved Examples

TYPED PROBLEMS

Type 1. *Based on Doppler's effect and Beats together*

Concept

(i) If there is a relative motion between source and observer, frequency changes.

(ii) Natural frequencies of a stretched wire are

$$f_n = n\left(\frac{v}{2l}\right) = n\left[\frac{\sqrt{T/\mu}}{2l}\right] \qquad \text{(where } n = 1, 2, \ldots)$$

If tension in the wire is changed, then frequency changes.

(iii) Natural frequencies of a closed pipe are

$$f_n = n\left(\frac{v}{4l}\right) \qquad \text{(where } n = 1, 3, \ldots)$$

and natural frequencies of an open pipe are

$$f_n = n\left(\frac{v}{2l}\right) \qquad \text{(where } n = 1, 2, \ldots)$$

Here, $$v = \sqrt{\frac{\gamma RT}{M}}$$

When temperature is increased, these sets of frequencies also increase.

(iv) Beat frequency is given by

$$f_b = f_1 - f_2 \qquad \text{(if } f_1 > f_2)$$

Example 1 *The first overtone of an open organ pipe beats with the first overtone of a closed organ pipe with a beat frequency of 2.2 Hz. The fundamental frequency of the closed organ pipe is 110 Hz. Find the lengths of the pipes. Speed of sound in air $v = 330$ m/s.*

Solution Let l_1 and l_2 be the lengths of closed and open pipes respectively.

Fundamental frequency of closed organ pipe is given by

$$f_1 = \frac{v}{4l}$$

where v is speed of sound in air and 330 m/s equal to

But, $$f_1 = 110 \text{ Hz (given)}$$

Therefore, $$\frac{v}{4l_1} = 110 \text{ Hz}$$

$\therefore$ $$l_1 = \frac{v}{4 \times 110} = \frac{330}{4 \times 110} \text{ m} = 0.75 \text{ m}$$

First overtone of closed organ pipe will be

$$f_3 = 3f_1 = 3\ (110)\ \text{Hz} = 330\ \text{Hz}$$

This produces a beat frequency of 2.2 Hz with first overtone of open organ pipe.

Therefore, first overtone frequency of open organ pipe is either

$$(330 + 2.2)\ \text{Hz} = 332.2\ \text{Hz}$$

or $$(330 - 2.2)\ \text{Hz} = 327.8\ \text{Hz}$$

If it is 332.2 Hz, then

$$2\left(\frac{v}{2l_2}\right) = 332.2\ \text{Hz}$$

or $$l_2 = \frac{v}{332.2} = \frac{330}{332.2}\ \text{m} = 0.99\ \text{m}$$

and if it is 327.8 Hz, then

$$2\left(\frac{v}{2l_2}\right) = 327.8\ \text{Hz}$$

or $$l_2 = \frac{v}{327.8}\ \text{m} = \frac{330}{327.8}\ \text{m} = 1.0067\ \text{m}$$

Therefore, length of the closed organ pipe is $l_1 = 0.75$ m while length of open pipe is either $l_2 = 0.99$ m or 1.0067 m. **Ans.**

Example 2 *A vibrating string of certain length l under a tension T resonates with a mode corresponding to the first overtone (third harmonic) of an air column of length 75 cm inside a tube closed at one end. The string also generates 4 beats/s with a tuning fork of frequency n. Now when the tension of the string is slightly increased the number of beats reduces to 2 per second. Assuming the velocity of sound in air to be 340 m/s, the frequency n of the tuning fork in Hz is* **(JEE 2008)**

(a) 344 *(b)* 336

(c) 117.3 *(d)* 109.3

Solution With increase in tension, frequency of vibrating string will increase. Since, number of beats are decreasing. Therefore, frequency of vibrating string or third harmonic frequency of closed pipe should be less than the frequency of tuning fork by 4.

$\therefore$ Frequency of tuning fork = third harmonic frequency of closed pipe + 4

$$= 3\left(\frac{v}{4l}\right) + 4 = 3\left(\frac{340}{4 \times 0.75}\right) + 4$$

$$= 344\ \text{Hz}$$

$\therefore$ Correct option is (a).

Example 3 *Two identical straight wires are stretched so as to produce 6 beats/s when vibrating simultaneously. On changing the tension slightly in one of them, the beat frequency remains unchanged. Denoting by T_1, T_2 the higher and the lower initial tension in the strings, then it could be said that while making the above changes in tension* **(JEE 1991)**

(a) T_2 was decreased *(b) T_2 was increased*

(c) T_1 was decreased *(d) T_1 was increased*

Solution $T_1 > T_2$

$\therefore$ $v_1 > v_2$

or $f_1 > f_2$

and $f_1 - f_2 = 6\text{ Hz}$

Now, if T_1 is increased, f_1 will increase or $f_1 - f_2$ will increase. Therefore, (d) option is wrong.

If T_1 is decreased, f_1 will decrease and it may be possible that now $f_2 - f_1$ become 6 Hz. Therefore, (c) option is correct.

Similarly, when T_2 is increased, f_2 will increase and again $f_2 - f_1$ may become equal to 6 Hz. So, (b) is also correct. But (a) is wrong.

Example 4 *A sonometer wire under tension of 64 N vibrating in its fundamental mode is in resonance with a vibrating tuning fork. The vibrating portion of the sonometer wire has a length of 10 cm and mass of 1 g. The vibrating tuning fork is now moved away from the vibrating wire with a constant speed and an observer standing near the sonometer hears one beat per second. Calculate the speed with which the tuning fork is moved, if the speed of sound in air is 300 m/s.* **(JEE 1983)**

Solution Fundamental frequency of sonometer wire,

$$f = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

$$= \frac{1}{2 \times 0.1}\sqrt{\frac{64 \times 0.1}{10^{-3}}}$$

$$= 400\text{ Hz}$$

Given beat frequency, $f_b = f - f' = 1\text{ Hz}$

$\therefore$ $f' = 399\text{ Hz}$

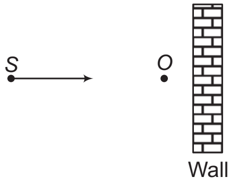
Using $f' = f\left(\dfrac{v}{v + v_s}\right)$

or $399 = 400\left(\dfrac{300}{300 + v_s}\right)$

Solving we get, $v_s = 0.75\text{ m/s}$ **Ans.**

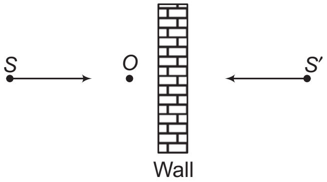
Type 2. *Reflection of a sound from a wall and beats*

Concept



A source of sound is approaching towards a wall. An observer hears two sounds. One is direct from the source and the other reflected from wall. If both frequencies are same, then the observer will hear no beats and if there is a frequency difference then beats are heard to the observer. This all depends on the position of observer. For finding the frequency of reflected sound we have the following two approaches.

Approach 1 Take wall as a plane mirror. Image of S on this plane mirror will behave like another source S' for the reflected sound.



For example, in the shown figure both direct sound source S and reflected sound source S' are approaching towards the observer.

Approach 2 First consider wall as an observer for the sound received to it from the source S. Now, consider this wall as a source for reflected sound.

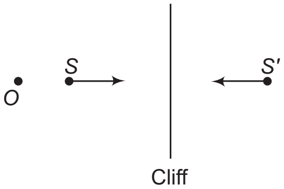
Example 5 *A siren emitting a sound of frequency* 1000 *Hz moves away from you toward a cliff at a speed of* 10 *m/s.*

(a) What is the frequency of the sound you hear coming directly from the siren?

(b) What is the frequency of the sound you hear reflected off the cliff?

(c) What beat frequency would you hear? Take the speed of sound in air as 330 *m/s.*

Solution The situation is as shown in figure.



(a) Frequency of sound reaching directly to us (by S)

$$f_1 = \left(\frac{v}{v + v_s}\right)$$

$$f = \left(\frac{330}{330 + 10}\right)(1000)$$

$$= 970.6 \text{ Hz}$$ **Ans.**

(b) Frequency of sound which is reflected from the cliff (from S')

$$f_2 = \left(\frac{v}{v - v_s}\right) f = \left(\frac{330}{330 - 10}\right)(1000)$$

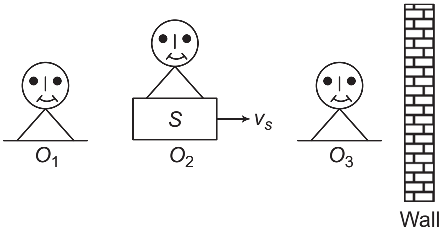
$$= 1031.3 \text{ Hz}$$ **Ans.**

(c) Beat frequency $= f_2 - f_1$

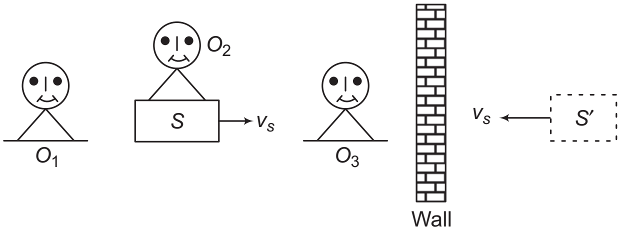
$$= 60.7 \text{ Hz}$$ **Ans.**

Note *Numerically the beat frequency comes out to be 60.7 Hz. But, beats between two tones can be detected by ear upto a frequency of about 7 per second. At higher frequencies beats cannot be distinguished in the sound produced. Hence, the correct answer of part (c) should be zero.*

Example 6 *A source of sound of frequency f is approaching towards a wall with speed v_s. Speed of sound is v. Three observers O_1, O_2 and O_3 are at different locations as shown. Find the beat frequency as observed by three different observers.*



Solution For reflected sound we can take wall as a plane mirror.



Observer O_1 S is receding from O_1 and S' is approaching towards him. So,

$$f_{s'} > f_s$$

$$f_b = f_{s'} - f_s$$

$$= f\left(\frac{v}{v - v_s}\right) - f\left(\frac{v}{v + v_s}\right)$$

$$= \left(\frac{2vv_s}{v^2 - v_s^2}\right) f \qquad \textbf{Ans.}$$

Observer O_2 O_2 has no relative motion with S. Therefore, there will be no change in the frequency of direct sound from S. Further, O_2 and S' are approaching towards each other. Therefore, the frequency of reflected sound will increase.

$$f_b = f_{s'} - f_s$$

$$= f\left(\frac{v + v_{o_2}}{v - v_s}\right) - f$$

But $v_{o_2} = v_s$

$$\therefore \qquad f_b = f\left(\frac{v + v_s}{v - v_s}\right) - f$$

$$= \left(\frac{2v_s}{v - v_s}\right) f \qquad \textbf{Ans.}$$

Observer O_3 S and S' both are approaching towards O_3. So, both frequencies will increase. But after the increase they are equal. So, beat frequency will be zero.

$$f_s = f_{s'} = f\left(\frac{v}{v - v_s}\right)$$

$$\therefore \qquad f_b = f_s - f_{s'} = 0$$

Example 7 *A sound wave of frequency f travels horizontally to the right. It is reflected from a large vertical plane surface moving to left with a speed v. The speed of sound in medium is c* **(JEE 1995)**

(a) The number of waves striking the surface per second is $f\dfrac{(c+v)}{c}$

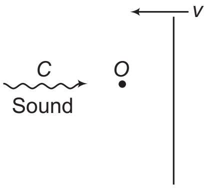
(b) The wavelength of reflected wave is $\dfrac{c\,(c-v)}{f\,(c+v)}$

(c) The frequency of the reflected wave is $f\dfrac{(c+v)}{(c-v)}$

(d) The number of beats heard by a stationary listener to the left of the reflecting surface is $\dfrac{vf}{c-v}$

Solution Moving plane is like a moving observer. Therefore, number of waves encountered by moving plane,

$$f_1 = f\left(\frac{v+v_0}{v}\right) = f\left(\frac{c+v}{c}\right)$$



Frequency of reflected wave, $f_2 = f_1\left(\dfrac{v}{v-v_s}\right) = f\left(\dfrac{c+v}{c-v}\right)$

Wavelength of reflected wave, $\lambda_2 = \dfrac{c}{f_2} = \dfrac{c}{f}\left(\dfrac{c-v}{c+v}\right)$

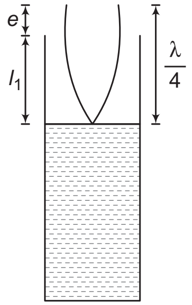
Beat frequency, $f_b = f_2 - f = f\left(\dfrac{c+v}{c-v}\right) - f = \dfrac{2fv}{c-v}$

Therefore, the correct options are (a), (b) and (c).

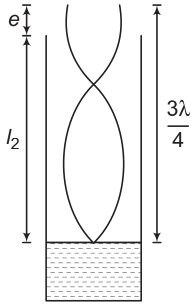
Type 3. *Based on the experiment of finding speed of sound using resonance tube*

Concept

(i) If a vibrating tuning fork (of known frequency) is held over the open end of a resonance tube, then resonance is obtained at some position as the level of water is lowered.



(i)



(ii)

If e is the end correction of the tube and l_1 is the length from the water level to the top of the tube, then

$$l_1 + e = \frac{\lambda}{4} \qquad \text{...(i)}$$

Now, the water level is further lowered until a resonance is again obtained. If l_2 is new length of the air column, then

$$l_2 + e = \frac{3\lambda}{4} \qquad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii),
we get

$$l_2 - l_1 = \frac{\lambda}{2}$$

or $$\lambda = 2(l_2 - l_1) = \frac{v}{f} \qquad \left(\lambda = \frac{v}{f}\right)$$

∴ $$\boxed{v = 2f(l_2 - l_1)}$$

(ii) The above result is independent of the end correction *e*.

(iii) In the above two figures v, λ, f and loop size $\left(= \frac{\lambda}{2}\right)$ are same in both figures. Length of closed organ pipe is different. In the second figure, length is more. So, the fundamental frequency $\left(f = \frac{v}{4l}\right)$ will be less.

Suppose frequency of given tuning fork is 300 Hz. Then, fundamental frequency of first pipe is also 300Hz. But fundamental frequency of the second pipe should be 100 Hz. So, that its next higher frequency (or first overtone frequency) which is three times the fundamental $(= 3 \times 100 = 300\,\text{Hz})$ should also be in resonance with the given tuning fork.

Example 8 *A tuning fork of 512 Hz is used to produce resonance in a resonance tube experiment. The level of water at first resonance is 30.7 cm and at second resonance is 63.2 cm. The error in calculating velocity of sound is* **(JEE 2005)**

(a) 204.1 cm/s (b) 110 cm/s

(c) 58 cm/s (d) 280 cm/s

Solution Actual speed of sound in air is 330 m/s

$$\frac{\lambda}{2} = (l_2 - l_1) = (63.2 - 30.7)\,\text{cm} = 32.5\,\text{cm} = 0.325\,\text{m}$$

or $$\lambda = 0.65\,\text{m}$$

∴ Speed of sound observed, $v_0 = f\lambda = 512 \times 0.65 = 332.8$ m/s

∴ Error in calculating velocity of sound = 2.8 m/s = 280 cm/s

∴ The correct option is (d).

Example 9 *In the experiment for the determination of the speed of sound in air using the resonance column method, the length of the air column that resonates in the fundamental mode, with a tuning fork is 0.1 m. When this length is changed to 0.35 m, the same tuning fork resonates with the first overtone. Calculate the end correction.* **(JEE 2003)**

(a) 0.012 m (b) 0.025 m (c) 0.05 m (d) 0.024 m

Solution Let e be the end correction.

Given that, fundamental tone for a length 0.1m = first overtone for the length 0.35 m.

$$\frac{v}{4(0.1+e)} = \frac{3v}{4(0.35+e)}$$

Solving this equation, we get $e = 0.025\,\text{m} = 2.5\,\text{cm}$.

∴ The correct option is (b).

Alternate method

$$l_2 + e = 3\lambda/4 \quad \text{and} \quad l_1 + e = \lambda/4$$

Solving these two equations, we get

$$e = \frac{l_2 - 3l_1}{2} = \frac{0.35 - 3\times 0.1}{2} = 0.025\text{ m}$$

Example 10 *A student is performing the experiment of resonance column. The diameter of the column tube is 4 cm. The frequency of the tuning fork is 512 Hz. The air temperature is 38° C in which the speed of sound is 336 m/s. The zero of the meter scale coincides with the top end of the resonance column tube. When the first resonance occurs, the reading of the water level in the column is* **(JEE 2012)**

(a) 14.0 *cm* *(b)* 15.2 *cm*

(c) 16.4 *cm* *(d)* 17.6 *cm*

Solution With end correction,

$$f = n\left[\frac{v}{4(l+e)}\right], \qquad \text{(where, } n = 1, 3, \ldots)$$

$$= n\left[\frac{v}{4(l+0.6r)}\right]$$

Because, $e = 0.6\,r$, where r is radius of pipe.

For first resonance, $n = 1$

$$\therefore \quad f = \frac{v}{4(l+0.6r)} \quad \text{or} \quad l = \frac{v}{4f} - 0.6r = \left[\left(\frac{336\times 100}{4\times 512}\right) - 0.6\times 2\right]\text{cm}$$

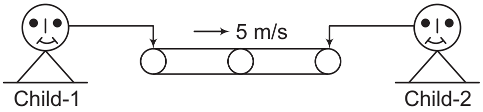
$$= 15.2\text{ cm}$$

∴ The correct option is (b).

Type 4. *Based on a situation which is similar to Doppler's effect*

Concept

Let us take an example.



A belt shown in figure is moving with velocity 5 m/s. Child-1 is placing some coins over the belt at a regular interval of 1sec. Child -2 is receiving the coins on the other hand. Now, let us compare this situation with Doppler's effect.

Child-1 is source, child-2 is observer, velocity of belt is net velocity of sound ($= v \pm v_m$) by which sound travels from source to observer. Actually, in this case

$$v = \text{velocity of coin (similar to velocity of sound)}$$
$$= 0$$
$$v_m = \text{velocity of medium (or velocity of belt)}$$
$$= 5\,\text{m/s}$$
$$T = \text{time interval of placing two successive coins}$$
$$= 1\,\text{sec}$$
$$\therefore \qquad f = \frac{1}{T} = \text{number of coins placed per second}$$

λ = distance between two coins $= v_m T = 5\,\text{m}$.
Now, let us consider the different cases.

Case 1 When source (child-1) and observer (child-2) both are at rest only belt (or medium) is in motion. In this case, distance between two coins moving towards child-2 with 5 m/s is 5m. If child-2 receives first coin at time $t_1 = 0$, then he will receive the second coin at $t_2 = \frac{5\,\text{m}}{5\,\text{m/s}} = 1\text{sec.}$

$\therefore$ T' = time interval of receival of two successive coins

$$= t_2 - t_1 = 1 \text{ sec}$$
$$\therefore \qquad f' = \frac{1}{T'} = 1\,\text{Hz}$$

Result By the motion of only medium, there is no change in frequency.

Case 2 If child-1 (or source) starts moving towards child-2 with 2 m/s. Then, in $T = 1$ sec, belt will move a distance of 5 m towards child-2 and child-1 will also travel a distance of 2m towards child-2 at the time of placing the second coin. Hence,
λ = new distance between two successive coins = $(5 - 2)$ m = 3 m
Now, if child-2 receives the first coin at time $t_1 = 0$, then he will receive the second coin (at a distance of 3m from first coin) moving towards him with 5 m/s at time $t_2 = \frac{3}{5}$ sec.

$$\therefore \qquad T' = t_2 - t_1 = \frac{3}{5}\,\text{sec} \quad \text{or} \quad f' = \frac{1}{T'} = \frac{5}{3}\,\text{Hz} \quad \text{or} \quad f' > f$$

Result By the motion of source only wavelength changes therefore frequency changes.

Case 3 Child-1 (or source) is stationary and child-2 starts moving towards child-1 with 2 m/s. This time,
λ = distance between two successive coins is unchanged or 5 m.
But coins and child-2 are moving towards each other with 5 m/s and 2 m/s, respectively. Therefore, relative velocity between them is 7m/s. So, if child-2 receives the first coin at time $t_1 = 0$, then he will receive the second coin at time $t_2 = \frac{5}{7}$ sec

$$\therefore \qquad T' = t_2 - t_1 = \frac{5}{7}\,\text{sec} \quad \text{or} \quad f' = \frac{1}{T'} = \frac{7}{5}\,\text{Hz} \quad \text{or} \quad f' > f$$

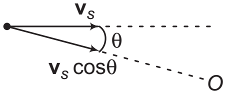
Result By the motion of observer relative velocity between sound and observer changes. Therefore, frequency changes.

Note *In case-2, $f' = \frac{5}{3}$ Hz and in case-3, $f' = \frac{7}{5}$ Hz. These two are not same, although speeds are same in both cases but change in frequency is different.*

Type 5. *When $\mathbf{v}_s$ or $\mathbf{v}_o$ are not along the line joining S and O but they are very far from each other.*

Concept

In this case, we take components of $\mathbf{v}_s$ and $\mathbf{v}_o$ along the line *SO*. For example in the figure shown below,



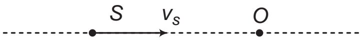
$$f' = f\left(\frac{v}{v - v_s \cos\theta}\right)$$

If source is very far from observer, then

$$\theta \to 0° \quad \text{or} \quad \cos\theta \to 1$$

$$\therefore \qquad f' = f\left(\frac{v}{v - v_s}\right)$$

Example 11 *A source of frequency f is moving towards the observer along the line SO with a constant velocity v_s as shown in figure. Plot f' versus t graph. Where f' is the changed frequency observed by the observer.*



Solution **During approach**

$$f' = f\left(\frac{v}{v - v_s}\right) = f_1 \qquad \text{(say)}$$

$$f' > f$$

but f' is constant with time.

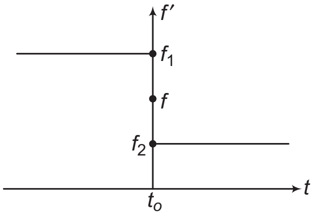
During recede

$$f' = f\left(\frac{v}{v + v_s}\right) = f_2 \;\; \text{(say)}$$

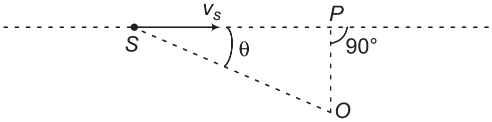
$$f' < f$$

but f' is constant with time. The correct f' *versus* t graph is as shown.

In the figure, t_o is the time when source is crossing the observer.



Example 12 *Repeat example-*11 *if source does not move along the line SO.*



Solution The changed frequency f' is now,

$$f' = f\left(\frac{v}{v - v_s\cos\theta}\right)$$

This time f' is not constant but it is a function of θ, which is variable. But, at a far distance, $\theta \to 0°$ or $\cos\theta \to 1$ or

$$\therefore \qquad f' = f\left(\frac{v}{v - v_s}\right)$$

and this is the maximum value of f' (say $f_{\max}$)

At point P, $\theta = 90°$

$$\therefore \qquad f' = f$$

During receding (to the right of P)

$$f' = f\left(\frac{v}{v + v_s\cos\theta}\right)$$

Again, at large distance from P,

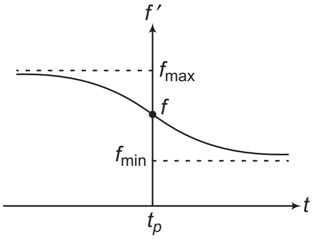
$\theta \to 0°$ or $\cos\theta \to 1$

$$\therefore \qquad f' = f\left(\frac{v}{v + v_s}\right)$$

This time, this is the minimum value of f' (say $f_{\min}$).

Hence, f' varies from $f_{\max}$ to $f_{\min}$, with $f' = f$ at the time of crossing P.

The correct f' *versus* t graph is as shown below.



Example 13 *A whistle emitting a sound of frequency* 440 *Hz is tied to a string of* 1.5 *m length and rotated with an angular velocity of* 20 *rad/s in the horizontal plane. Calculate the range of frequencies heard by an observer stationed at a large distance from the whistle. (Speed of sound* = 330 *m/s).* **(JEE 1996)**

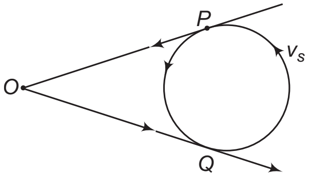
Solution v_s = Speed of source (whistle) = $R\omega$ = (1.5) (20) m/s, $v_s = 30$ m/s

Maximum frequency will be heard by the observer in position P and minimum in position Q.

Now,

$$f_{\max} = f\left(\frac{v}{v - v_s}\right)$$

where, v = speed of sound in air = 330 m/s



$$= (440)\left(\frac{330}{330 - 30}\right)\text{ Hz}$$

or $$f_{\max} = 484\text{ Hz}$$

and $$f_{\min} = f\left(\frac{v}{v + v_s}\right) = (440)\left(\frac{330}{330 + 30}\right)$$

$$f_{\min} = 403.33\text{ Hz}$$

Therefore, range of frequencies heard by observer is from 484 Hz to 403.33 Hz.

Miscellaneous Examples

Example 14 *The water level in a vertical glass tube 1.0 m long can be adjusted to any position in the tube. A tuning fork vibrating at 660 Hz is held just over the open top end of the tube. At what positions of the water level will there be in resonance? Speed of sound is 330 m/s.*

Solution Resonance corresponds to a pressure antinode at closed end and pressure node at open end. Further, the distance between a pressure node and a pressure antinode is $\frac{\lambda}{4}$, the condition of resonance would be,

Length of air column $l = n\frac{\lambda}{4} = n\left(\frac{v}{4f}\right)$

Here, $n = 1, 3, 5, \ldots$

$$l_1 = (1)\left(\frac{330}{4 \times 660}\right) = 0.125\text{ m}$$

$$l_2 = 3l_1 = 0.375\text{ m}$$

$$l_3 = 5l_1 = 0.625\text{ m}$$

$$l_4 = 7l_1 = 0.875\text{ m}$$

$$l_5 = 9l_1 = 1.125\text{ m}$$

Since, $l_5 > 1$ m (the length of tube), the length of air columns can have the values from l_1 to l_4 only.

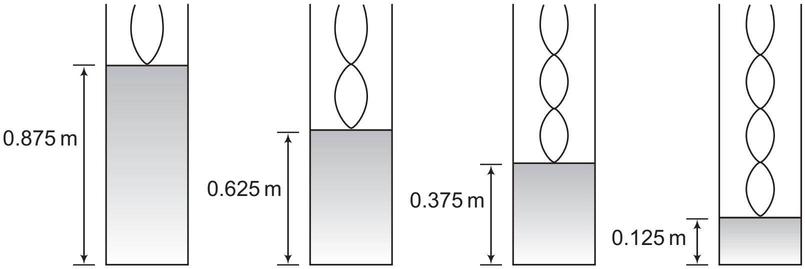
Therefore, level of water at resonance will be

$$(1.0-0.125)\text{ m}=0.875\text{ m}$$

$$(1.0-0.375)\text{ m}=0.625\text{ m}$$

$$(1.0-0.625)\text{ m}=0.375\text{ m}$$

and $$(1.0-0.875)\text{ m}=0.125\text{ m}$$ **Ans.**



In all the four cases shown in figure, the resonance frequency is 660 Hz but first one is the fundamental tone or first harmonic. Second is first overtone or third harmonic and so on.

Example 15 *A tube 1.0 m long is closed at one end. A stretched wire is placed near the open end. The wire is 0.3 m long and has a mass of 0.01 kg. It is held fixed at both ends and vibrates in its fundamental mode. It sets the air column in the tube into vibration at its fundamental frequency by resonance. Find*

(a) the frequency of oscillation of the air column and

(b) the tension in the wire.

Speed of sound in air = 330 m/s.

Solution (a) Fundamental frequency of closed pipe $=\dfrac{v}{4l}$

$$=\frac{330}{4\times 1}=82.5\text{ Hz}$$ **Ans.**

(b) At resonance, given that :

fundamental frequency of stretched wire (at both ends)

= fundamental frequency of air column

$$\therefore \quad \frac{v}{2l}=82.5\text{ Hz}$$

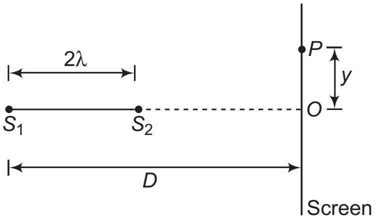
$$\therefore \quad \frac{\sqrt{T/\mu}}{2l}=82.5$$

or $$T=\mu\,(2\times 0.3\times 82.5)^2$$

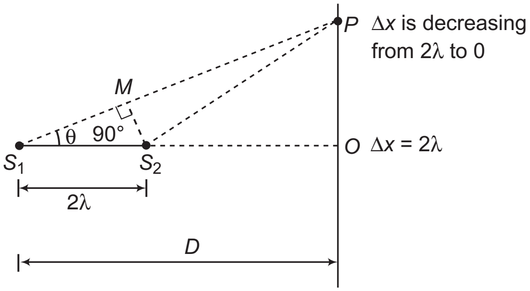
$$=\left(\frac{0.01}{0.3}\right)(2\times 0.3\times 82.5)^2$$

$$=81.675\text{ N}$$ **Ans.**

Example 16 *Two coherent narrow slits emitting sound of wavelength λ in the same phase are placed parallel to each other at a small separation of 2λ. The sound is detected by moving a detector on the screen at a distance $D (>> \lambda)$ from the slit S_1 as shown in figure. Find the distance y such that the intensity at P is equal to intensity at O.*



Solution At point O on the screen the path difference between the sound waves reaching from S_1 and S_2 is 2λ, i.e. constructive interference is obtained at O. At a very large distance from point O on the screen the path difference is zero.



Thus, we can conclude that as we move away from point O on the screen path difference decreases from 2λ to zero. At O constructive interference is obtained (where $\Delta x = 2\lambda$). So, next constructive interference will be obtained where $\Delta x = \lambda$. Hence,

$$S_1P - S_2P = \lambda$$

or $$\sqrt{D^2 + y^2} - \sqrt{y^2 + (D - 2\lambda)^2} = \lambda$$

$\therefore$ $$\sqrt{D^2 + y^2} - \lambda = \sqrt{y^2 + (D - 2\lambda)^2}$$

On squaring both sides, we get

$$D^2 + y^2 + \lambda^2 - 2\lambda\sqrt{D^2 + y^2} = y^2 + D^2 + 4\lambda^2 - 4\lambda D$$

or $$2\sqrt{D^2 + y^2} = 4D - 3\lambda$$

as $D >> \lambda$, $$4D - 3\lambda \approx 4D$$

$\therefore$ $$2\sqrt{D^2 + y^2} = 4D$$

or $$\sqrt{D^2 + y^2} = 2D$$

Again squaring both sides, we get

$$D^2 + y^2 = 4D^2$$

or $$y = \sqrt{3}\, D$$ **Ans.**

Alternate method Let $\Delta x = \lambda$ at angle θ as shown. Path difference between the waves is $S_1M = 2\lambda \cos\theta$ because $S_2P \approx MP$

$\therefore$ $$2\lambda \cos\theta = \lambda \qquad (\Delta x = \lambda)$$

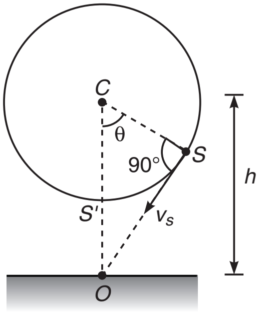
or $$\theta = 60^\circ$$

Now, $$PO = S_1O \tan\theta = S_1O \tan 60^\circ$$

or $$y = \sqrt{3}\, D$$ **Ans.**

Example 17 *A fighter plane moving in a vertical loop with constant speed of radius R. The centre of the loop is at a height h directly overhead of an observer standing on the ground. The observer receives maximum frequency of the sound produced by the plane when it is nearest to him. Find the speed of the plane. Velocity of sound in air is v.*

Solution Let the speed of the plane (source) be v_s. Maximum frequency will be observed by the observer when v_s is along SO. The observer receives maximum frequency when the plane is nearest to him. That is as soon as the wave pulse reaches from S to O with speed v the plane reaches from S to S' with speed v_s. Hence,



$$t = \frac{SO}{v} = \frac{SS'}{v_s} \quad \text{or} \quad v_s = \left(\frac{SS'}{SO}\right) v$$

$$= \frac{R\theta}{\sqrt{h^2 - R^2}}\, v \qquad \textbf{Ans.}$$

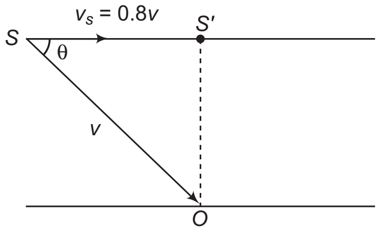
Here, $$\cos\theta = \frac{R}{h} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{R}{h}\right)$$

Example 18 *A source of sound of frequency* 1000 *Hz moves uniformly along a straight line with velocity* 0.8 *times velocity of sound. An observer is located at a distance* $l = 250$ *m from this line. Find*

(a) the frequency of the sound at the instant when the source is closest to the observer.

(b) the distance of the source when he observes no change in the frequency.

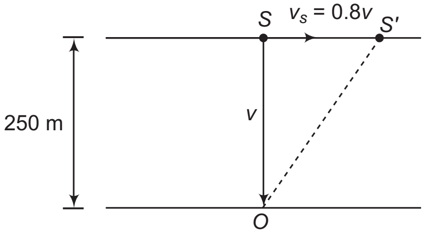
Solution (a) Suppose the pulse which is emitted when the source is at S reaches the observer O in the same time in which the source reaches from S to S', then



$$\cos\theta = \frac{SS'}{SO} = \frac{v_s t}{vt} = \frac{v_s}{v} = 0.8$$

Now,
$$f' = \left(\frac{v}{v - v_s \cos\theta}\right) f$$
$$= \left\{\frac{v}{v - (0.8v)(0.8)}\right\}(1000)$$
$$= \left(\frac{1}{1 - 0.64}\right)(1000)$$
$$= 2777.7 \text{ Hz}$$ **Ans.**

(b) The observer will observe no change in the frequency when the source is at S as shown in figure. In the time when the wave pulse reaches from S to O, the source will reach from S to S'. Hence,



$$t = \frac{SO}{v} = \frac{SS'}{v_s}$$

∴ $$SS' = \left(\frac{v_s}{v}\right) SO$$
$$= (0.8)(250) = 200 \text{ m}$$

Therefore, distance of observer from source at this instant is
$$S'O = \sqrt{(SO)^2 + (SS')^2}$$
$$= \sqrt{(250)^2 + (200)^2}$$
$$\approx 320 \text{ m}$$ **Ans.**

Example 19 *The air column in a pipe closed at one end is made to vibrate in its second overtone by a tuning fork of frequency 440 Hz. The speed of sound in air is 330 m/s. End corrections may be neglected. Let* p_0 *denotes the mean pressure at any point in the pipe, and* Δp_0 *the maximum amplitude of pressure variation.*

(a) Find the length L of the air column.

(b) What is the amplitude of pressure variation at the middle of the column?

(c) What are the maximum and minimum pressures at the open end of the pipe?

(d) What are the maximum and minimum pressures at the closed end of the pipe?

Solution (a) Frequency of second overtone of the closed pipe
$$= 5\left(\frac{v}{4L}\right) = 440 \text{ Hz}$$ (Given)

∴ $$L = \frac{5v}{4 \times 440} \text{ m}$$

Substituting

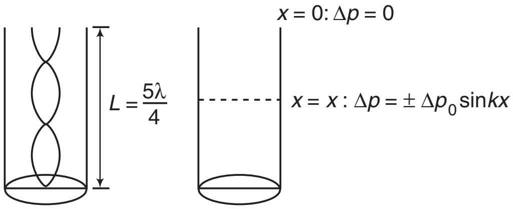
$$v = \text{speed of sound in air} = 330 \text{ m/s}$$

$$L = \frac{5 \times 330}{4 \times 440} = \frac{15}{16} \text{ m}$$ **Ans.**

(b) $$\lambda = \frac{4L}{5} = \frac{4\,(15/16)}{5} = \frac{3}{4} \text{ m}$$

Open end is displacement antinode. Therefore, it would be a pressure node

or $$\text{at } x = 0; \quad \Delta p = 0$$



Pressure amplitude at $x = x$ can be written as

$$\Delta p = \pm \Delta p \sin kx$$

where, $$k = \frac{2\pi}{\lambda} = \frac{2\pi}{3/4} = \frac{8\pi}{3} \text{ m}^{-1}$$

Therefore, pressure amplitude at $x = \frac{L}{2} = \frac{15/16}{2}$ m or (15/32) m will be

$$\Delta p = \pm \Delta p_0 \sin\left(\frac{8\pi}{3}\right)\left(\frac{15}{32}\right) = \pm \Delta p_0 \sin\left(\frac{5\pi}{4}\right)$$

$$\Delta p = \pm \frac{\Delta p_0}{\sqrt{2}}$$ **Ans.**

(c) Open end is a pressure node, i.e. $\Delta p = 0$

Hence, $$p_{\text{max}} = p_{\text{min}} = \text{Mean pressure } (p_0)$$

(d) Closed end is a displacement node or pressure antinode.

Therefore, $$p_{\text{max}} = p_0 + \Delta p_0$$

and $$p_{\text{min}} = p_0 - \Delta p_0$$

Example 20 *At a distance 20 m from a point source of sound the loudness level is 30 dB. Neglecting the damping, find*

(a) the loudness at 10 m from the source

(b) the distance from the source at which sound is not heard.

Solution (a) Intensity due to a point source varies with distance r from it as

$$I \propto \frac{1}{r^2}$$

or $$\frac{I_1}{I_2} = \left(\frac{r_2}{r_1^2}\right)^2$$

Now, $$L_1 = 10 \log \frac{I_1}{I_0}$$

and $$L_2 = 10 \log \frac{I_2}{I_0}$$

$\therefore$ $$L_1 - L_2 = 10\left[\log \frac{I_1}{I_0} - \log \frac{I_2}{I_0}\right]$$

$$= 10 \log \frac{I_1}{I_2} = 10 \log \left(\frac{r_2}{r_1}\right)^2$$

Substituting, $L_1 = 30\,\text{dB}$, $r_1 = 20\,\text{m}$ and $r_2 = 10\,\text{m}$

we have $$30 - L_2 = 10 \log \left(\frac{10}{20}\right)^2 = -6.0$$

or $$L_2 = 36\,\text{dB}$$ **Ans.**

(b) $L_1 - L_2 = 10 \log \left(\frac{r_2}{r_1}\right)^2$

Sound is not heard at a point where $L_2 = 0$

or $$30 = 10 \log \left(\frac{r_2}{r_1}\right)^2$$

$\therefore$ $$\left(\frac{r_2}{r_1}\right)^2 = 1000 \quad \text{or} \quad \frac{r_2}{r_1} = 31.62$$

$\therefore$ $$r_2 = (31.62)(20) \approx 632\,\text{m}$$ **Ans.**

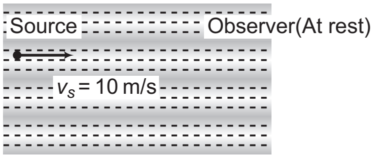
Example 21 *A boat is travelling in a river with a speed 10 m/s along the stream flowing with a speed 2 m/s. From this boat, a sound transmitter is lowered into the river through a rigid support. The wavelength of the sound emitted from the transmitter inside the water is 14.45 mm. Assume that attenuation of sound in water and air is negligible.*

(a) What will be the frequency detected by a receiver kept inside the river downstream?

(b) The transmitter and the receiver are now pulled up into air. The air is blowing with a speed 5 m/s in the direction opposite the river stream. Determine the frequency of the sound detected by the receiver.

(Temperature of the air and water = 20° C; Density of river water = $10^3\ kg/m^3$; Bulk modulus of the water = 2.088×10^9 Pa; Gas constant, $R = 8.31\ J/mol\text{-}K$; Mean molecular mass of air = 28.8×10^{-3} kg/mol; C_p/C_V for air = 1.4) **(JEE 2001)**

Solution Velocity of sound in water is



$$v_w = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.088 \times 10^9}{10^3}}$$

$$= 1445\,\text{m/s}$$

Frequency of sound in water will be

$$f_0 = \frac{v_w}{\lambda_w} = \frac{1445}{14.45 \times 10^{-3}} \text{ Hz}$$

$$f_0 = 10^5 \text{ Hz}$$

(a) Frequency of sound detected by receiver (observer) at rest would be

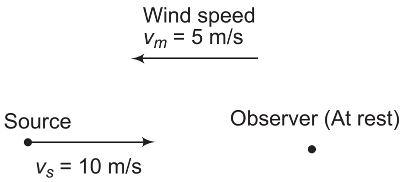
$$f_1 = f_0 \left(\frac{v_w + v_r}{v_w + v_r - v_s} \right)$$

$$= (10^5) \left(\frac{1445 + 2}{1445 + 2 - 10} \right) \text{ Hz}$$

$$f_1 = 1.0069 \times 10^5 \text{ Hz}$$ **Ans.**

(b) Velocity of sound in air is

$$v_a = \sqrt{\frac{\gamma RT}{M}}$$



$$= \sqrt{\frac{(1.4)\,(8.31)\,(20 + 273)}{28.8 \times 10^{-3}}}$$

$$= 344 \text{ m/s}$$

Frequency does not depend on the medium. Therefore, frequency in air is also $f_0 = 10^5$ Hz.

∴ Frequency of sound detected by receiver (observer) in air would be

$$f_2 = f_0 \left(\frac{v_a - v_w}{v_a - v_w - v_s} \right) = 10^5 \left[\frac{344 - 5}{344 - 5 - 10} \right] \text{Hz}$$

$$f_2 = 1.0304 \times 10^5 \text{ Hz}$$ **Ans.**

Example 22 *Three sound sources A, B and C have frequencies* 400, 401 *and* 402 *Hz, respectively. Calculate the number of beats noted per second.*

Solution Let us make the following table.

Group	Beat frequency $(f_1 - f_2)$ Hz	Beat time period $\left(\frac{1}{f_1 - f_2}\right)$ second
A and B	1	1
B and C	1	1
A and C	2	0.5

Beat time period for A and B is 1 s. It implies that if A and B are in phase at time $t = 0$, they are again in phase after 1 s. Same is the case with B and C. But beat time period for A and C is 0.5 s.

Therefore, beat time period for all together A, B and C will be 1 s. Because if, at $t = 0$, A, B and C all are in phase then after 1 s. (A and B) and (B and C) will again be in phase for the first time while (A and C) will be in phase for the second time. Or we can say that all A, B and C are again in phase after 1 s.

$\therefore$ Beat time period, $T_b = 1$ s

or Beat frequency, $f_b = \dfrac{1}{T_b} = 1$ Hz **Ans.**

Alternate method Suppose at time t, the equations of waves are

$$y_1 = A_1 \sin 2\pi f_A t \qquad (\omega = 2\pi f)$$

$$y_2 = A_2 \sin 2\pi f_B t$$

and

$$y_3 = A_3 \sin 2\pi f_C t$$

If they are in phase at some given instant of time t, then

$$2\pi f_A t = 2\pi f_B t = 2\pi f_C t \qquad \text{...(i)}$$

Let T_b be the beat time period, i.e. after time T_b they all are again in phase. As $f_C > f_B > f_A$, so

$$2\pi f_C\ (t + T_b) = 2\pi f_A\ (t + T_b) + 2m\pi \qquad \text{...(ii)}$$

and

$$2\pi f_B\ (t + T_b) = 2\pi f_A\ (t + T_b) + 2n\pi \qquad \text{...(iii)}$$

Here, m and $n\ (< m)$ are positive integers.

From Eqs. (i) and (ii),

$$(f_C - f_A)\ T_b = m \qquad \text{...(iv)}$$

Similarly, from Eqs. (i) and (iii)

$$(f_B - f_A)\ T_b = n \qquad \text{...(v)}$$

Dividing Eq. (iv) by Eq. (v),

$$\frac{m}{n} = \frac{f_C - f_A}{f_B - f_A} = \frac{402 - 400}{401 - 400} = \frac{2}{1}$$

Thus, letting $m = 2$ and $n = 1$

$$T_b = \frac{m}{f_C - f_A} \qquad \text{[from Eq. (iv)]}$$

$$= \frac{2}{2} = 1 \text{ Hz}$$

$\therefore$ Beat frequency, $f_b = \dfrac{1}{T_b} = 1$ Hz **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** A closed pipe and an open organ pipe are of same length. Then, neither of their frequencies can be same.

Reason : In the above case fundamental frequency of closed organ pipe will be two times the fundamental frequency of open organ pipe.

2. **Assertion :** A sound source is approaching towards a stationary observer along the line joining them. Then, apparent frequency to the observer will go on increasing.

Reason : If there is no relative motion between source and observer, apparent frequency is equal to the actual frequency.

3. **Assertion :** In longitudinal wave pressure is maximum at a point where displacement is zero.

Reason : There is a phase difference of $\frac{\pi}{2}$ between $y(x, t)$ and $\Delta P(x, t)$ equation in case of longitudinal wave.

4. **Assertion :** A train is approaching towards a hill. The driver of the train will hear beats.

Reason : Apparent frequency of reflected sound observed by driver will be more than the frequency of direct sound observed by him.

5. **Assertion :** Sound level increases linearly with intensity of sound.

Reason : If intensity of sound is doubled, sound level increases approximately 3 dB.

6. **Assertion :** Speed of sound in gases is independent of pressure of gas.

Reason : With increase in temperature of gas speed of sound will increase.

7. **Assertion :** Beat frequency between two tuning forks A and B is 4 Hz. Frequency of A is greater than the frequency of B. When A is loaded with wax, beat frequency may increase or decrease.

Reason : When a tuning fork is loaded with wax, its frequency decreases.

8. **Assertion :** Two successive frequencies of an organ pipe are 450 Hz and 750 Hz. Then, this pipe is a closed pipe.

Reason : Fundamental frequency of this pipe is 150 Hz.

9. **Assertion :** Fundamental frequency of a narrow pipe is more.

Reason : According to Laplace end correction if radius of pipe is less, frequency should be more.

10. **Assertion :** In the experiment of finding speed of sound by resonance tube method, as the level of water is lowered, wavelength increases.

Reason : By lowering the water level number of loops increases.

Objective Questions

1. Velocity of sound in vacuum is

(a) equal to 330 m/s (b) greater than 330 m/s

(c) less than 330 m/s (d) None of these

2. Longitudinal waves are possible in

(a) solids (b) liquids

(c) gases (d) All of these

3. If the fundamental frequency of a pipe closed at one end is 512 Hz. The frequency of a pipe of the same dimension but open at both ends will be

(a) 1024 Hz (b) 512 Hz

(c) 256 Hz (d) 128 Hz

4. The temperature at which the velocity of sound in oxygen will be the same as that of nitrogen at 15°C is

(a) 112°C (b) 72°C

(c) 56°C (d) 17°C

5. A closed organ pipe is excited to vibrate in the third overtone. It is observed that there are

(a) three nodes and three antinodes (b) three nodes and four antinodes

(c) four nodes and three antinodes (d) four nodes and four antinodes

6. When temperature is increased, the frequency of organ pipe

(a) increases (b) decreases

(c) remains same (d) Nothing can be said

7. When a sound wave travels from water to air, it

(a) bends towards normal (b) bends away from normal

(c) may bend in any direction (d) data insufficient

8. A closed organ pipe and an open organ pipe are tuned to the same fundamental frequency. The ratio of their lengths is

(a) 1 : 2 (b) 2 : 1

(c) 1 : 4 (d) 4 : 1

9. A sonometer wire under a tension of 10 kg weight is in unison with a tuning fork of frequency 320 Hz. To make the wire vibrate in unison with a tuning fork of frequency 256 Hz, the tension should be altered by

(a) 3.6 kg decreased (b) 3.6 kg increased

(c) 6.4 kg decreased (d) 6.4 kg increased

10. A tuning fork of frequency 256 Hz is moving towards a wall with a velocity of 5 m/s. If the speed of sound is 330 m/s, then the number of beats heard per second by a stationary observer lying between tuning fork and the wall is

(a) 2 (b) 4 (c) zero (d) 8

11. Two sound waves of wavelength 1 m and 1.01 m in a gas produce 10 beats in 3 s. The velocity of sound in the gas is

(a) 330 m/s (b) 337 m/s (c) 360 m/s (d) 300 m/s

12. When a source is going away from a stationary observer with the velocity equal to that of sound in air, then the frequency heard by observer is n times the original frequency. The value of n is

(a) 0.5
(b) 0.25
(c) 1.0
(d) No sound is heard

13. When interference is produced by two progressive waves of equal frequencies, then the maximum intensity of the resulting sound are N times the intensity of each of the component waves. The value of N is

(a) 1
(b) 2
(c) 4
(d) 8

14. A tuning fork of frequency 500 Hz is sounded on a resonance tube. The first and second resonances are obtained at 17 cm and 52 cm. The velocity of sound is

(a) 170 m/s
(b) 350 m/s
(c) 520 m/s
(d) 850 m/s

15. A vehicle, with a horn of frequency n is moving with a velocity of 30 m/s in a direction perpendicular to the straight line joining the observer and the vehicle. The observer perceives the sound to have a frequency $(n + n_1)$. If the sound velocity in air is 300 m/s, then

(a) $n_1 = 10\,n$
(b) $n_1 = 0$
(c) $n_1 = 0.1\,n$
(d) $n_1 = -0.1\,n$

16. How many frequencies below 1 kHz of natural oscillations of air column will be produced if a pipe of length 1 m is closed at one end? [velocity of sound in air is 340 m/s]

(a) 3
(b) 6
(c) 4
(d) 8

17. A sound source emits frequency of 180 Hz when moving towards a rigid wall with speed 5 m/s and an observer is moving away from wall with speed 5 m/s. Both source and observer moves on a straight line which is perpendicular to the wall. The number of beats per second heard by the observer will be [speed of sound = 355 m/s]

(a) 5 beats/s
(b) 10 beats/s
(c) 6 beats/s
(d) 8 beats/s

18. Two sound waves of wavelengths λ_1 and λ_2 $(\lambda_2 > \lambda_1)$ produce n beats/s, the speed of sound is

(a) $\dfrac{n\lambda_1\lambda_2}{\lambda_2 - \lambda_1}$
(b) $n\left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_2}\right)$
(c) $n(\lambda_2 - \lambda_1)$
(d) $n(\lambda_2 + \lambda_1)$

19. A, B and C are three tuning forks. Frequency of A is 350 Hz. Beats produced by A and B are 5/s and by B and C are 4/s. When a wax is put on A beat frequency between A and B is 2 Hz and between A and C is 6 Hz. Then, frequency of B and C respectively, are

(a) 355 Hz, 349 Hz
(b) 345 Hz, 341 Hz
(c) 355 Hz, 341 Hz
(d) 345 Hz, 349 Hz

20. The first resonance length of a resonance tube is 40 cm and the second resonance length is 122 cm. The third resonance length of the tube will be

(a) 200 cm (b) 202 cm (c) 203 cm (d) 204 cm

21. Two identical wires are stretched by the same tension of 100 N and each emits a note of frequency 200 Hz. If the tension in one wire is increased by 1 N, then the beat frequency is

(a) 2 Hz
(b) $\frac{1}{2}$ Hz
(c) 1 Hz
(d) None of these

22. A tuning fork of frequency 340 Hz is sounded above an organ pipe of length 120 cm. Water is now slowly poured in it. The minimum height of water column required for resonance is (speed of sound in air = 340 m/s)

(a) 25 cm
(b) 95 cm
(c) 75 cm
(d) 45 cm

23. In a closed end pipe of length 105 cm, standing waves are set up corresponding to the third overtone. What distance from the closed end, amongst the following is a pressure node?

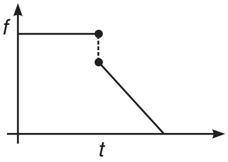
(a) 20 cm
(b) 60 cm
(c) 85 cm
(d) 45 cm

24. Oxygen is 16 times heavier than hydrogen. At NTP equal volume of hydrogen and oxygen are mixed. The ratio of speed of sound in the mixture to that in hydrogen is

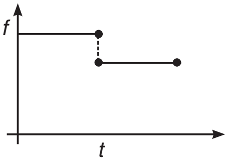
(a) $\sqrt{8}$
(b) $\sqrt{\frac{1}{8}}$
(c) $\sqrt{\frac{2}{17}}$
(d) $\sqrt{\frac{32}{17}}$

25. A train is moving towards a stationary observer. Which of the following curve best represents the frequency received by observer f as a function of time?

(a)



(b)



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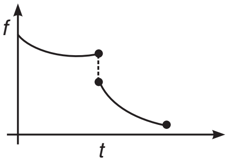
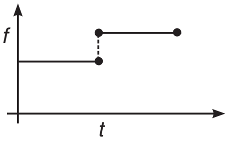


(d)



26. A closed organ pipe and an open organ pipe of same length produce 4 beats when they are set into vibrations simultaneously. If the length of each of them were twice their initial lengths, the number of beats produced will be

(a) 2
(b) 4
(c) 1
(d) 8

27. One train is approaching an observer at rest and another train is receding from him with the same velocity 4 m/s. Both trains blow whistles of same frequency of 243 Hz. The beat frequency in Hz as heard by the observer is (speed of sound in air = 320 m/s)

(a) 10
(b) 6
(c) 4
(d) 1

28. Speed of sound in air is 320 m/s. A pipe closed at one end has a length of 1 m and there is another pipe open at both ends having a length of 1.6 m. Neglecting end corrections, both the air columns in the pipes can resonate for sound of frequency

(a) 80 Hz (b) 240 Hz (c) 320 Hz (d) 400 Hz

29. Four sources of sound each of sound level 10 dB are sounded together in phase, the resultant intensity level will be ($\log_{10} 2 = 0.3$)

(a) 40 dB (b) 26 dB (c) 22 dB (d) 13 dB

30. A longitudinal sound wave given by $p = 2.5 \sin \frac{\pi}{2}(x - 600\,t)$ (p is in N/m^2, x is in metre and t is in second) is sent down a closed organ pipe. If the pipe vibrates in its second overtone, the length of the pipe is

(a) 6 m (b) 8 m (c) 5 m (d) 10 m

31. Sound waves of frequency 600 Hz fall normally on perfectly reflecting wall. The distance from the wall at which the air particles have the maximum amplitude of vibration is

(speed of sound in air = 330 m/s)

(a) 13.75 cm (b) 40.25 cm (c) 70.5 cm (d) 60.75 cm

32. The wavelength of two sound waves are 49 cm and 50 cm, respectively. If the room temperature is 30°C, then the number of beats produced by them is approximately
(velocity of sound in air at 30°C = 332 m/s)

(a) 6 (b) 10 (c) 13 (d) 18

33. Two persons A and B, each carrying a source of frequency 300 Hz, are standing a few metre apart. A starts moving towards B with velocity 30 m/s. If the speed of sound is 300 m/s, which of the following is true?

(a) Number of beats heard by A is higher than that heard by B
(b) The number of beats heard by B are 30 Hz
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

34. A fixed source of sound emitting a certain frequency appears as f_a when the observer is approaching the source with speed v_0 and f_r when the observer recedes from the source with the same speed. The frequency of the source is

(a) $\frac{f_r + f_a}{2}$ (b) $\frac{f_r - f_a}{2}$ (c) $\sqrt{f_a f_r}$ (d) $\frac{2 f_r f_a}{f_r + f_a}$

Subjective Questions

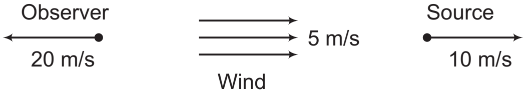
1. Determine the speed of sound waves in water, and find the wavelength of a wave having a frequency of 242 Hz. Take $B_{\text{water}} = 2 \times 10^9$ Pa.

2. If the source and receiver are at rest relative to each other but the wave medium is moving relative to them, will the receiver detect any wavelength or frequency shift.

3. Using the fact that hydrogen gas consists of diatomic molecules with $M = 2$ kg/K-mol. Find the speed of sound in hydrogen at 27° C.

4. About how many times more intense will the normal ear perceive a sound of 10^{-6} W/m^2 than one of 10^{-9} W/m^2?

5. A 300 Hz source, an observer and a wind are moving as shown in the figure with respect to the ground. What frequency is heard by the observer? Take speed of sound in air = 340 m/s.



6. A person standing between two parallel hills fires a gun. He hears the first echo after $\frac{3}{2}$ s, and a second echo after $\frac{5}{5}$ s. If speed of sound is 332 m/s, calculate the distance between the hills. When will he hear the third echo?

7. Helium is a monoatomic gas that has a density of $0.179\ \text{kg/m}^3$ at a pressure of 76 cm of mercury and a temperature of 0° C. Find the speed of compressional waves (sound) in helium at this temperature and pressure.

8. (a) In a liquid with density $1300\ \text{kg/m}^3$, longitudinal waves with frequency 400 Hz are found to have wavelength 8.00 m. Calculate the bulk modulus of the liquid.

(b) A metal bar with a length of 1.50 m has density $6400\ \text{kg/m}^3$. Longitudinal sound waves take 3.90×10^{-4} s to travel from one end of the bar to the other. What is Young's modulus for this metal?

9. What must be the stress (F/A) in a stretched wire of a material whose Young's modulus is Y for the speed of longitudinal waves equal to 30 times the speed of transverse waves?

10. A gas is a mixture of two parts by volume of hydrogen and one part by volume of nitrogen at STP. If the velocity of sound in hydrogen at 0° C is 1300 m/s. Find the velocity of sound in the gaseous mixture at 27° C.

11. The explosion of a fire cracker in the air at a height of 40 m produces a 100 dB sound level at ground below. What is the instantaneous total radiated power? Assuming that it radiates as a point source.

12. (a) What is the intensity of a 60 dB sound?

(b) If the sound level is 60 dB close to a speaker that has an area of $120\ \text{cm}^2$. What is the acoustic power output of the speaker?

13. (a) By what factor must the sound intensity be increased to increase the sound intensity level by 13.0 dB?

(b) Explain why you do not need to know the original sound intensity?

14. The speed of a certain compressional wave in air at standard temperature and pressure is 330 m/s. A point source of frequency 300 Hz radiates energy uniformly in all directions at the rate of 5 Watt.

(a) What is the intensity of the wave at a distance of 20 m from the source?

(b) What is the amplitude of the wave there? [Density of air at STP = $1.29\ \text{kg/m}^3$]

15. What is the amplitude of motion for the air in the path of a 60 dB, 800 Hz sound wave? Assume that $\rho_{\text{air}} = 1.29\ \text{kg/m}^3$ and $v = 330$ m/s.

16. A rock band gives rise to an average sound level of 102 dB at a distance of 20 m from the centre of the band. As an approximation, assume that the band radiates sound equally into a sphere. What is the sound power output of the band?

17. If it were possible to generate a sinusoidal 300 Hz sound wave in air that has a displacement amplitude of 0.200 mm. What would be the sound level of the wave? (Assume $v = 330$ m/s and $\rho_{\text{air}} = 1.29\ \text{kg/m}^3$)

18. (a) A longitudinal wave propagating in a water-filled pipe has intensity $3.00 \times 10^{-6}\ \text{W/m}^2$ and frequency 3400 Hz. Find the amplitude A and wavelength λ of the wave. Water has density $1000\ \text{kg/m}^3$ and bulk modulus 2.18×10^9 Pa.

(b) If the pipe is filled with air at pressure 1.00×10^5 Pa and density $1.20\ \text{kg/m}^3$, what will be the amplitude A and wavelength λ of a longitudinal wave with the same intensity and frequency as in part (a)?

(c) In which fluid is the amplitude larger, water or air?

What is the ratio of the two amplitudes? Why is this ratio so different from one? Consider air as diatomic.

19. For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about 6.0×10^{-5} Pa. Calculate the corresponding intensity and sound intensity level at 20° C. (Assume $v = 330$ m/s and $\rho_{\text{air}} = 1.29\ \text{kg/m}^3$).

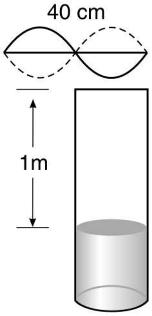
20. Find the fundamental frequency and the frequency of the first two overtones of a pipe 45.0 cm long. (a) If the pipe is open at both ends. (b) If the pipe is closed at one end. Use $v = 344$ m/s.

21. A uniform tube of length 60 cm stands vertically with its lower end dipping into water. First two air column lengths above water are 15 cm and 45 cm, when the tube responds to a vibrating fork of frequency 500 Hz. Find the lowest frequency to which the tube will respond when it is open at both ends.

22. Write the equation for the fundamental standing sound waves in a tube that is open at both ends. If the tube is 80 cm long and speed of the wave is 330 m/s. Represent the amplitude of the wave at an antinode by A.

23. A long glass tube is held vertically, dipping into water, while a tuning fork of frequency 512 Hz is repeatedly struck and held over the open end. Strong resonance is obtained, when the length of the tube above the surface of water is 50 cm and again 84 cm, but not at any intermediate point. Find the speed of sound in air and next length of the air column for resonance.

24. A wire of length 40 cm which has a mass of 4 g oscillates in its second harmonic and sets the air column in the tube to vibrations in its fundamental mode as shown in figure. Assuming the speed of sound in air as 340 m/s, find the tension in the wire.



25. In a resonance tube experiment to determine the speed of sound in air, a pipe of diameter 5 cm is used. The column in pipe resonates with a tuning fork of frequency 480 Hz when the minimum length of the air column is 16 cm. Find the speed of sound in air column at room temperature.

26. On a day when the speed of sound is 345 m/s, the fundamental frequency of a closed organ pipe is 220 Hz. (a) How long is this closed pipe? (b) The second overtone of this pipe has the same wavelength as the third harmonic of an open pipe. How long is the open pipe?

27. A closed organ pipe is sounded near a guitar, causing one of the strings to vibrate with large amplitude. We vary the tension of the string until we find the maximum amplitude. The string is 80% as long as the closed pipe. If both the pipe and the string vibrate at their fundamental frequency, calculate the ratio of the wave speed on the string to the speed of sound in air.

28. A police siren emits a sinusoidal wave with frequency $f_S = 300\,\text{Hz}$. The speed of sound is 340 m/s. (a) Find the wavelength of the waves if the siren is at rest in the air. (b) If the siren is moving at 30 m/s, find the wavelength of the waves ahead of and behind the source.

29. Two identical violin strings, when in tune and stretched with the same tension, have a fundamental frequency of 440.0 Hz. One of the string is retuned by adjusting its tension. When this is done, 1.5 beats per second are heard when both strings are plucked simultaneously. (a) What are the possible fundamental frequencies of the retuned string? (b) By what fractional amount was the string tension changed if it was (i) increased (ii) decreased?

30. A swimming duck paddles the water with its feet once every 1.6 s, producing surface waves with this period. The duck is moving at constant speed in a pond where the speed of surface waves is 0.32 m/s, and the crests of the waves ahead of the duck are spaced 0.12 m apart. (a) What is the duck's speed? (b) How far apart are the crests behind the duck?

31. A railroad train is travelling at 30.0 m/s in still air. The frequency of the note emitted by the train whistle is 262 Hz. What frequency is heard by a passenger on a train moving in the opposite direction to the first at 18.0 m/s and (a) approaching the first? (b) receding from the first? Speed of sound in air = 340 m/s.

32. A boy is walking away from a wall at a speed of 1.0 m/s in a direction at right angles to the wall. As he walks, he blows a whistle steadily. An observer towards whom the boy is walking hears 4.0 beats per second. If the speed of sound is 340 m/s, what is the frequency of the whistle?

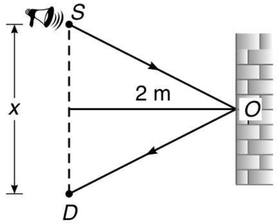
33. A tuning fork P of unknown frequency gives 7 beats in 2 seconds with another tuning fork Q. When Q runs towards a wall with a speed of 5 m/s it gives 5 beats per second with its echo. On loading P with wax, it gives 5 beats per second with Q. What is the frequency of P? Assume speed of sound = 332 m/s.

34. A stationary observer receives sonic oscillations from two tuning forks one of which approaches and the other recedes with the same velocity. As this takes place, the observer hears the beats of frequency $f = 2.0\,\text{Hz}$. Find the velocity of each tuning fork if their oscillation frequency is $f_0 = 680\,\text{Hz}$ and the velocity of sound in air is $v = 340$ m/s.

35. Sound waves from a tuning fork A reach a point P by two separate paths ABP and ACP. When ACP is greater than ABP by 11.5 cm, there is silence at P. When the difference is 23 cm the sound becomes loudest at P and when 34.5 cm there is silence again and so on. Calculate the minimum frequency of the fork if the velocity of sound is taken to be 331.2 m/s.

36. Two loudspeakers S_1 and S_2 each emit sounds of frequency 220 Hz uniformly in all directions. S_1 has an acoustic output of 1.2×10^{-3} W and S_2 has 1.8×10^{-3} W. S_1 and S_2 vibrate in phase. Consider a point P such that $S_1P = 0.75$ m and $S_2P = 3$ m. How are the phases arriving at P related? What is the intensity at P when both S_1 and S_2 are on? Speed of sound in air is 330 m/s.

37. A source of sound emitting waves at 360 Hz is placed in front of a vertical wall, at a distance 2 m from it. A detector is also placed in front of the wall at the same distance from it. Find the minimum distance between the source and the detector for which the detector detects a maximum of sound. Take speed of sound in air = 360 m/s. Assume that there is no phase change in reflected wave.



38. The atomic mass of iodine is 127 g/mol. A standing wave in iodine vapour at 400 K has nodes that are 6.77 cm apart when the frequency is 1000 Hz. At this temperature, is iodine vapour monoatomic or diatomic.

39. A tuning fork whose natural frequency is 440 Hz is placed just above the open end of a tube that contains water. The water is slowly drained from the tube while the tuning fork remains in place and is kept vibrating. The sound is found to be enhanced when the air column is 60 cm long and when it is 100 cm long. Find the speed of sound in air.

40. A piano wire A vibrates at a fundamental frequency of 600 Hz. A second identical wire B produces 6 beats per second with it when the tension in A is slightly increased. Find the ratio of the tension in A to the tension in B.

41. A tuning fork of frequency 256 Hz produces 4 beats per second with a wire of length 25 cm vibrating in its fundamental mode. The beat frequency decreases when the length is slightly shortened. What could be the minimum length by which the wire be shortened so that it produces no beats with the tuning fork?

42. Show that when the speed of the source and the observer are small compared to the speed of sound in the medium, the change in frequency becomes independent of the fact whether the source is moving or the observer.

43. A sound source moves with a speed of 80 m/s relative to still air toward a stationary listener. The frequency of sound is 200 Hz and speed of sound in air is 340 m/s.

(a) Find the wavelength of the sound between the source and the listener.

(b) Find the frequency heard by the listener.

44. A railroad train is travelling at 30 m/s in still air. The frequency of the note emitted by the locomotive whistle is 500 Hz. What is the wavelength of the sound waves :

(a) in front of the locomotive? (b) behind the locomotive?

What is the frequency of the sound heard by a stationary listener

(c) in front of the locomotive? (d) behind the locomotive?

Speed of sound in air is 344 m/s.

45. For a certain organ pipe, three successive resonance frequencies are observed at 425, 595 and 765 Hz respectively. Taking the speed of sound in air to be 340 m/s,

(a) explain whether the pipe is closed at one end or open at both ends.

(b) determine the fundamental frequency and length of the pipe.

46. Two tuning forks A and B sounded together give 8 beats per second. With an air resonance tube closed at one end, the two forks give resonances when the two air columns are 32 cm and 33 cm respectively. Calculate the frequencies of forks.

LEVEL 2

Single Correct Option

1. A plane wave of sound travelling in air is incident upon a plane water surface. The angle of incidence is 60°. If velocity of sound in air and water are 330 m/s and 1400 m/s, then the wave undergoes

(a) refraction only
(b) reflection only
(c) Both reflection and refraction
(d) neither reflection nor refraction

2. An organ pipe of $(3.9\,\pi)$ m long, open at both ends is driven to third harmonic standing wave. If the amplitude of pressure oscillation is 1% of mean atmospheric pressure $[p_0 = 10^5\ \text{N/m}^2]$. The maximum displacement of particle from mean position will be
[Given, velocity of sound = 200 m/s and density of air = 1.3 kg/m^3]

(a) 2.5 cm
(b) 5 cm
(c) 1 cm
(d) 2 cm

3. A plane sound wave passes from medium 1 into medium 2. The speed of sound in medium 1 is 200 m/s and in medium 2 is 100 m/s. The ratio of amplitude of the transmitted wave to that of incident wave is

(a) $\frac{3}{4}$
(b) $\frac{4}{5}$
(c) $\frac{5}{6}$
(d) $\frac{2}{3}$

4. Two sources of sound are moving in opposite directions with velocities v_1 and v_2 $(v_1 > v_2)$. Both are moving away from a stationary observer. The frequency of both the sources is 1700 Hz. What is the value of $(v_1 - v_2)$ so that the beat frequency observed by the observer is 10 Hz? $v_{\text{sound}} = 340$ m/s and assume that v_1 and v_2 both are very much less than v_{sound}.

(a) 1 m/s
(b) 2 m/s
(c) 3 m/s
(d) 4 m/s

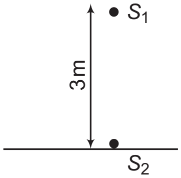
5. A sounding body emitting a frequency of 150 Hz is dropped from a height. During its fall under gravity it crosses a balloon moving upwards with a constant velocity of 2 m/s one second after it started to fall. The difference in the frequency observed by the man in balloon just before and just after crossing the body will be (velocity of sound = 300 m/s, $g = 10$ m/s^2)

(a) 12
(b) 6
(c) 8
(d) 4

6. A closed organ pipe has length L. The air in it is vibrating in third overtone with maximum amplitude a. The amplitude at distance $\frac{L}{7}$ from closed end of the pipe is

(a) 0
(b) a
(c) $\frac{a}{2}$
(d) Data insufficient

7. S_1 and S_2 are two coherent sources of sound having no initial phase difference. The velocity of sound is 330 m/s. No maxima will be formed on the line passing through S_2 and perpendicular to the line joining S_1 and S_2. If the frequency of both the sources is



(a) 330 Hz (b) 120 Hz (c) 100 Hz (d) 220 Hz

8. A source is moving with constant speed $v_s = 20$ m/s towards a stationary observer due east of the source. Wind is blowing at the speed of 20 m/s at 60° north of east. The source has frequency 500 Hz. Speed of sound = 300 m/s. The frequency registered by the observer is approximately

(a) 541 Hz (b) 552 Hz
(c) 534 Hz (d) 517 Hz

9. A car travelling towards a hill at 10 m/s sounds its horn which has a frequency 500 Hz. This is heard in a second car travelling behind the first car in the same direction with speed 20 m/s. The sound can also be heard in the second car by reflection of sound from the hill. The beat frequency heard by the driver of the second car will be (speed of sound in air = 340 m/s)

(a) 31 Hz (b) 24 Hz
(c) 21 Hz (d) 34 Hz

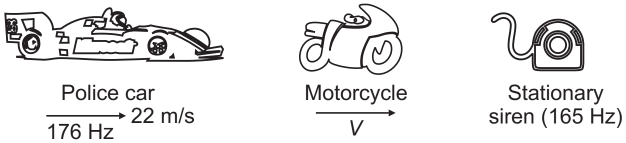
10. Two sounding bodies are producing progressive waves given by $y_1 = 2\sin(400\pi t)$ and $y_2 = \sin(404\,\pi t)$ where t is in second, which superpose near the ears of a person. The person will hear

(a) 2 beats/s with intensity ratio 9/4 between maxima and minima
(b) 2 beats/s with intensity ratio 9 between maxima and minima
(c) 4 beats/s with intensity ratio 16 between maxima and minima
(d) 4 beats/s with intensity ratio 16/9 between maxima and minima

11. The air in a closed tube 34 cm long is vibrating with two nodes and two antinodes and its temperature is 51°C. What is the wavelength of the waves produced in air outside the tube, when the temperature of air is 16°C?

(a) 42.8 cm (b) 68 cm (c) 17 cm (d) 102 cm

12. A police car moving at 22 m/s, chase a motorcyclist. The police man sounds his horn at 176 Hz, while both of them move towards a stationary siren of frequency 165 Hz. Calculate the speed of the motorcyclist, if he does not observe any beats. (velocity of sound in air = 330 m/s) **(JEE 2003)**

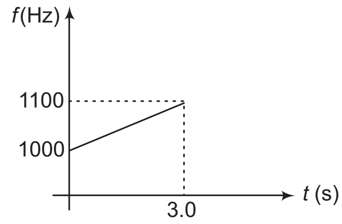


(a) 33 m/s (b) 22 m/s (c) zero (d) 11 m/s

13. A closed organ pipe resonates in its fundamental mode at a frequency of 200 Hz with O_2 in the pipe at a certain temperature. If the pipe now contains 2 moles of O_2 and 3 moles of ozone, then what will be the fundamental frequency of same pipe at same temperature?

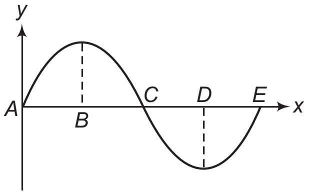
(a) 268.23 Hz (b) 175.4 Hz (c) 149.45 Hz (d) None of these

14. A detector is released from rest over a source of sound of frequency $f_0 = 10^3$ Hz. The frequency observed by the detector at time t is plotted in the graph. The speed of sound in air is ($g = 10\ \text{m/s}^2$)



(a) 330 m/s (b) 350 m/s (c) 300 m/s (d) 310 m/s

15. Sound waves are travelling along positive x-direction. Displacement of particle at any time t is as shown in figure. Select the wrong statement.



(a) Particle located at E has its velocity in negative x-direction
(b) Particle located at D has zero velocity
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

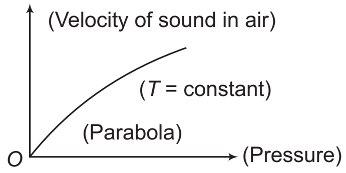
More than One Correct Options

1. An air column in a pipe, which is closed at one end, is in resonance with a vibrating tuning fork of frequency 264 Hz. If $v = 330$ m/s, the length of the column in cm is (are)

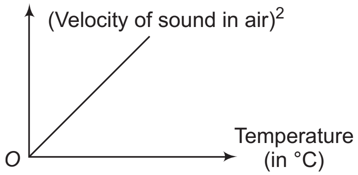
(a) 31.25 (b) 62.50
(c) 93.75 (d) 125

2. Which of the following is/are correct?

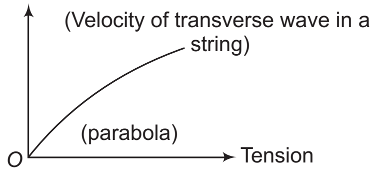
(a)



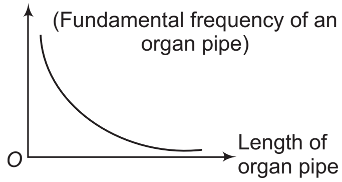
(b)



(c)



(d)



3. Choose the correct options for longitudinal wave

(a) maximum pressure variation is BAk
(b) maximum density variation is ρAk
(c) pressure equation and density equation are in phase
(d) pressure equation and displacement equation are out of phase

4. Second overtone frequency of a closed pipe and fourth harmonic frequency of an open pipe are same. Then, choose the correct options.

(a) Fundamental frequency of closed pipe is more than the fundamental frequency of open pipe
(b) First overtone frequency of closed pipe is more than the first overtone frequency of open pipe
(c) Fifteenth harmonic frequency of closed pipe is equal to twelfth harmonic frequency of open pipe
(d) Tenth harmonic frequency of closed pipe is equal to eighth harmonic frequency of open pipe

5. For fundamental frequency f of a closed pipe, choose the correct options.

(a) If radius of pipe is increased, f will decrease
(b) If temperature is increased, f will increase
(c) If molecular mass of the gas filled in the pipe is increased, f will decrease.
(d) If pressure of gas (filled in the pipe) is increased without change in temperature, f will remain unchanged

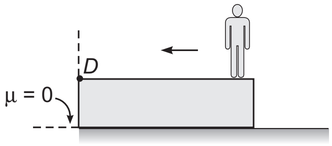
6. A source is approaching towards an observer with constant speed along the line joining them. After crossing the observer, source recedes from observer with same speed. Let f is apparent frequency heard by observer. Then,

(a) f will keep on increasing during approaching
(b) f will keep on decreasing during receding
(c) f will remain constant during approaching
(d) f will remain constant during receding

Comprehension Based Questions

Passage

A man of mass 50 *kg is running on a plank of mass* 150 *kg with speed of* 8 *m/s relative to plank as shown in the figure (both were initially at rest and the velocity of man with respect to ground any how remains constant). Plank is placed on smooth horizontal surface. The man, while running, whistles with frequency* f_0. *A detector* (D) *placed on plank detects frequency. The man jumps off with same velocity (w.r.t. to ground) from point* D *and slides on the smooth horizontal surface [Assume coefficient of friction between man and horizontal is zero]. The speed of sound in still medium is* 330 *m/s. Answer the following questions on the basis of above situations.*



1. The frequency of sound detected by detector D, before man jumps off the plank is

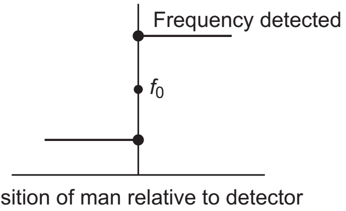
(a) $\dfrac{332}{324} f_0$ (b) $\dfrac{330}{322} f_0$
(c) $\dfrac{328}{336} f_0$ (d) $\dfrac{330}{338} f_0$

2. The frequency of sound detected by detector D, after man jumps off the plank is

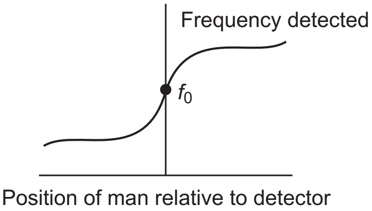
(a) $\dfrac{332}{324} f_0$ (b) $\dfrac{330}{322} f_0$
(c) $\dfrac{328}{336} f_0$ (d) $\dfrac{330}{338} f_0$

3. Choose the correct plot between the frequency detected by detector *versus* position of the man relative to detector.

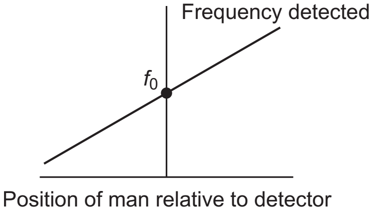
(a)



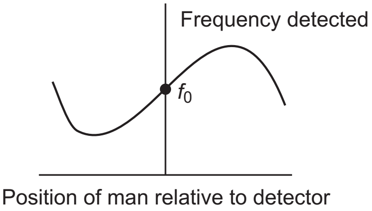
(b)



(c)



(d)

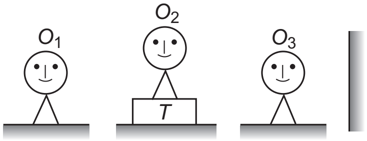


Match the Columns

1. Fundamental frequency of an open organ pipe is f. Match the following two columns for a closed pipe of double the length.

Column I	Column II
(a) Fundamental frequency	(p) $1.25\,f$
(b) Second overtone frequency	(q) f
(c) Third harmonic frequency	(r) $0.75\,f$
(d) First overtone frequency	(s) None of these

2. A train T horns a sound of frequency f. It is moving towards a wall with speed $\frac{1}{4}$th the speed of sound. There are three observers O_1, O_2 and O_3 as shown. Match the following two columns.



Column I	Column II
(a) Beat frequency observed to O_1	(p) $\frac{2}{3}f$
(b) Beat frequency observed to O_2	(q) $\frac{8}{15}f$
(c) Beat frequency observed to O_3	(r) None of these
(d) If train moves in opposite direction with same speed, then beat frequency observed to O_3	(s) Zero

3. A tuning fork is placed near a vibrating stretched wire. A boy standing near the two hears a beat frequency f. It is known that frequency of tuning fork is greater than frequency of stretched wire. Match the following two columns.

Column I	Column II
(a) If tuning fork is loaded with wax,	(p) beat frequency must increase.
(b) If prongs of tuning fork are filed,	(q) beat frequency must decrease.
(c) If tension in stretched wire is increased,	(r) beat frequency may increase.
(d) If tension in stretched wire is decreased,	(s) beat frequency may decrease.

4. I represents intensity of sound wave, A the amplitude and r the distance from the source. Then, match the following two columns.

Column I	Column II
(a) Intensity due to a point source.	(p) proportional to $r^{-1/2}$
(b) Amplitude due to a point source.	(q) proportional to r^{-1}
(c) Intensity due to a line source.	(r) proportional to r^{-2}
(d) Amplitude due to a line source.	(s) proportional to r^{-4}

5. The equation of longitudinal stationary wave in second overtone mode in a closed organ pipe is

$$y = (4 \text{ mm}) \sin \pi x \cos \pi t$$

Here, x is in metre and t in second. Then, match the following two columns.

Column I	Column II
(a) Length of pipe	(p) 1 m
(b) Wavelength	(q) 1.5 m
(c) Distance of displacement node from the closed end	(r) 2.0 m
(d) Distance of pressure node from the closed end	(s) None of these

Subjective Questions

1. A window whose area is $2\,m^2$ opens on a street where the street noise results at the window an intensity level of 60 dB. How much acoustic power enters the window through sound waves? Now, if a sound absorber is fitted at the window, how much energy from the street will it collect in a day?

2. A point A is located at a distance $r = 1.5$ m from a point source of sound of frequency 600 Hz. The power of the source is 0.8 W. Speed of sound in air is 340 m/s and density of air is 1.29 kg/m^3. Find at the point A,

(a) the pressure oscillation amplitude $(\Delta p)_m$

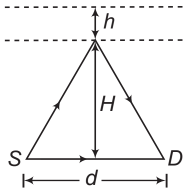
(b) the displacement oscillation amplitude A.

3. A flute which we treat as a pipe open at both ends is 60 cm long.

(a) What is the fundamental frequency when all the holes are covered?

(b) How far from the mouthpiece should a hole be uncovered for the fundamental frequency to be 330 Hz? Take speed of sound in air as 340 m/s.

4. A source S and a detector D of high frequency waves are a distance d apart on the ground. The direct wave from S is found to be in phase at D with the wave from S that is reflected from a horizontal layer at an altitude H. The incident and reflected rays make the same angle with the reflecting layer. When the layer rises a distance h, no signal is detected at D. Neglect absorption in the atmosphere and find the relation between d, h, H and the wavelength λ of the waves.



5. Two sound speakers are driven in phase by an audio amplifier at frequency 600 Hz. The speed of sound is 340 m/s. The speakers are on the y-axis, one at $y = +1.0$ m and the other at $y = -1.0$ m. A listener begins at $y = 0$ and walks along a line parallel to the y-axis at a very large distance x away.

(a) At what angle θ (between the line from the origin to the listener at the x-axis) will she first hear a minimum sound intensity?

(b) At what angle will she first hear a maximum (after $\theta = 0°$) sound intensity?

(c) How many maxima can she possibly hear if she keeps walking in the same direction?

6. Two speakers separated by some distance emit sound of the same frequency. At some point P the intensity due to each speaker separately is I_0. The path difference from P to one of the speakers is $\frac{1}{2}\lambda$ greater than that from P to the other speaker. What is the intensity at P if

(a) the speakers are coherent and in phase;

(b) the speakers are incoherent; and

(c) the speakers are coherent but have a phase difference of 180°?

7. Two loudspeakers radiate in phase at 170 Hz. An observer sits at 8 m from one speaker and 11m from the other. The intensity level from either speaker acting alone is 60 dB. The speed of sound is 340 m/s.

(a) Find the observed intensity when both speakers are on together.

(b) Find the observed intensity level when both speakers are on together but one has its leads reversed so that the speakers are 180° out of phase.

(c) Find the observed intensity level when both speakers are on and in phase but the frequency is 85 Hz.

8. Two identical speakers emit sound waves of frequency 680 Hz uniformly in all directions with a total audio output of 1 mW each. The speed of sound in air is 340 m/s. A point P is a distance 2.00 m from one speaker and 3.00 m from the other.

(a) Find the intensities I_1 and I_2 from each speaker at point P separately.

(b) If the speakers are driven coherently and in phase, what is the intensity at point P?

(c) If they are driven coherently but out of phase by 180°, what is the intensity at point P?

(d) If the speakers are incoherent, what is the intensity at point P?

9. A train of length l is moving with a constant speed v along a circular track of radius R. The engine of the train emits a sound of frequency f. Find the frequency heard by a guard at the rear end of the train.

10. A 3 m long organ pipe open at both ends is driven to third harmonic standing wave. If the amplitude of pressure oscillations is 1 per cent of mean atmospheric pressure ($p_0 = 10^5$ Nm^2). Find the amplitude of particle displacement and density oscillations. Speed of sound $v = 332$ m / s and density of air $\rho = 1.03$ kg / m^3.

11. A siren creates a sound level of 60 dB at a location 500 m from the speaker. The siren is powered by a battery that delivers a total energy of 1.0 kJ. Assuming that the efficiency of siren is 30%, determine the total time the siren can sound.

12. A cylinder of length 1 m is divided by a thin perfectly flexible diaphragm in the middle. It is closed by similar flexible diaphragms at the ends. The two chambers into which it is divided contain hydrogen and oxygen. The two diaphragms are set in vibrations of same frequency. What is the minimum frequency of these diaphragms for which the middle diaphragm will be motionless? Velocity of sound in hydrogen is 1100 m/ s and that in oxygen is 300 m/ s.

13. A conveyor belt moves to the right with speed $v = 300$ m/ min. A very fast pieman puts pies on the belt at a rate of 20 per minute and they are received at the other end by a pieeater.

(a) If the pieman is stationary find the spacing x between the pies and the frequency with which they are received by the stationary pieeater.

(b) The pieman now walks with speed 30 m/min towards the receiver while continuing to put pies on the belt at 20 per minute. Find the spacing of the pies and the frequency with which they are received by the stationary pieeater.

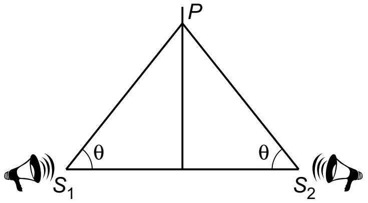
14. A point sound source is situated in a medium of bulk modulus 1.6×10^5 N/ m^2. An observer standing at a distance 10 m from the source writes down the equation for the wave as $y = A \sin (15\pi x - 6000\, \pi t)$. Here y and x are in metres and t is in second. The maximum pressure amplitude received to the observer's ear is (24π) Pa, then find.

(a) the density of the medium,

(b) the displacement amplitude A of the waves received by the observer and

(c) the power of the sound source.

15. Two sources of sound S_1 and S_2 vibrate at the same frequency and are in phase. The intensity of sound detected at a point P (as shown in figure) is I_0.



(a) If $\theta = 45°$ what will be the intensity of sound detected at this point if one of the sources is switched off ?

(b) What will be intensity of sound detected at P if $\theta = 60°$ and both the sources are now switched on?

16. Two narrow cylindrical pipes A and B have the same length. Pipe A is open at both ends and is filled with a monoatomic gas of molar mass M_A. Pipe B is open at one end and closed at the other end and is filled with a diatomic gas of molar mass M_B. Both gases are at the same temperature. **(JEE 2002)**

(a) If the frequency of the second harmonic of the fundamental mode in pipe A is equal to the frequency of the third harmonic of the fundamental mode in pipe B, determine the value of $\frac{M_A}{M_B}$.

(b) Now, the open end of pipe B is also closed (so that the pipe is closed at both ends.) Find the ratio of the fundamental frequency in pipe A to that in pipe B.

17. A boat is travelling in a river with a speed 10 m/s along the stream flowing with a speed 2 m/s. From this boat a sound transmitter is lowered into the river through a rigid support. The wavelength of the sound emitted from the transmitter inside the water is 14.45 mm. Assume that attenuation of sound in water and air is negligible.

(a) What will be the frequency detected by a receiver kept inside the river downstream?

(b) The transmitter and the receiver are now pulled up into the air. The air is blowing with a speed 5 m/s in the direction opposite to the river stream. Determine the frequency of the sound detected by the receiver.

(Temperature of the air and water $= 20°$ C; Density of river water $= 10^3$ kg/m^3; bulk modulus the water $= 2.088 \times 10^9$ Pa; Gas constant $R = 8.31$ J/mol-K; Mean molecular mass of air $= 28.8 \times 10^{-3}$ kg per mol and C_p/C_V for air $= 1.4$)

18. A string 25 cm long and having a mass of 2.5 g is under tension. A pipe closed at one end is 40 cm long. When the string is set vibrating in its first overtone and the air in the pipe in its fundamental frequency, 8 beats per second are heard. It is observed that decreasing the tension in the string decrease the beat frequency. If the speed of sound in air is 320 m/s, find the tension in the string.

19. A source emits sound waves of frequency 1000 Hz. The source moves to the right with a speed of 32 m/s relative to ground. On the right a reflecting surface moves towards left with a speed of 64 m/s relative to the ground. The speed of sound in air is 332 m/s. Find

(a) the wavelength of sound in ahead of the source,

(b) the number of waves arriving per second which meets the reflecting surface,

(c) the speed of reflected waves and

(d) the wavelength of reflected waves.

Answers

Introductory Exercise 19.1

1. 1.4×10^5 N/m^2 **2.** 7.25 cm, 72.5 m **3.** (a) Zero (b) 3.63×10^{-6} m

4. 1.04×10^{-5} m

Introductory Exercise 19.2

1. 819°C **2.** 19.6 m/s **3.** 3.6×10^9 Pa **4.** 315 m/s

Introductory Exercise 19.3

1. (a) 4.67 Pa (b) 2.64×10^{-2} W/m^2 (c) 104 dB **2.** 7.9 **3.** 20 dB

4. Faintest (a) 4.49×10^{-13} W/m^2, –3.48 dB (b) 1.43×10^{-11} m

Loudest (a) 0.881 W/m^2, +119 dB (b) 2.01×10^{-5} m

Introductory Exercise 19.4

1. 1375 Hz **2.** (a) 11.7 cm (b) 180°

Introductory Exercise 19.5

1. (a) **2.** (c) **3.** (a) **4.** (a) 0.392 m (b) 0.470 m

5. (a) Fundamental 0.8 m, first overtone 0.267 m, 0.8 m, second overtone 0.16 m, 0.48 m, 0.8 m
(b) Fundamental 0, first overtone 0, 0.533 m. Second overtone 0, 0.32 m, 0.64 m

6. (a) closed (b) 5, 7 (c) 1.075 m

Introductory Exercise 19.6

1. 252 Hz **2.** 387 Hz

Introductory Exercise 19.7

1. (d) **2.** (d) **3.** (b) **4.** (a)

Exercises

LEVEL 1

Assertion and Reason

1. (c) **2.** (d) **3.** (d) **4.** (a) **5.** (d) **6.** (d) **7.** (b) **8.** (b) **9.** (a) **10.** (d)

Objective Questions

1.(d) **2.**(d) **3.**(a) **4.**(c) **5.**(d) **6.**(a) **7.**(a) **8.**(a) **9.**(a) **10.**(c)
11.(b) **12.**(a) **13.**(c) **14.**(b) **15.**(b) **16.**(b) **17.**(a) **18.**(a) **19.**(b) **20.**(d)
21.(c) **22.**(d) **23.**(d) **24.**(c) **25.**(b) **26.**(a) **27.**(b) **28.**(d) **29.**(c) **30.**(c)
31.(a) **32.**(c) **33.**(d) **34.**(a)

Subjective Questions

1. 1414 m/s, 5.84 m **2.** No **3.** 1321 m/s **4.** Two times **5.** 274 Hz

6. 664 m, 4 s **7.** 972 m/s **8.** (a) 1.33×10^{10} Pa (b) 9.47×10^{10} Pa **9.** $Y/900$

10. 591 m/s **11.** 201 W **12.** (a) 10^{-6} W/m^2 (b) 1.2×10^{-8} W **13.** (a) 20
14. (a) 9.95×10^{-4} W/m^2 (b) 1.15×10^{-6} m **15.** 13.6 nm **16.** 80 W **17.** 134.4 dB
18. (a) 9.44×10^{-11} m, 0.43 m (b) 5.66×10^{-9} m, 0.1 m (c) air, $\dfrac{A_{\text{air}}}{A_{\text{water}}} = 60$
19. 4.2×10^{-12} W/m^2, 6.23 dB **20.** (a) 382.2Hz, 764.4 Hz, 1146.6 Hz (b) 191.1 Hz, 573.3 Hz, 955.5 Hz
21. 250 Hz **22.** $y = A\cos(3.93x)\sin(1297t)$ **23.** 348.16 m/s, 118 cm **24.** 11.56 N
25. 336 m/s **26.** (a) 0.392 m (b) 0.470 m **27.** 0.40 **28.** (a) 1.13 m (b) 1.03 m, 1.23 m
29. (a) 441.5 Hz, 438.5 Hz (b) (i) + 0.68% (ii) –0.68% **30.** (a) 0.245 m/s (b) 0.904 m
31. (a) 302 Hz (b) 228 Hz **32.** 680 Hz **33.** 160 Hz **34.** 0.5 m/s
35. 1440 Hz **36.** $\Delta\phi_1 = \pi$, $\Delta\phi_2 = 4\pi$, Resultant intensity = 8.2×10^{-5} Wm^{-2} **37.** 7.5 m
38. Diatomic **39.** 352 m/s **40.** 1.02 **41.** 0.4 cm
43. (a) 1.3 m (b) 262 Hz **44.** (a) 0.628 m (b) 0.748 m (c) 548 Hz (d) 460 Hz
45. (a) closed (b) 85 Hz, 1 m **46.** $f_1 = 264$ Hz and $f_2 = 256$ Hz

LEVEL 2

Single Correct Option

1.(b) **2.**(a) **3.**(d) **4.**(b) **5.**(a) **6.**(b) **7.**(c) **8.**(c) **9.**(a) **10.**(b)
11.(a) **12.**(b) **13.**(b) **14.**(c) **15.**(c)

More than One Correct Options

1. (a,c) **2.** (c,d) **3.** (a,b,c,d) **4.** (b,c,d) **5.** (a,b,c,d) **6.** (c,d)

Comprehension Based Questions

1. (a) **2.** (c) **3.** (a)

Match the Columns

1. (a) → s (b) → p (c) → r (d) → r
2. (a) → q (b) → p (c) → s (d) → s
3. (a) → r,s (b) → p (c) → r, s (d) → p
4. (a) → r (b) → q (c) → q (d) → p
5. (a) → s (b) → r (c) → p,r (d) → q

Subjective Questions

1. 2 μW, 0.173 J **2.** (a) 4.98 N/m^2 (b) 3.0×10^{-6} m **3.** (a) 283.33 Hz (b) 51.5 cm
4. $\lambda = 2\sqrt{4(H+h)^2 + d^2} - 2\sqrt{4H^2 + d^2}$
5. (a) 8.14° (b) 16.5° (c) Three maxima beyond the maximum corresponding to $\theta = 0°$
6. (a) 0 (b) $2I_0$ (c) $4I_0$ **7.** (a) 0 (b) 66 dB (c) 63 dB
8. (a) $I_1 = 19.9\,\mu$W/m^2, $I_2 = 8.84\,\mu$W/m^2 (b) 55.3 μW/m^2 (c) 2.2 μW/m^2 (d) 28.7 μW/m^2
9. f **10.** 0.28 cm, 9.0×10^{-3} kg/m^3 **11.** 95.5 s **12.** 1650 Hz
13. (a) 15 m, 20 min^{-1} (b) 13.5 m, 22.22 min^{-1} **14.** (a) 1 kg/m^3 (b) 10 μm (c) 288 π^3 W
15. (a) $\dfrac{I_0}{4}$ (b) I_0 **16.** (a) $\dfrac{400}{189}$ (b) $\dfrac{3}{4}$ **17.** (a) 1.0069×10^5 Hz (b) 1.0304×10^5 Hz
18. 27.04 N **19.** (a) 0.3 m (b) 1320 (c) 332 m/s (d) 0.2 m

20

Thermometry, Thermal Expansion and Kinetic Theory of Gases

Chapter Contents

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20.2 The Constant Volume Gas Thermometer and The Absolute Temperature Scale
20.3 Heat and Temperature
20.4 Thermal Expansion
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20.9 Molar Heat Capacity
20.10 Kinetic Theory of Gases

20.1 Thermometers and The Celsius Temperature Scale

Thermometers are devices that are used to measure temperatures. All thermometers are based on the principle that some physical properties of a system change as the system's temperature changes. Some physical properties that change with temperature are

1. the volume of a liquid
2. the length of a solid
3. the pressure of a gas at constant volume
4. the volume of a gas at constant pressure and
5. the electric resistance of a conductor.

A common thermometer in everyday use consists of a mass of liquid, usually mercury or alcohol that expands in a glass capillary tube when heated. In this case, the physical property is the **change in volume of the liquid**. Any temperature change is proportional to the change in length of the liquid column. The thermometer can be calibrated accordingly. On the celsius temperature scale, a thermometer is usually calibrated between 0°C (called the ice point of water) and 100°C (called the steam point of water). Once the liquid levels in the thermometer have been established at these two points, the distance between the two points is divided into 100 equal segments to create the celsius scale.

Thus, each segment denotes a change in temperature of one celsius degree (1°C). A practical problem in this type of thermometer is that readings may vary for two different liquids. When one thermometer reads a temperature, for example 40°C the other may indicate a slightly different value. This discrepancies between thermometers are especially large at temperatures far from the calibration points. To surmount this problem we need a universal thermometer whose readings are independent of the substance used in it. The gas thermometer used in the next article meets this requirement.

20.2 The Constant Volume Gas Thermometer and The Absolute Temperature Scale

The physical property used by the constant volume gas thermometer is the **change in pressure of a gas at constant volume**.

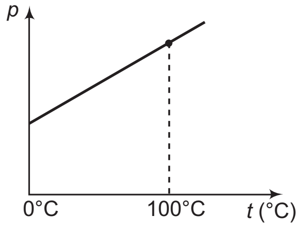


Fig. 20.1

The pressure *versus* temperature graph for a typical gas taken with a constant volume is shown in figure. The two dots represent the two reference temperatures namely, the ice and steam points of water. The line connecting them serves as a calibration curve for unknown temperatures. Experiments show that the thermometer readings are nearly independent of the type of gas used as long as the gas pressure is low and the temperature is well above the point at which the gas liquefies.

If you extend the curves shown in figure toward negative temperatures, you find, in every case, that the pressure is zero when the temperature is –273.15° C. This significant temperature is used as the basis for the **absolute temperature scale**, which sets –273.15° C as its zero point.

This temperature is often referred to as **absolute zero**. The size of a degree on the absolute temperature scale is identical to the size of a degree on the celsius scale. Thus, the conversion between these temperatures is

$$\boxed{T_C = T - 273.15} \qquad \dots(i)$$

In 1954, by the International committee on weights and measures, the **triple point of water** was chosen as the reference temperature for this new scale. The triple point of water is the single combination of temperature and pressure at which liquid water, gaseous water and ice (solid water) coexist in equilibrium. This triple point occurs at a temperature of approximately 0.01° C and a pressure of 4.58 mm of mercury. On the new scale, which uses the unit **kelvin**, the temperature of water at the triple point was set at 273.16 kelvin, abbreviated as 273.16 K. (No degree sign is used with the unit kelvin).

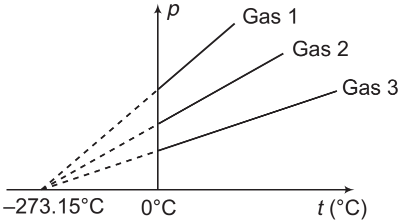


Fig. 20.2

This new absolute temperature scale (also called the kelvin scale) employs the SI unit of absolute temperature, the kelvin which is defined to be "$\frac{1}{273.16}$ of the difference between absolute zero and the temperature of the triple point of water".

The Celsius, Fahrenheit and Kelvin Temperature Scales

Eq. (i) shows the relation between the temperatures in celsius scales and kelvin scale. Because the size of a degree is the same on the two scales, a temperature difference of 10°C is equal to a temperature difference of 10 K. The two scales differ only in the choice of the zero point. The ice point temperature on the kelvin scale, 273.15 K, corresponds to 0.00°C and the kelvin steam point 373.15 K, is equivalent to 100.00°C.

A common temperature scale in everyday use in US is the **Fahrenheit scale**. The ice point in this scale is 32°F and the steam point is 212°F. The distance between these two points are divided in 180 equal parts. The relation between celsius scale and Fahrenheit scale is as derived below.

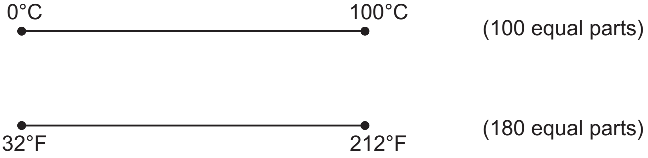
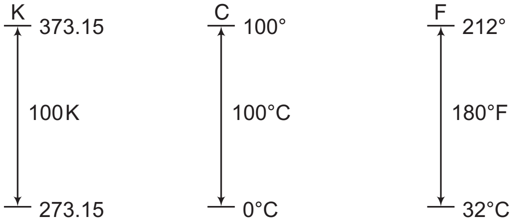


Fig. 20.3

100 parts of celsius scale = 180 parts of Fahrenheit scale

$\therefore$ 1 part of celsius scale $= \frac{9}{5}$ parts of Fahrenheit scale



Relation among Kelvin, Celsius and Fahrenheit temperature scales

Fig. 20.4

Hence,
$$T_F = 32 + \frac{9}{5} T_C \qquad \ldots\text{(ii)}$$

Further,
$$\Delta T_C = \Delta T = \frac{5}{9} \Delta T_F \qquad \ldots\text{(iii)}$$

Extra Points to Remember

- **Different Thermometers**

  Thermometric property It is the property that can be used to measure the temperature. It is represented by any physical quantity such as length, volume, pressure and resistance etc., which varies linearly with a certain range of temperature. Let X denotes the thermometric physical quantity and X_0, X_{100} and X_t be its values at 0°C, 100°C and t °C respectively. Then,

  $$\boxed{t = \left(\frac{X_t - X_0}{X_{100} - X_0}\right) \times 100°\text{C}}$$

  (i) Constant volume gas thermometer The pressure of a gas at constant volume is the thermometric property. Therefore,

  $$\boxed{t = \left(\frac{p_t - p_0}{p_{100} - p_0}\right) \times 100°\text{C}}$$

  (ii) Platinum resistance thermometer The resistance of a platinum wire is the thermometric property. Hence,

  $$\boxed{t = \left(\frac{R_t - R_0}{R_{100} - R_0}\right) \times 100°\text{C}}$$

  (iii) Mercury thermometer In this thermometer, the length of a mercury column from some fixed point is taken as thermometric property. Thus,

  $$\boxed{t = \left(\frac{l_t - l_0}{l_{100} - l_0}\right) \times 100°\text{C}}$$

- Two other thermometers, commonly used are **thermocouple thermometer** and **total radiation pyrometer.**

- **Total radiation pyrometer** is used to measure very high temperatures. When a body is at a high temperature, it glows brightly and the radiation emitted per second from unit area of the surface of the body is proportional to the fourth power of the absolute temperature of the body. If this radiation is measured by some device, the temperature of the body is calculated. This is the principle of a total radiation pyrometer. The main advantage of this thermometer is that the experimental body is not kept in contact with it. Hence, there is no definite higher limit of its temperature range. It can measure temperature from 800°C to 3000°C–4000°C. However, it cannot be used to measure temperatures below 800°C because at low temperatures the emission of radiation is so poor that it cannot be measured directly.
- **Ranges of different thermometers**

Table 20.1

Thermometer	Lower limit	Upper limit
Mercury thermometer	–30°C	300°C
Gas thermometer	–268°C	1500°C
Platinum resistance thermometer	–200°C	1200°C
Thermocouple thermometer	–200°C	1600°C
Radiation thermometer	800°C	No limit

- **Reaumur's scale** Other than Celsius, Fahrenheit and Kelvin temperature scales **Reaumur's scale** was designed by Reaumur in 1730. The lower fixed point is 0°R representing melting point of ice. The upper fixed point is 80°R, which represents boiling point of water. The distance between the two fixed points is divided into 80 equal parts. Each part represents 1°R. If T_C, T_F and T_R are temperature values of a body on Celsius scale, Fahrenheit scale and Reaumur scale respectively, then

$$\boxed{\frac{T_C - 0}{100} = \frac{T_F - 32}{180} = \frac{T_R - 0}{80}}$$

- A substance is found to exist in three states solid, liquid and gas. For each substance, there is a set of temperature and pressure at which all the three states may coexist. This is called **triple point** of that substance. For water, the values of pressure and temperature corresponding to triple point are 4.58 mm of Hg and 273.16°K.

Example 20.1 *Express a temperature of 60°F in degrees Celsius and in Kelvin.*

Solution Substituting $T_F = 60°$ F in Eq. (ii),

$$T_C = \frac{5}{9}(T_F - 32) = \frac{5}{9}(60° - 32°)$$

$$= 15.55° \text{ C}$$ **Ans.**

From Eq. (i),

$$T = T_C + 273.15 = 15.55° \text{ C} + 273.15$$

$$= 288.7 \text{ K}$$ **Ans.**

Example 20.2 *The temperature of an iron piece is heated from 30°C to 90°C. What is the change in its temperature on the Fahrenheit scale and on the kelvin scale?*

Solution $\Delta T_C = 90°\text{ C} - 30°\text{ C} = 60°\text{ C}$

Using Eq. (iii),

$$\Delta T_F = \frac{9}{5}\Delta T_C = \frac{9}{5}(60°\text{ C})$$

$$= 108°\text{ F}$$ **Ans.**

and $$\Delta T = \Delta T_C = 60\text{ K}$$ **Ans.**

INTRODUCTORY EXERCISE 20.1

1. What is the value of
   (a) 0°F in Celsius scale?
   (b) 0 K on Fahrenheit scale?
2. At what temperature is the Fahrenheit scale reading equal to
   (a) twice (b) half of Celsius?
3. A faulty thermometer reads 5°C in melting ice and 99°C in steam. Find the correct temperature in °F when this faulty thermometer reads 52°C.
4. At what temperature the Fahrenheit and Kelvin scales of temperature give the same reading?
5. At what temperature the Fahrenheit and Celsius scales of temperature give the same reading?

20.3 Heat and Temperature

The word heat is always used during transfer of thermal energy from hot body to cold body (due to temperature difference between them). Following are given some statements. Some of them are right and some are wrong. From those statements, you will be able to find the difference between heat and temperature.

Wrong Statements

(i) This body has large quantity of heat.

(ii) Temperature transfer is taking place from body *A* to body *B*.

Correct Statements

(i) Temperature of body *A* is more than temperature of body *B*.

(ii) Heat transfer is taking place from body *A* to body *B* because *A* is at higher temperature than body *B*.

Note *One calorie (1 cal) is defined as the amount of heat required to raise the temperature of one gram of water from 14.5°C to 15.5°C.*

Experiments have shown that

$$1\ cal = 4.186\ J$$

Similarly, $$1\ kcal = 1000\ cal = 4186\ J$$

The calorie is not a fundamental SI unit.

20.4 Thermal Expansion

Most substances expand when they are heated. Thermal expansion is a consequence of the change in average separation between the constituent atoms of an object. Atoms of an object can be imagined to be connected to one another by stiff springs as shown in Fig. 20.5. At ordinary temperatures, the atoms in a solid oscillate about their equilibrium positions with an amplitude of approximately 10^{-11} m. The average spacing between the atoms is about 10^{-10} m. As the temperature of solid increases, the atoms oscillate with greater amplitudes, as a result the average separation between them increases, consequently the object expands.

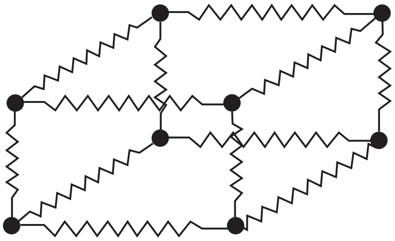


Fig. 20.5

More precisely, thermal expansion arises from the asymmetrical nature of the potential energy curve.

At the atomic level, thermal expansion may be understood by considering how the potential energy of the atoms varies with distance. The equilibrium position of an atom will be at the minimum of the potential energy well if the well is symmetric. At a given temperature, each atom vibrates about its equilibrium position and its average position remains at the minimum point. If the shape of the well is not symmetrical, as shown in figure, the average position of an atom will not be at the minimum point. When the temperature is raised the amplitude of the vibrations increases and the average position is located at a greater interatomic separation. This increased separation is manifested as expansion of the material.

Linear Expansion

Suppose that the temperature of a thin rod of length l is changed from T to $T + \Delta T$. It is found experimentally that, if ΔT is not too large, the corresponding change in length Δl of the rod is directly proportional to ΔT and l. Thus,

$$\Delta l \propto \Delta T$$

and

$$\Delta l \propto l$$

Introducing a proportionality constant α (which is different for different materials) we may write Δl as

$$\Delta l = l\alpha\Delta T \qquad \text{...(i)}$$

Here, the constant α is called the **coefficient of linear expansion** of the material of the rod and its units are K^{-1} or $[(°C)^{-1}]$. Remember that $\Delta T = \Delta T_C$.

Actually, α does depend slightly on the temperature, but its variation is usually small enough to be negligible, even over a temperature range of 100°C. We will always assume that α is a constant.

Volume Expansion

Because the linear dimensions of an object change with temperature, it follows that surface area and volume change as well. Just as with linear expansion, experiments show that if the temperature change ΔT is not too great (less than 100°C or so), the increase in volume ΔV is proportional to both, the temperature change ΔT and the initial volume V. Thus,

$$\Delta V \propto \Delta T$$

and
$$\Delta V \propto V$$

Introducing a proportionality constant γ, we may write ΔV as

$$\Delta V = V \times \gamma \times \Delta T \qquad \text{... (ii)}$$

Here, γ is called the **coefficient of volume expansion**. The units of γ are K^{-1} or $(°C)^{-1}$.

Relation Between γ and α

For an isotropic solid (which has the same value of α in all directions) $\gamma = 3\alpha$. To see that $\gamma = 3\alpha$ for a solid, consider a cube of length l and volume $V = l^3$.

When the temperature of the cube is increased by dT, the side length increases by dl and the volume increases by an amount dV given by

$$dV = \left(\frac{dV}{dl}\right) \cdot dl = 3l^2 \cdot dl$$

Now,
$$dl = l\alpha dT$$

$\therefore$
$$dV = 3l^3 \alpha dT = (3\alpha) V dT$$

This is consistent with Eq. (ii),

$$dV = \gamma V dT, \text{ only if}$$

$$\gamma = 3\alpha \qquad \text{...(iii)}$$

Average values of α and γ for some materials are listed in Table 20.2. You can check the relation $\gamma = 3\alpha$, for the materials given in the table.

Table 20.2

Material	α [K^{-1} or $(°C)^{-1}$]	γ [K^{-1} or $(°C)^{-1}$]
Steel	1.2×10^{-5}	3.6×10^{-5}
Copper	1.7×10^{-5}	5.1×10^{-5}
Brass	2.0×10^{-5}	6.0×10^{-5}
Aluminium	2.4×10^{-5}	7.2×10^{-5}

The Anomalous Expansion of Water

Most liquids also expand when their temperatures increase. Their expansion can also be described by Eq. (ii). The volume expansion coefficients for liquids are about 100 times larger than those for solids.

Some substances contract when heated over a certain temperature range. The most common example is water.

Figure shows how the volume of 1 g of water varies with temperature at atmospheric pressure. The volume decreases as the temperature is raised from 0°C to about 4°C, at 4°C the volume is a minimum and the density is a maximum (1000 kg/m^3). Above 4° C, water expands with increasing temperature like most substances.

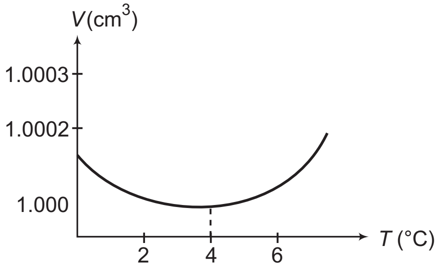


Fig. 20.6

This anomalous behaviour of water causes ice to form first at the surface of a lake in cold weather. As winter approaches, the water temperature decreases initially at the surface. The water there sinks because of its increased density. Consequently, the surface reaches 0°C first and the lake becomes covered with ice. Now ice is bad conductor of heat so water at the bottom remains at 4°C, the highest density of water. Aquatic life is able to survive the cold winter as the bottom of the lake remains unfrozen at a temperature of about 4°C.

Extra Points to Remember

- If a solid object has a hole in it, what happens to the size of the hole, when the temperature of the object increases? A common misconception is that if the object expands, the hole will shrink because material expands into the hole. But the truth is that if the object expands, the hole will expand too, because every linear dimension of an object changes in the same way when the temperature changes.

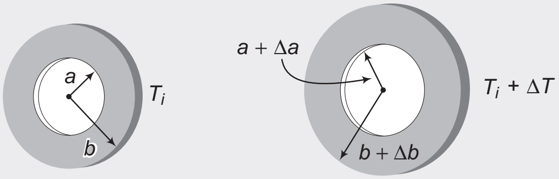
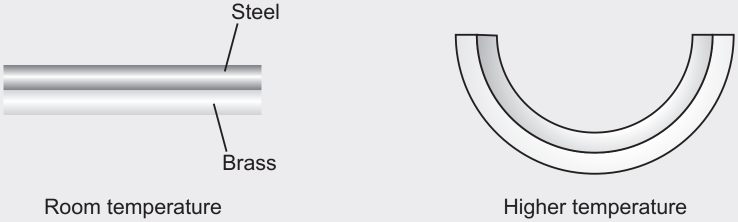


Fig. 20.7

- **Expansion of a bimetallic strip** As Table 20.2 indicates, each substance has its own characteristic average coefficient of expansion.



Room temperature

Higher temperature

Fig. 20.8

For example, when the temperatures of a brass rod and a steel rod of equal length are raised by the same amount from some common initial value, the brass rod expands more than the steel rod because brass has a greater average coefficient of expansion than steel. Such type of bimetallic strip is found in practical devices such as **thermostats** to break or make electrical contact.

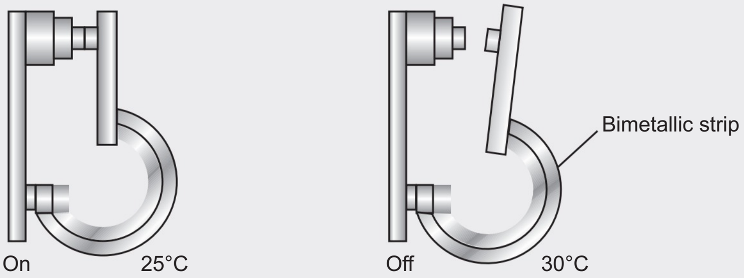


Fig. 20.9

- **Variation of density with temperature** Most substances expand when they are heated, i.e. volume of a given mass of a substance increases on heating, so the density should decrease $\left(\text{as } \rho \propto \frac{1}{V}\right)$. Let us see how the density (ρ) varies with increase in temperature.

$$\rho = \frac{m}{V}$$

or $$\rho \propto \frac{1}{V} \qquad \text{(for a given mass)}$$

$\therefore$ $$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + \Delta V} = \frac{V}{V + \gamma V \Delta T} = \frac{1}{1 + \gamma \Delta T}$$

$\therefore$ $$\boxed{\rho' = \frac{\rho}{1 + \gamma \Delta T}}$$

This expression can also be written as

$$\rho' = \rho (1 + \gamma \Delta T)^{-1}$$

If γ is very small, $$(1 + \gamma \Delta T)^{-1} \approx 1 - \gamma \Delta T$$

$\therefore$ $$\boxed{\rho' \approx \rho (1 - \gamma \Delta T)}$$

- **Effect of temperature on apparent weight when immersed in a liquid** When a solid body is completely immersed in a liquid, its apparent weight gets decreased due to an upthrust acting on it by the liquid. The apparent weight is given by

$$w_{\text{app}} = w - F$$

Here, F = upthrust = $V_S \rho_L g$

where, V_S = volume of solid and ρ_L = density of liquid

Now, as the temperature is increased V_S increases while ρ_L decreases. So, F may increase or decrease (or may remain constant also) depending upon the condition that which factor dominates on the other. We can write

$$F \propto V_S \rho_L$$

or $$\frac{F'}{F} = \frac{V_S'}{V_S} \cdot \frac{\rho_L'}{\rho_L} = \frac{(V_S + \Delta V_S)}{V_S} \cdot \left(\frac{1}{1 + \gamma_L \Delta T}\right)$$

$$= \left(\frac{V_S + \gamma_S V_S \Delta T}{V_S}\right)\left(\frac{1}{1 + \gamma_L \Delta T}\right)$$

or $$F' = F\left(\frac{1+\gamma_S \Delta T}{1+\gamma_L \Delta T}\right)$$

Now, if $\gamma_S > \gamma_L, F' > F$

or $w'_{\text{app}} < w_{\text{app}}$ and *vice-versa*.

And if $\gamma_S = \gamma_L, \quad F' = F$

or $w'_{\text{app}} = w_{\text{app}}$

- **Effect of temperature on immersed fraction of a solid in floating condition**
  When a solid, whose density is less than the density of liquid is floating in it, then

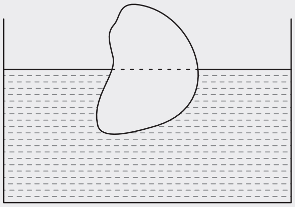


Fig. 20.10

Weight of the solid = Upthrust on solid from liquid

$$\therefore \quad V\rho_s g = V_i \rho_l g \qquad \text{...(i)}$$

Here, ρ_s = density of solid

ρ_l = density of liquid

V_i = immersed volume of solid and

V = total volume of solid

From Eq. (i),

$$\frac{V_i}{V} = \frac{\rho_s}{\rho_l} = f \qquad \text{...(ii)}$$

where, f = immersed fraction of solid.

With increase in temperature, ρ_s and ρ_l both will decrease. Therefore, this fraction may increase, decrease or remain constant. At some higher temperature,

$$f' = \frac{\rho_s'}{\rho_l'} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we have

$$\frac{f'}{f} = \left(\frac{\rho_s'}{\rho_s}\right)\left(\frac{\rho_l}{\rho_l'}\right)$$

$$= \left(\frac{1}{1+\gamma_s \Delta\theta}\right)(1+\gamma_l \Delta\theta)$$

or $$\frac{f'}{f} = \left(\frac{1+\gamma_l \Delta\theta}{1+\gamma_s \Delta\theta}\right)$$

Now, if $\gamma_l > \gamma_s$, $f' > f$ or immersed fraction will increase.

If $\gamma_l = \gamma_s$, $f' = f$ or immersed fraction will remain unchanged and if,

$\gamma_l < \gamma_s$, then $f' < f$ or immersed fraction will decrease.

- **Effect of temperature on the time period of a pendulum** The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

or $$T \propto \sqrt{l}$$

As the temperature is increased, length of the pendulum and hence, time period gets increased or a pendulum clock becomes slow and it loses the time.

$$\frac{T'}{T} = \sqrt{\frac{l'}{l}} = \sqrt{\frac{l + \Delta l}{l}}$$

Here, we put $\Delta l = l\alpha\Delta\theta$ in place of $l\alpha\Delta T$ so as to avoid the confusion with change in time period. Thus,

$$\frac{T'}{T} = \sqrt{\frac{l + l\alpha\Delta\theta}{l}} = (1 + \alpha\Delta\theta)^{1/2}$$

or $$T' \approx T\left(1 + \frac{1}{2}\alpha\Delta\theta\right) \qquad (\text{if } \alpha\Delta\theta << 1)$$

or $$\boxed{\Delta T = T' - T = \frac{1}{2}T\alpha\Delta\theta}$$

Time lost in time t (by a pendulum clock whose actual time period is T and the changed time period at some higher temperature is T') is

$$\Delta t = \left(\frac{\Delta T}{T'}\right)t$$

- At some higher temperature a scale will expand and scale reading will be lesser than true value. However, at lower temperatures scale reading will be more or true value will be less.
- When a rod whose ends are rigidly fixed such as to prevent from expansion or contraction undergoes a change in temperature, thermal stresses are developed in the rod. This is because, if the temperature is increased, the rod has a tendency to expand but since it is fixed at two ends it is not allowed to expand. So, the rod exerts a force on supports to expand.

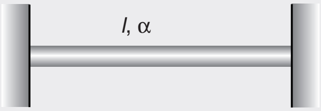


Fig. 20.11

Thermal strain $= \frac{\Delta l}{l} = \alpha\Delta T$

So thermal stress = (Y) (termal strain) = $Y\alpha\Delta T$

or force on supports $F = A$ (stress) = $YA\alpha\Delta T$

Here, Y = Young's modulus of elasticity of the rod.

$$\boxed{F = YA\alpha\Delta T}$$

- **Expansion of liquids** For heating a liquid it has to be put in some container. When the liquid is heated, the container will also expand. We define coefficient of apparent expansion of a liquid as the apparent increase in volume per unit original volume per °C rise in temperature. It is represented by γ_a. Thus,

$$\gamma_a = \gamma_r - \gamma_g$$

Here, γ_r = coefficient of volume expansion of liquid

and γ_g = coefficient of volume expansion of the container

Example 20.3 *A steel ruler exactly* 20 *cm long is graduated to give correct measurements at* 20°C.

(a) *Will it give readings that are too long or too short at lower temperatures?*

(b) *What will be the actual length of the ruler when it is used in the desert at a temperature of* 40° C ? $\alpha_{steel} = 1.2 \times 10^{-5}\ (°C)^{-1}$.

Solution (a) If the temperature decreases, the length of the ruler also decreases through thermal contraction. Below 20°C, each centimetre division is actually somewhat shorter than 1.0 cm, so the steel ruler gives readings that are too long.

(b) At 40°C, the increase in length of the ruler is

$$\Delta l = l\alpha\Delta T$$

$$= (20)(1.2 \times 10^{-5})(40° - 20°)$$

$$= 0.48 \times 10^{-2} \text{ cm}$$

∴ The actual length of the ruler is

$$l' = l + \Delta l$$

$$= 20.0048 \text{ cm}$$ **Ans.**

Example 20.4 *The scale on a steel meter stick is calibrated at* 15°C. *What is the error in the reading of* 60 *cm at* 27°C? $\alpha_{steel} = 1.2 \times 10^{-5}\ (°C)^{-1}$.

Solution The error in the reading will be

$$\Delta l = (\text{scale reading})(\alpha)(\Delta T)$$

$$= (60)(1.2 \times 10^{-5})(27° - 15°)$$

$$= 0.00864 \text{ cm}$$ **Ans.**

Example 20.5 *A second's pendulum clock has a steel wire. The clock is calibrated at* 20°C. *How much time does the clock lose or gain in one week when the temperature is increased to* 30°C? $\alpha_{steel} = 1.2 \times 10^{-5} (°C)^{-1}$.

Solution The time period of second's pendulum is 2 second. As the temperature increases length and hence, time period increases. Clock becomes slow and it loses the time. The change in time period is

$$\Delta T = \frac{1}{2} T\alpha\Delta\theta$$

$$= \left(\frac{1}{2}\right)(2)(1.2 \times 10^{-5})(30° - 20°)$$

$$= 1.2 \times 10^{-4} \text{ s}$$

∴ New time period is

$$T' = T + \Delta T = (2 + 1.2 \times 10^{-4})$$

$$= 2.00012 \text{ s}$$

∴ Time lost in one week

$$\Delta t = \left(\frac{\Delta T}{T'}\right) t$$

$$= \frac{(1.2 \times 10^{-4})}{(2.00012)} (7 \times 24 \times 3600)$$

$$= 36.28 \, \text{s}$$ **Ans.**

Example 20.6 *A sphere of diameter 7 cm and mass 266.5 g floats in a bath of liquid. As the temperature is raised, the sphere just sinks at a temperature of 35° C. If the density of the liquid at 0° C is 1.527 g/cm³, find the coefficient of cubical expansion of the liquid. Ignore expansion of sphere.*

Solution The sphere will sink in the liquid at 35°C, when its density becomes equal to the density of liquid at 35° C.

The density of sphere,

$$\rho_{35} = \frac{266.5}{\frac{4}{3} \times \left(\frac{22}{7}\right) \times \left(\frac{7}{2}\right)^3}$$

$$= 1.483 \, \text{g/cm}^3$$

Now,

$$\rho_0 = \rho_{35} \, [1 + \gamma \Delta T)$$

$$1.527 = 1.483 \, [1 + \gamma \times 35]$$

$$1.029 = 1 + \gamma \times 35$$

$$\gamma = \frac{1.029 - 1}{35}$$

$$= 0.00083/^\circ \text{C}$$ **Ans.**

Example 20.7 *A glass beaker holds exactly 1 L at 0°C*

(a) What is its volume at 50°C?

(b) If the beaker is filled with mercury at 0°C, what volume of mercury overflows when the temperature is 50°C ? $\alpha_g = 8.3 \times 10^{-6}$ *per ° C and* $\gamma_{Hg} = 1.82 \times 10^{-4}$ *per ° C.*

Solution (a) The volume of beaker after the temperature change is

$$V_{\text{beaker}} = V_0 \, (1 + 3\alpha_g \Delta\theta)$$

$$= (1) [1 + 3 \times 8.3 \times 10^{-6} \times 50]$$

$$= 1.001 \text{ L}$$ **Ans.**

(b) Volume of mercury at 50°C is

$$V_{\text{mercury}} = V_0 \, (1 + \gamma_{\text{Hg}} \Delta\theta)$$

$$= (1) [1 + 1.82 \times 10^{-4} \times 50]$$

$$= 1.009 \text{ L}$$

The overflow is thus $1.009 - 1.001 = 0.008$ L or 8 mL **Ans.**

INTRODUCTORY EXERCISE 20.2

Take the values of α from Table 20.2.

1. A pendulum clock of time period 2 s gives the correct time at 30°C. The pendulum is made of iron. How many seconds will it lose or gain per day when the temperature falls to 0°C? $\alpha_{Fe} = 1.2 \times 10^{-5}\,(^\circ\text{C})^{-1}$.
2. A block of wood is floating in water at 0°C. The temperature of water is slowly raised from 0°C to 10°C. How will the percentage of volume of block above water level change with rise in temperature?
3. A piece of metal floats on mercury. The coefficient of volume expansion of metal and mercury are γ_1 and γ_2, respectively. If the temperature of both mercury and metal are increased by an amount ΔT, by what factor does the fraction of the volume of the metal submerged in mercury changes?
4. A brass disc fits snugly in a hole in a steel plate. Should you heat or cool the system to losen the disc from the hole? Given that $\alpha_B > \alpha_{Fe}$.
5. An iron ball has a diameter of 6 cm and is 0.010 mm too large to pass through a hole in a brass plate when the ball and plate are at a temperature of 30°C. At what temperature, the same for ball and plate, will the ball just pass through the hole?

   Take the values of α from Table 20.2.
6. (a) An aluminium measuring rod which is correct at 5°C, measures a certain distance as 88.42 cm at 35 $^\circ$C. Determine the error in measuring the distance due to the expansion of the rod. (b) If this aluminium rod measures a length of steel as 88.42 cm at 35°C, what is the correct length of the steel at 35°C?
7. A steel tape is calibrated at 20°C . On a cold day when the temperature is -15°C, what will be the percentage error in the tape?

20.5 Behaviour of Gases

Gases are the most diffused form of matter. In a gas, the molecules are highly energetic and they can be widely separated from each other. The behaviour of gases and their properties derive from these facts.

Sometimes the term vapour is used to describe a gas. Strictly speaking, a gas is a substance at a temperature above its boiling point. A vapour is the gaseous phase of a substance that under ordinary conditions, exists as a liquid or solid.

One of the more obvious characteristics of gases is their ability to expand and fill any volume they are placed in. This contrasts with the behaviour exhibited by solids (fixed shape and volume) and liquids (fixed volume but indeterminate shape).

Ideal and Real Gas Laws

Gases, unlike solids and liquids have indefinite shape and indefinite volume. As a result, they are subjected to pressure changes, volume changes and temperature changes. Real gas behaviour is simpler. By understanding ideal gas behaviour, real gas behaviour becomes more tangible.

Ideal Gases

How do we describe an ideal gas? An ideal gas has the following properties :

1. An ideal gas is considered to be a "point mass". A point mass is a particle so small, its volume is very nearly zero. This means an ideal gas particle has virtually no volume.
2. Collisions between ideal gases are "elastic". This means that no attractive or repulsive forces are involved during collisions. Also, the kinetic energy of the gas molecules remains constant since these intermolecular forces are lacking.

   Volume and temperature are by now familiar concepts. Pressure, however, may need some explanation. Pressure is defined as a force per unit area. When gas molecules collide with the sides of a exert, they exert a force over that area of the container. This gives rise to the pressure inside the container.

Gas Laws

Assuming permanent (or real) gases to be ideal, through experiments, it was established that gases irrespective of their nature obey the following laws :

Boyle's Law

According to this law, for a given mass of a gas, the volume of a gas at constant temperature (called **isothermal process**) is inversely proportional to its pressure, i.e.

$$V \propto \frac{1}{p} \qquad (T = \text{constant})$$

or

$$pV = \text{constant}$$

or

$$p_i V_i = p_f V_f$$

Thus, p-V graph in an isothermal process is a rectangular hyperbola. Or pV *versus* p or V graph is a straight line parallel to p or V axis.

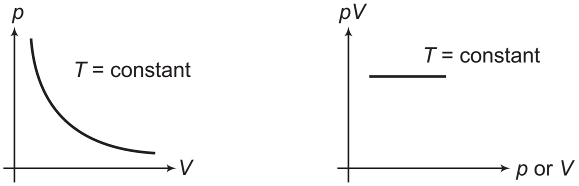


Fig. 20.12

Charles' Law

According to this law, for a given mass of a gas the volume of a gas at constant pressure (called **isobaric** process) is directly proportional to its absolute temperature, i.e.

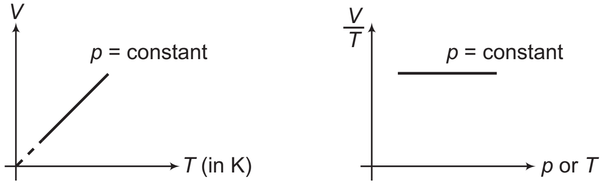


Fig. 20.13

$$V \propto T \qquad (p = \text{constant})$$

or
$$\frac{V}{T} = \text{constant} \quad \text{or} \quad \frac{V_i}{T_i} = \frac{V_f}{T_f}$$

Thus, V-T graph in an isobaric process is a straight line passing through origin. Or V/T *versus* V or T graph is a straight line parallel to V or T axis.

Gay Lussac's Law or Pressure Law

According to this law, for a given mass of a gas the pressure of a gas at constant volume (called **isochoric process**) is directly proportional to its absolute temperature, i.e.

$$p \propto T \quad (V = \text{constant}) \quad \text{or} \quad \frac{p}{T} = \text{constant} \quad \text{or} \quad \frac{p_i}{T_i} = \frac{p_f}{T_f}$$

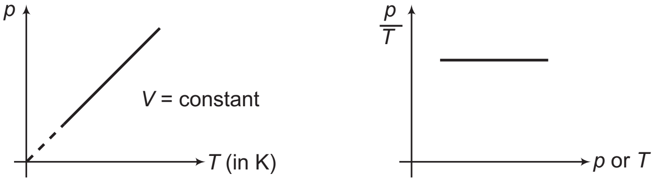


Fig. 20.14

Thus, p - T graph in an isochoric process is a straight line passing through origin or p/T *versus* p or T graph is a straight line parallel to p or T axis.

Ideal Gas Equation

All the above four laws can be written in one single equation known as ideal gas equation. According to this equation.

$$\boxed{pV = nRT = \frac{m}{M}RT}$$

In this equation, n = number of moles of the gas

$$= \frac{m}{M}$$

m = total mass of the gas

M = molecular mass of the gas

and R = universal gas constant

$= 8.31$ J/mol-K

$= 2.0$ cal/mol-K

The above three laws can be derived from this single equation. For example, for a given mass of a gas (m = constant)

pV = constant at constant temperature (Boyle's law)

$\frac{p}{T}$ = constant at constant volume (Pressure law)

$\frac{V}{T}$ = constant at constant pressure (Charles' law)

Avogadro's Law

The next empirical gas law, we will look at, is called Avogadro's law. This law deals with the relationship between the volume and moles of a gas at constant pressure and temperature. According to this law, at same temperature and pressure equal volumes of all gases contain equal number of molecules. Let's derive this law from the ideal gas law. Give the moles and volume subscripts, since their conditions will change.

$$pV_1 = n_1RT \quad \text{and} \quad pV_2 = n_2RT$$

Collect terms. Divide each equation by pressure, p and divide each equation by their respective mole term

$$\frac{n_1}{V_1} = \frac{n_2}{V_2} = \frac{p}{RT}$$

Thus, the number of molecules per unit volume is same for all gases at a fixed temperature and pressure. The number in 22.4 litres of any gas is 6.02×10^{23}. This is known as Avogadro number and is denoted by N_A. The mass of 22.4 litres of any gas is equal to its molecular weight in grams at S.T.P. (standard temperature 273 K and pressure 1 atm). This amount of substance is called a mole.

Note $k = \frac{R}{N_A}$ *is called* ***Boltzmann constant****. Its value in SI unit is* 1.38×10^{-23} *J/K.*

Extra Points to Remember

- In our previous discussion, we have discussed Charles' law and pressure law in absolute temperature scale. In centigrade scale, these laws are as under :

Charles' law When a given mass of a gas is heated at constant pressure then for each 1°C rise in temperature the volume of the gas increases by a fraction α of its volume at 0°C. Thus, if the volume of a given mass of a gas at 0°C is V_0, then on heating at constant pressure to t°C its volume will increase by $V_0\alpha t$. Therefore, if its volume at t°C be V_t, then

$$V_t = V_0 + V_0\alpha t \quad \text{or} \quad V_t = V_0(1 + \alpha t)$$

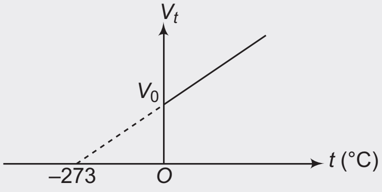


Fig. 20.15

Here, α is called the 'volume coefficient' of the gas. For all gases, the experimental value of α is nearly $\frac{1}{273}$ per °C.

$$\therefore \qquad V_t = V_0\left(1 + \frac{t}{273}\right)$$

Thus, V_t *versus* t graph is a straight line with slope $\frac{V_0}{273}$ and positive intercept V_0. Further $V_t = 0$ at

$$t = -273°\text{C}.$$

Pressure law According to this law, when a given mass of a gas is heated at constant volume then for each 1°C rise in temperature, the pressure of the gas increases by a fraction β of its pressure at 0°C. Thus, if the pressure of a given mass of a gas at 0°C be p_0, then on heating at constant volume to t°C, its pressure will increase by $p_0\beta t$. Therefore, if its pressure at t°C be p_t, then

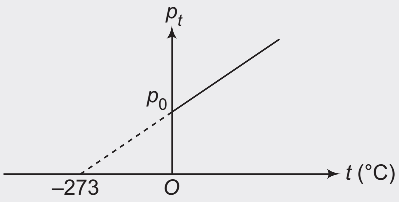


Fig. 20.16

$$p_t = p_0 + p_0\beta t \quad \text{or} \quad p_t = p_0(1 + \beta t)$$

Here, β is called the 'pressure coefficient' of the gas. For all gases the experimental value of β is also

$$\frac{1}{273} \text{ per }^\circ\text{C} \quad \Rightarrow \quad p_t = p_0\left(1+\frac{t}{273}\right)$$

p_t *versus* t graph is as shown in Fig 20.16.

- The above forms of Charles' law and pressure law can be simply expressed in terms of absolute temperature.

  Let at constant pressure, the volume of a given mass of a gas at 0°C, t_1°C and t_2°C be V_0, V_1 and V_2 respectively. Then,

  $$V_1 = V_0\left(1+\frac{t_1}{273}\right) = V_0\left(\frac{273+t_1}{273}\right)$$

  $$V_2 = V_0\left(1+\frac{t_2}{273}\right) = V_0\left(\frac{273+t_2}{273}\right)$$

  $$\therefore \qquad \frac{V_1}{V_2} = \frac{273+t_1}{273+t_2} = \frac{T_1}{T_2}$$

  where, T_1 and T_2 are the absolute temperatures corresponding to t_1°C and t_2°C . Hence,

  $$\frac{V_1}{T_1} = \frac{V_2}{T_2} \quad \text{or} \quad \frac{V}{T} = \text{constant} \quad \text{or} \quad V \propto T$$

  This is the form of Charles' law which we have already studied in article 20.5. In the similar manner, we can prove the pressure law.

- Under isobaric conditions (p = constant), V-T graph is a straight line passing through origin (where, T is in kelvin). The slope of this line is $\left(\frac{nR}{p}\right)$ as $V = \left(\frac{nR}{p}\right)T$ or slope of the line is directly proportional to $\frac{n}{p}$.

  $$\text{slope} = \frac{nR}{p} \quad \text{or} \quad \text{slope} \propto \frac{n}{p}$$

  Similarly, under isochoric conditions (V = constant), p-T graph is a straight line passing through origin whose slope is $\frac{nR}{V}$ or slope is directly proportional to $\frac{n}{V}$.

- **Density of a gas** The ideal gas equation is

  $$pV = nRT = \frac{m}{M}RT$$

  $$\therefore \qquad \frac{m}{V} = \rho = \frac{pM}{RT} \qquad (\rho = \text{density})$$

  $$\therefore \qquad \rho = \frac{pM}{RT}$$

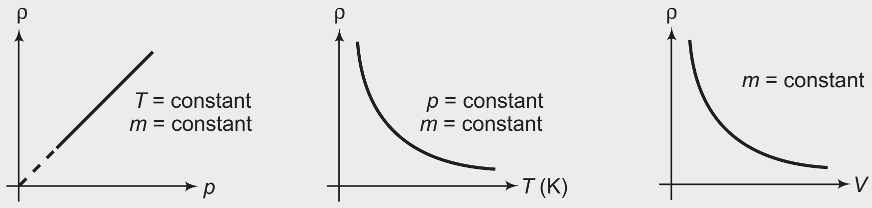


Fig. 20.17

From this equation, we can see that ρ-p graph is straight line passing through origin at constant temperature $(\rho \propto p)$ and ρ-T graph is a rectangular hyperbola at constant pressure $\left(\rho \propto \frac{1}{T}\right)$. Similarly, for a given mass of a gas ρ-V graph is a rectangular hyperbola $\left(\rho \propto \frac{1}{V}\right)$.

Example 20.8 *p-V diagrams of same mass of a gas are drawn at two different temperatures T_1 and T_2. Explain whether $T_1 > T_2$ or $T_2 > T_1$.*

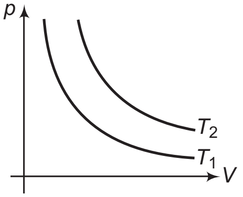


Fig. 20.18

Solution The ideal gas equation is

$$pV = nRT \quad \text{or} \quad T = \frac{pV}{nR}$$

$T \propto pV$ if number of moles of the gas are kept constant. Here, mass of the gas is constant, which implies that number of moles are constant, i.e. $T \propto pV$. In the given diagram, product of p and V for T_2 is more than T_1 at all points (keeping either p or V same for both graphs). Hence,

$$T_2 > T_1$$ **Ans.**

Example 20.9 *The p-V diagram of two different masses m_1 and m_2 are drawn (as shown) at constant temperature T. State whether $m_1 > m_2$ or $m_2 > m_1$.*

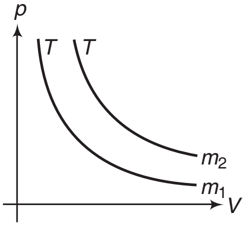


Fig. 20.19

Solution

$$pV = nRT = \frac{m}{M}RT$$

$$\therefore \quad m = (pV)\left(\frac{M}{RT}\right) \quad \text{or} \quad m \propto pV \text{ if } T = \text{constant}$$

From the graph, we can see that $p_2V_2 > p_1V_1$ (for same p or V). Therefore,

$$m_2 > m_1$$ **Ans.**

Example 20.10 *The p-T graph for the given mass of an ideal gas is shown in figure. What inference can be drawn regarding the change in volume (whether it is constant, increasing or decreasing)?*

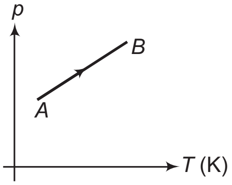


Fig. 20.20

How to Proceed *Definitely, it is not constant. Because when volume of the gas is constant p-T graph is a straight line passing through origin. The given line does not pass through origin, hence volume is not constant.*

$$V = (nR)\left(\frac{T}{p}\right)$$

Now, to see the volume of the gas we will have to see whether $\frac{T}{p}$ *is increasing or decreasing.*

Solution From the given graph, we can write the p-T equation as

$$p = aT + b \qquad (y = mx + c)$$

Here, a and b are positive constants. Further,

$$\frac{p}{T} = a + \frac{b}{T}$$

Now, $$T_B > T_A$$

$\therefore$ $$\frac{b}{T_B} < \frac{b}{T_A} \quad \text{or} \quad \left(\frac{p}{T}\right)_B < \left(\frac{p}{T}\right)_A$$

or $$\left(\frac{T}{p}\right)_B > \left(\frac{p}{T}\right)_A$$

or $$V_B > V_A$$ **Ans.**

Thus, as we move from A to B, volume of the gas is increasing.

Example 20.11 *A gas at* $27°C$ *in a cylinder has a volume of* 4 *litre and pressure* $100\ Nm^{-2}$.

(i) *Gas is first compressed at constant temperature so that the pressure is* $150\ Nm^{-2}$. *Calculate the change in volume.*

(ii) *It is then heated at constant volume so that temperature becomes* $127°C$. *Calculate the new pressure.*

Solution (i) Using Boyle's law for constant temperature,

$$p_1V_1 = p_2V_2$$

$\therefore$ $$V_2 = \frac{p_1V_1}{p_2} = \frac{100 \times 4}{150} = 2.667\ \text{L}$$

$\therefore$ $$\text{Change in volume} = V_2 - V_1 = 2.667 - 4$$

$$= -1.333\ \text{L}$$

(ii) Using Gay Lussac's law for constant volume,

$$\frac{p_2}{p_1} = \frac{T_2}{T_1} \quad \text{or} \quad p_2 = \frac{T_2}{T_1} \times p_1 = \frac{(127 + 273) \times 150}{(27 + 273)}$$

$$= 200\ \text{Nm}^{-2}$$

Example 20.12 *A balloon partially filled with helium has a volume of* $30\ m^3$, *at the earth's surface, where pressure is* 76 *cm of Hg and temperature is* $27°C$. *What will be the increase in volume of gas if balloon rises to a height, where pressure is* 7.6 *cm of Hg and temperature is* $-54°C$?

Solution As $\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$

$$\therefore \quad V_2 = \frac{p_1V_1T_2}{T_1\,p_2}$$

$$= \frac{76 \times 30 \times (-54 + 273)}{(27 + 273) \times 7.6}$$

$$= 219\,\text{m}^3$$

∴ Increase in volume of gas

$$= V_2 - V_1 = 219 - 30$$

$$= 189\ \text{m}^3$$

INTRODUCTORY EXERCISE 20.3

1. From the graph for an ideal gas, state whether m_1 or m_2 is greater.

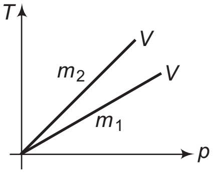


Fig. 20.21

2. A vessel is filled with an ideal gas at a pressure of 20 atm and is at a temperature of 27°C. One-half of the mass is removed from the vessel and the temperature of the remaining gas is increased to 87°C. At this temperature, find the pressure of the gas.
3. A vessel contains a mixture of 7 g of nitrogen and 11 g of carbon dioxide at temperature $T = 290\,\text{K}$. If pressure of the mixture is 1 atm $(= 1.01 \times 10^5\ \text{N/m}^2)$, calculate its density $(R = 8.31\,\text{J/mol-K})$.
4. An electric bulb of volume $250\,\text{cm}^3$ was sealed off during manufacture at a pressure of 10^{-3} mm of mercury at 27°C. Compute the number of air molecules contained in the bulb. Given that $R = 8.31\,\text{J/mol-K}$ and $N_A = 6.02 \times 10^{23}$ per mol.
5. State whether $p_1 > p_2$ or $p_2 > p_1$ for given mass of a gas?

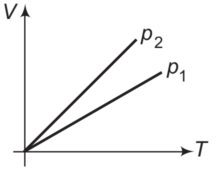


Fig. 20.22

6. For a given mass of a gas what is the shape of p *versus* $\frac{1}{V}$ graph at constant temperature?
7. For a given mass of a gas, what is the shape of pV *versus* T graph in isothermal process?

20.6 Degree of Freedom (*f*)

The minimum number of ways in which motion of a body (or a system) can be described completely is called its degree of freedom.

For example In Fig. (a), block has one degree of freedom, because it is confined to move in a straight line and has only one translational degree of freedom.

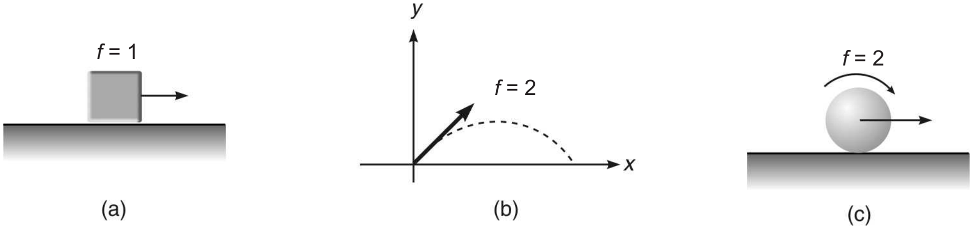


Fig. 20.23

In Fig. (b), the particle has two degrees of freedom because it is confined to move in a plane and so it has two translational degrees of freedom.

In Fig. (c), the sphere has two degrees of freedom one rotational and another translational.

Similarly, a particle free to move in space will have three translational degrees of freedom.

Degree of Freedom of Gas Molecules

A gas molecule can have, the following types of energies

(i) translational kinetic energy
(ii) rotational kinetic energy
(iii) vibrational energy (potential + kinetic)

Vibrational Energy

The forces between different atoms of a gas molecule (interatomic force) may be visualized by imagining every atom as being connected to its neighbours by springs. Each atom can vibrate along the line joining the atoms. Energy associated with this is called vibrational energy.

Degree of Freedom of Monoatomic Gas

A monoatomic gas molecule (like He) consists of a single atom. It can have translational motion in any direction in space. Thus, it has 3 translational degrees of freedom.

$$f = 3 \quad \text{(all translational)}$$

It can also rotate but due to its small moment of inertia, rotational kinetic energy is neglected.

Degree of Freedom of a Diatomic and Linear Polyatomic Gas

The molecules of a diatomic and linear polyatomic gas (like O_2, CO_2 and H_2) cannot only move bodily but also rotate about anyone of the three coordinate axes as shown in figure. However, its moment of inertia about the axis joining the two atoms (x-axis) is negligible. Hence, it can have only two rotational degrees of freedom. Thus, a diatomic molecule has 5 degrees of freedom : 3 translational and 2 rotational. At sufficiently high temperatures, it has vibrational energy as well providing it two more degrees of freedom (one vibrational kinetic energy and another vibrational potential energy). Thus, at high temperatures a diatomic molecule has 7 degrees of freedom, 3 translational, 2 rotational and 2 vibrational. Thus,

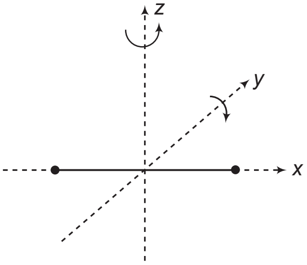


Fig. 20.24

$f = 5$ (3 translational + 2 rotational) at room temperatures and

$f = 7$ (3 translational + 2 rotational + 2 vibrational) at high temperatures.

Degree of Freedom of Non-linear Polyatomic Gas

A non-linear polyatomic molecule (such as NH_3) can rotate about any of three coordinate axes. Hence, it has 6 degrees of freedom 3 translational and 3 rotational. At room temperatures, a polyatomic gas molecule has insignificant vibrational energy. But at high enough temperatures it is also significant. So, it has 8 degrees of freedom 3 rotational, 3 translational and 2 vibrational. Thus,

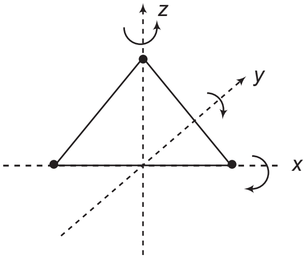


Fig. 20.25

$f = 6$ (3 translational + 3 rotational) at room temperatures and

$f = 8$ (3 translational + 3 rotational + 2 vibrational) at high temperatures.

Degree of Freedom of a Solid

An atom in a solid has no degrees of freedom for translational and rotational motion. At high temperatures, due to vibration along 3 axes it has $3 \times 2 = 6$ degrees of freedom.

$f = 6$ (all vibrational) at high temperatures

Note *(i) Degrees of freedom of a diatomic and polyatomic gas depends on temperature and since there is no clear cut demarcation line above which vibrational energy become significant. Moreover, this temperature varies from gas to gas. On the other hand, for a monoatomic gas there is no such confusion. Degree of freedom here is 3 at all temperatures. Unless and until stated in the question you can take $f = 3$ for monoatomic gas, $f = 5$ for a diatomic gas and $f = 6$ for a non-linear polyatomic gas.*

(ii) When a diatomic or polyatomic gas dissociates into atoms it behaves as a monoatomic gas. Whose degrees of freedom are changed accordingly.

20.7 Internal Energy of an Ideal Gas

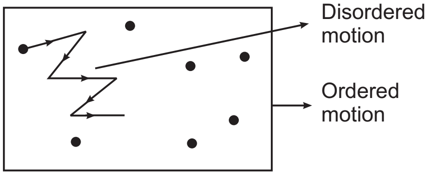


Fig. 20.26

Suppose a gas is contained in a closed vessel as shown in figure. If the container as a whole is moving with some speed, then this motion is called the **ordered motion** of the gas. Source of this motion is some external force. The *zig-zag* motion of gas molecules within the vessel is known as the **disordered motion**. This motion is directly related to the temperature of the gas. As the temperature is increased, the disordered motion of the gas molecules gets fast. The internal energy (U) of the gas is concerned only with its disordered motion. It is in no way concerned with its ordered motion. When the temperature of the gas is increased, its disordered motion and hence its internal energy is increased.

Intermolecular forces in an ideal gas is zero. Thus, PE due to intermolecular forces of an ideal gas is zero. A monoatomic gas is having a single atom. Hence, its vibrational energy is zero. For dia and polyatomic gases vibrational energy is significant only at high temperatures. So, they also have only translational and rotational KE. We may thus conclude that at room temperature the internal energy of an ideal gas (whether it is mono, dia or poly) consists of only translational and rotational KE. Thus,

U (of an ideal gas) $= K_T + K_R$ at room temperatures.

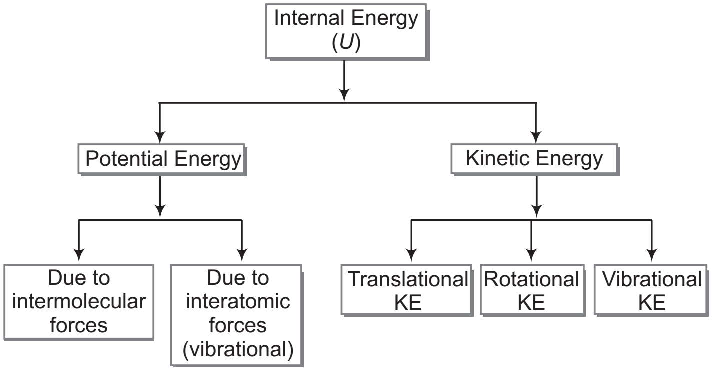


Fig. 20.27

Later in the next article, we will see that K_T (translational KE) and K_R (rotational KE) depends on T only. They are directly proportional to the absolute temperature of the gas. Thus, **internal energy of an ideal gas depends only on its absolute temperature (*T*) and is directly proportional to *T*.**

$$\boxed{U \propto T}$$

20.8 Law of Equipartition of Energy

An ideal gas is just like an ideal father. An ideal father distributes whole of its properties and assets equally among his children. Same is the case with an ideal gas. It distributes its internal energy equally in all degrees of freedom. In each degree of freedom energy of one mole of an ideal gas is

$\frac{1}{2}RT$, where T is the absolute temperature of the gas. Thus, if f be the number of degrees of freedom, the internal energy of 1 mole of the gas will be $\frac{f}{2}RT$ or internal energy of n moles of the gas will be $\frac{n}{2}fRT$. Thus,

$$\boxed{U = \frac{n}{2}fRT} \qquad \ldots\text{(i)}$$

For a monoatomic gas, $f = 3$.

Therefore, $$U = \frac{3}{2}RT \qquad \text{(for 1 mole of a monoatomic gas)}$$

For a dia and linear polyatomic gas at low temperatures, $f = 5$, so,

$$U = \frac{5}{2}RT \qquad \text{(for 1 mole)}$$

and for non-linear polyatomic gas at low temperatures, $f = 6$, so

$$U = \frac{6}{2}RT = 3RT \qquad \text{(for 1 mole)}$$

Note *(i) For one molecule, R (the gas constant) is replaced by k (the Boltzmann constant). For example, internal energy of one molecule in one degree of freedom will be* $\frac{1}{2}kT$.

(ii) Ignoring the vibrational effects we can summarise the above results in tabular form as below.

Table 20.3

Nature of gas	Degree of freedom			Internal energy of 1 mole			Internal energy of 1 molecule		
	Total	Translational	Rotational	Total	Translational	Rotational	Total	Translational	Rotational
Monoatomic	3	3	0	$\frac{3}{2}RT$	$\frac{3}{2}RT$	0	$\frac{3}{2}kT$	$\frac{3}{2}kT$	0
Dia or linear polyatomic	5	3	2	$\frac{5}{2}RT$	$\frac{3}{2}RT$	RT	$\frac{5}{2}kT$	$\frac{3}{2}kT$	kT
Non-linear polyatomic	6	3	3	$3RT$	$\frac{3}{2}RT$	$\frac{3}{2}RT$	$3kT$	$\frac{3}{2}kT$	$\frac{3}{2}kT$

(iii) From the above table, we can see that translational kinetic energy of all types of gases is same. The difference is in rotational kinetic energy.

Example 20.13 *Find total internal energy of 3 moles of hydrogen gas at temperature T.*

Solution Using the relation, $$U = \frac{nf}{2}RT$$

we have, $n = 3$, $f = 5$ for diatomic H_2 gas.

$$\therefore \qquad U = \frac{3 \times 5}{2}RT = 7.5\,RT \qquad \textbf{Ans.}$$

Example 20.14 *Ten moles of O_2 gas are kept at temperature T. At some higher temperature $2T$, forty percent of molecular oxygen breaks into atomic oxygen. Find change in internal energy of the gas.*

Solution **Initial energy** Using the relation, $U = \frac{nf}{2} RT$

we have, $n = 10$, $f = 5$ for diatomic O_2 gas

$$U_i = \frac{10 \times 5}{2} (RT) = 25\, RT$$

Final energy Forty percent means 4 moles O_2 breaks into O. So, it will become 8 moles of monoatomic gas O. Remaining 6 moles are of diatomic gas O_2. But now the new temperature is $2T$.

$$\therefore \qquad U_f = \frac{8 \times 3}{2} (R)(2T) + \frac{6 \times 5}{2} (R)(2T) = 54\, RT$$

So, change in internal energy, $\Delta U = U_f - U_i = 29\, RT$ **Ans.**

Example 20.15 *At a given temperature internal energy of a monoatomic, diatomic and non-linear polyatomic gas is U_0 each. Find their translational and rotational kinetic energies separately.*

Solution **Monoatomic gas** Translational degree of freedom of monoatomic gas is 3 and rotational degree of freedom is zero. Therefore, whole internal energy is in the form of translational kinetic energy.

$$\therefore \qquad K_T = U_0 \quad \text{and} \quad K_R = 0$$

Diatomic gas Translational degree of freedom of diatomic gas is 3 and rotational degree of freedom is 2. Hence, ratio of translational and rotational kinetic energy will be 3 : 2

$$\therefore \qquad K_T = \frac{3}{5} U_0 \quad \text{and} \quad K_R = \frac{2}{5} U_0$$

Non-linear polyatomic gas In non-linear polyatomic gas translational and rotational both degrees of freedom are 3 each. So, translational and rotational kinetic energy are equal. Hence,

$$K_T = K_R = \frac{U_0}{2}$$

INTRODUCTORY EXERCISE 20.4

1. A gas mixture consists of 2 moles of oxygen and 4 moles of argon at temperature T. Neglecting all vibrational modes, the total internal energy of the system is **(JEE 1999)**
   (a) 4 RT (b) 15 RT (c) 9 RT (d) 11RT
2. The average translational kinetic energy of O_2 (molar mass 32) molecules at a particular temperature is 0.048 eV. The translational kinetic energy of N_2 (molar mass 28) molecules in eV at the same temperature is **(JEE 1997)**
   (a) 0.0015 (b) 0.003 (c) 0.048 (d) 0.768
3. At a given temperature, rotational kinetic energy of diatomic gas is K_0. Find its translational and total kinetic energy.

20.9 Molar Heat Capacity

"Molar heat capacity C is the heat required to raise the temperature of 1 mole of a gas by 1°C (or 1 K)." Thus,

$$\boxed{C = \frac{\Delta Q}{n\Delta T} \quad \text{or} \quad \Delta Q = nC\Delta T}$$

For a gas, the value of C depends on the process through which its temperature is raised.
For example, in an isothermal process $\Delta T = 0$ or $C_{\text{iso}} = \infty$. In an adiabatic process, (we will discuss it later) $\Delta Q = 0$. Hence, $C_{\text{adi}} = 0$. Thus, molar heat capacity of a gas varies from 0 to ∞ depending on the process. In general, experiments are made either at constant volume or at constant pressure. In case of solids and liquids, due to small thermal expansion, the difference in measured values of molar heat capacities is very small and is usually neglected. However, in case of gases molar heat capacity at constant volume C_V is quite different from that at constant pressure C_p. Later in the next chapter, we will derive the following relations, (for an ideal gas)

$$\boxed{\begin{aligned} C_V &= \frac{dU}{dT} = \frac{f}{2}R = \frac{R}{\gamma - 1} \\ C_p &= C_V + R \\ \gamma &= \frac{C_p}{C_V} = 1 + \frac{2}{f} \end{aligned}}$$

Here, U is the internal energy of one mole of the gas. The most general expression for C in the process $pV^x =$ constant is

$$\boxed{C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}} \qquad \text{(we will derive it later)}$$

For example : For isobaric process $p =$ constant or $x = 0$ and

$$C = C_p = \frac{R}{\gamma - 1} + R = C_V + R$$

For an isothermal process, $pV =$ constant or $x = 1$

$\therefore$ $C = \infty$ and

For an adiabatic process $pV^{\gamma} =$ constant or $x = \gamma$

$\therefore$ $C = 0$

Values of f, U, C_V, C_p and γ for different gases are given in Table 20.4.

Table 20.4

Nature of gas	f	$U = \frac{f}{2}RT$	$C_V = dU/dT = \frac{f}{2}R$	$C_p = C_V + R$	$\gamma = \frac{C_p}{C_V} = 1 + \frac{2}{f}$
Monoatomic	3	$\frac{3}{2}RT$	$\frac{3}{2}R$	$\frac{5}{2}R$	$\frac{5}{3} = 1.67$
Dia and linear polyatomic	5	$\frac{5}{2}RT$	$\frac{5}{2}R$	$\frac{7}{2}R$	$\frac{7}{5} = 1.4$
Non-linear polyatomic	6	$3\,RT$	$3\,R$	$4\,R$	$\frac{4}{3} = 1.33$

Extra Points to Remember

- Mixture of non-reactive gases

(a) $n = n_1 + n_2$

(b) $p = p_1 + p_2$

(c) $U = U_1 + U_2$

(d) $\Delta U = \Delta U_1 + \Delta U_2$

(e) $C_V = \dfrac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2}$

(f) $C_p = \dfrac{n_1 C_{p_1} + n_2 C_{p_2}}{n_1 + n_2} = C_V + R$

(g) $\gamma = \dfrac{C_p}{C_V}$ or $\dfrac{n}{\gamma - 1} \Rightarrow \dfrac{n_1 + n_2}{\gamma - 1} = \dfrac{n_1}{\gamma_1 - 1} + \dfrac{n_2}{\gamma_2 - 1}$

(h) $M = \dfrac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$

Example 20.16 *Two moles of helium (He) are mixed with four moles of hydrogen* (H_2). *Find*

(a) C_V *of the mixture*

(b) C_p *of the mixture and*

(c) γ *of the mixture.*

Solution Helium is a monoatomic gas.

$$\therefore \qquad C_V = \frac{3}{2}R,\ C_P = \frac{5}{2}R \text{ and } \gamma = \frac{5}{3}$$

Hydrogen is a diatomic gas.

$$\therefore \qquad C_V = \frac{5}{2}R \text{ and } C_V = \frac{7}{2}R \text{ and } \gamma = \frac{7}{5}$$

(a) **C_V of the mixture**

$$C_V = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2} = \frac{(2)\left(\frac{3}{2}R\right) + 4\left(\frac{5}{2}R\right)}{2+4}$$

$$= \frac{13}{6}R \qquad \textbf{Ans.}$$

(b) **C_p of the mixture**

$$C_p = C_V + R = \frac{13}{6}R + R$$

$$= \frac{19}{6}R \qquad \textbf{Ans.}$$

Alternate method

$$C_p = \frac{n_1 C_{p_1} + n_2 C_{p_2}}{n_1 + n_2}$$

$$= \frac{(2)\left(\frac{5}{2}R\right) + (4)\left(\frac{7}{2}R\right)}{2+4}$$

$$= \frac{19}{6}R \qquad \textbf{Ans.}$$

(c) **γ of the mixture**

$$\gamma = \frac{C_p}{C_V} = \frac{(19/6)R}{(13/6)R}$$

$$= \frac{19}{13}$$ **Ans.**

Alternate method

$$\frac{n_1 + n_2}{\gamma - 1} = \frac{n_1}{\gamma_1 - 1} + \frac{n_2}{\gamma_2 - 1}$$

$$\therefore \qquad \frac{2+4}{\gamma - 1} = \frac{2}{\frac{5}{3} - 1} + \frac{4}{\frac{7}{5} - 1}$$

Solving this equation, we get

$$\gamma = \frac{19}{13}$$

Example 20.17 *Temperature of two moles of a monoatomic gas is increased by 300 K in the process $p \propto V$. Find*

(a) molar heat capacity of the gas in the given process

(b) heat required in the given process.

Solution (a) Molar heat capacity C is given by

$$C = C_V + \frac{R}{1 - x} \qquad \text{...(i)}$$

in the process pV^x = constant.

Now, $C_V = \frac{3}{2} R$ for monoatomic gas.

Further, the given process can be written as

$$PV^{-1} = \text{constant}$$

$$\therefore \qquad x = -1$$

Substituting the values in Eq. (i), we have

$$C = \frac{3}{2} R + \frac{R}{1+1} = 2R$$ **Ans.**

(b) Q or $\Delta Q = nC\Delta T$

Substituting the values, we have

$$\Delta Q = (2)(2R)(300)$$

$$= 1200R$$ **Ans.**

20.10 Kinetic Theory of Gases

We have studied the mechanics of single particles. When we approach the mechanics associated with the many particles in systems such as gases, liquids and solids, we are faced with analyzing the dynamics of a huge number of particles. The dynamics of such many particle systems is called statistical mechanics.

The game involved in studying a system with a large number of particles is similar to what happens after every physics test. Of course we are interested in our individual marks, but we also want to know the class average.

The kinetic theory that we study in this article is a special aspect of the statistical mechanics of large number of particles. We begin with the simplest model for a monatomic ideal gas, a dilute gas whose particles are single atoms rather than molecules.

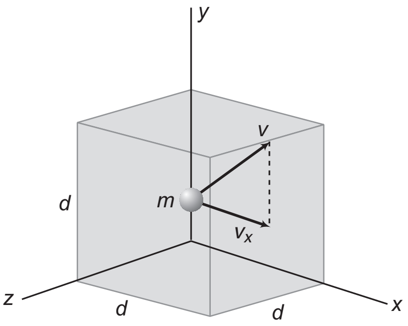
Macroscopic variables of a gas are pressure, volume and temperature and microscopic properties are speed of gas molecules, momentum of molecules, etc. Kinetic theory of gases relates the microscopic properties to macroscopic properties. Furthermore, the kinetic theory provides us with a physical basis for our understanding of the concept of pressure and temperature.

The Ideal Gas Approximation

We make the following assumptions while describing an ideal gas :

1. The number of particles in the gas is very large.
2. The volume V containing the gas is much larger than the total volume actually occupied by the gas particles themselves.
3. The dynamics of the particles is governed by Newton's laws of motion.
4. The particles are equally likely to be moving in any direction.
5. The gas particles interact with each other and with the walls of the container only *via* elastic collisions.
6. The particles of the gas are identical and indistinguishable.

The Pressure of an Ideal Gas



A cubical box with sides of length *d* containing an ideal gas. The molecule shown moves with velocity *v*.

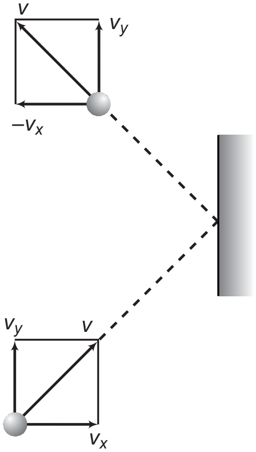
Fig. 20.28

Consider an ideal gas consisting of N molecules in a container of volume V. The container is a cube with edges of length d. Consider the collision of one molecule moving with a velocity **v** toward the right hand face of the cube. The molecule has velocity components v_x, v_y and v_z. Previously, we used m to represent the mass of a sample, but in this article we shall use m to represent the mass of one molecule. As the molecule collides with the wall elastically its x-component of velocity is reversed, while its y and z-components of velocity remain unaltered. Because the x-component of the momentum of the molecule is mv_x before the collision and $-mv_x$ after the collision, the change in momentum of the molecule is

$$\Delta p_x = -mv_x - (mv_x) = -2mv_x$$

Applying impulse = change in momentum to the molecule

$$F\Delta t = \Delta p_x = -2mv_x$$



A molecule makes an elastic collision with the wall of the container. Its x-component of momentum is reversed, while its y-component remains unchanged. In this construction, we assume that the molecule moves in the x-y plane.

Fig. 20.29

where, F is the magnitude of the average force exerted by the wall on the molecule in time Δt. For the molecules to collide second time with the same wall, it must travel a distance $2d$ in the x-direction. Therefore, the time interval between two collisions with the same wall is $\Delta t = \frac{2d}{v_x}$. Over a time interval that is long compared with Δt, the average force exerted on the molecules for each collision is

$$F = \frac{-2mv_x}{\Delta t} = \frac{-2mv_x}{2d/v_x} = \frac{-mv_x^2}{d}$$

According to Newton's third law, the average force exerted by the molecule on the wall is, $\frac{mv_x^2}{d}$.

Each molecule of the gas exerts a force on the wall. We find the total force exerted by all the molecules on the wall by adding the forces exerted by the individual molecules.

$$\therefore \qquad F_{\text{wall}} = \frac{m}{d}(v_{x_1}^2 + v_{x_2}^2 + \ldots + v_{x_N}^2)$$

$$= \frac{mN}{d}\left(\frac{v_{x_1}^2 + v_{x_2}^2 + \ldots + v_{x_N}^2}{N}\right)$$

This can also be written as

$$F_{\text{wall}} = \frac{Nm}{d}\bar{v}_x^2$$

where, $$\bar{v}_x^2 = \frac{v_{x_1}^2 + v_{x_2}^2 + \ldots + v_{x_N}^2}{N}$$

Since, the velocity has three components v_x, v_y and v_z, we can have

$$\bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2 \qquad (\text{as } v^2 = v_x^2 + v_y^2 + v_z^2)$$

Because the motion is completely random, the average values $\bar{v}_x^2$, $\bar{v}_y^2$ and $\bar{v}_z^2$ are equal to each other. So,

$$\bar{v}^2 = 3\bar{v}_x^2 \quad \text{or} \quad \bar{v}_x^2 = \frac{1}{3}\bar{v}^2$$

Therefore, $$F_{\text{wall}} = \frac{N}{3}\left(\frac{m\bar{v}^2}{d}\right)$$

∴ Pressure on the wall

$$p = \frac{F_{\text{wall}}}{A} = \frac{F_{\text{wall}}}{d^2} = \frac{1}{3}\left(\frac{N}{d^3}m\bar{v}^2\right)$$

$$= \frac{1}{3}\left(\frac{N}{V}\right)m\bar{v}^2 = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{1}{2}m\bar{v}^2\right)$$

$$\therefore \qquad p = \frac{1}{3}\frac{mN}{V}\bar{v}^2 = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{1}{2}m\bar{v}^2\right) \qquad \ldots\text{(i)}$$

This result indicates that the pressure is proportional to the number of molecules per unit volume (N/V) and to the average translational kinetic energy of the molecules $\frac{1}{2}m\bar{v}^2$. This result relates the large scale quantity (macroscopic) of pressure to an atomic quantity (microscopic)—the average value of the square of the molecular speed. The above equation verifies some features of pressure with which you are probably familiar. One way to increase the pressure inside a container is to increase the number of molecules per unit volume in the container.

The Meaning of the Absolute Temperature

Rewriting Eq. (i) in the more familiar form

$$pV = \frac{2}{3}N\left(\frac{1}{2}m\bar{v}^2\right)$$

Let us now compare it with the ideal gas equation

$$pV = nRT$$

$$nRT = \frac{2}{3} N \left(\frac{1}{2} m\bar{v}^2\right)$$

Here, $$n = \frac{N}{N_A} \qquad (N_A = \text{Avogadro number})$$

$\therefore$ $$T = \frac{2}{3}\left(\frac{N_A}{R}\right)\left(\frac{1}{2} m\bar{v}^2\right)$$

or $$T = \frac{2}{3k}\left(\frac{1}{2} m\bar{v}^2\right) \qquad \text{...(ii)}$$

where, k is **Boltzmann's constant** which has the value

$$k = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J/K}$$

By rearranging Eq. (ii) we can relate the translational molecular kinetic energy to the temperature

$$\frac{1}{2} mv^2 = \frac{3}{2} kT$$

That is, the average translational kinetic energy per molecule is $\frac{3}{2} kT$. Further,

$$\bar{v}_x^2 = \frac{1}{3}\bar{v}^2, \quad \text{it follows that}$$

$$\frac{1}{2} m\bar{v}_x^2 = \frac{1}{2} kT$$

In the similar manner, it follows that

$$\frac{1}{2} m\bar{v}_y^2 = \frac{1}{2} kT$$

and $$\frac{1}{2} m\bar{v}_z^2 = \frac{1}{2} kT$$

Thus, in each translational degree of freedom one gas molecule has an energy $\frac{1}{2} kT$. One mole of a gas has N_A number of molecules. Thus, one mole of the gas has an energy $\frac{1}{2}(kN_A)T = \frac{1}{2} RT$ in each degree of freedom. Which is nothing but the law of equipartition of energy. The total translational kinetic energy of one mole of an ideal gas is therefore, $\frac{3}{2} RT$.

$$\boxed{(\text{KE})_{\text{Trans}} = \frac{3}{2} RT \quad \text{(of one mole)}}$$

Root Mean Square Speed

The square root of $\overline{\mathbf{v}}^2$ is called the root mean square (rms) speed of the molecules. From Eq. (ii), we obtain, for the rms speed

$$v_{\text{rms}} = \sqrt{\bar{v}^2} = \sqrt{\frac{3kT}{m}}$$

Using $$k = \frac{R}{N_A}, \qquad mN_A = M$$

and $$\frac{RT}{M} = \frac{p}{\rho}$$

we can write,

$$\boxed{v_{\text{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3p}{\rho}}}$$

Mean Speed or Average Speed

The particles of a gas have a range of speeds. The average speed is found by taking the average of the speeds of all the particles at a given instant. Remember that the speed is a positive scalar since it is the magnitude of the velocity.

$$v_{\text{av}} = \frac{v_1 + v_2 + \ldots + v_N}{N}$$

From Maxwellian speed distribution law, we can show that

$$\boxed{v_{\text{av}} = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8p}{\pi\rho}}}$$

Most Probable Speed

This is defined as the speed which is possessed by maximum fraction of total number of molecules of the gas. For example, if speeds of 10 molecules of a gas are, 1, 2, 2, 3, 3, 3, 4, 5, 6, 6 km/s, then the most probable speed is 3 km/s, as maximum fraction of total molecules possess this speed. Again from Maxwellian speed distribution law :

$$\boxed{v_{\text{mp}} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2p}{\rho}}}$$

Note 1. *In the above expressions of v_{rms}, v_{av} and v_{mp}, M is the molar mass in kilogram per mole. For example, molar mass of hydrogen is 2×10^{-3} kg/mol.*

2. $v_{rms} > v_{av} > v_{mp}$ *(RAM)*

3. $v_{rms} : v_{av} : v_{mp} = \sqrt{3} : \sqrt{\frac{8}{\pi}} : \sqrt{2}$

and since, $\frac{8}{\pi} \approx 2.5$, *we have* $v_{rms} : v_{av} : v_{mp} = \sqrt{3} : \sqrt{2.5} : \sqrt{2}$

Extra Points to Remember

- Pressure exerted by an ideal gas is numerically equal to two-third of the mean kinetic energy of translation per unit volume of (E) the gas. Thus,

$$\boxed{p = \frac{2}{3}E = \frac{1}{3}\frac{mN}{V}v_{\text{rms}}^2}$$

Here, m = mass of one gas molecule

N = total number of molecules

- **Mean Free Path** Every gas consists of a very large number of molecules. These molecules are in a state of continuous rapid and random motion. They undergo perfectly elastic collisions against one another. Therefore, path of a single gas molecule consists of a series of short *zig-zag* paths of different lengths. The mean free path of a gas molecule is the average distance between two successive collisions. It is represented by λ.

$$\boxed{\lambda = \frac{kT}{\sqrt{2}\pi\sigma^2 p}}$$

Here, σ = diameter of the molecule

k = Boltzmann's constant

- **Avogadro's Hypothesis** At constant temperature and pressure equal volumes of different gases contain equal number of molecules. In 1 g-mole of any gas there are 6.02×10^{23} molecules of that gas. This is called Avogadro's number. Thus,

$$\boxed{N = 6.02 \times 10^{23} \text{ per g-mole}}$$

Therefore, the number of molecules in mass m of the substance

$$\boxed{\text{Number of molecules } = nN = \frac{m}{M} \times N}$$

- **Dalton' Law of Partial Pressure** According to this law if the gases filled in a vessel do not react chemically, then the combined pressure of all the gases is due to the partial pressure of the molecules of the individual gases. If $p_1, p_2, \ldots$ represent the partial pressures of the different gases, then the total pressure is,

$$p = p_1 + p_2 \ldots$$

Example 20.18 *Find the rms speed of hydrogen molecules at room temperature* (= 300 *K*).

Solution Mass of 1 mole of hydrogen gas

$$= 2\text{g} = 2 \times 10^{-3} \text{ kg}$$

$$\Rightarrow \qquad v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

$$= \sqrt{\frac{3 \times 8.31 \times 300}{2 \times 10^{-3}}}$$

$$= 1.93 \times 10^3 \text{ m/s}$$

Ans.

Example 20.19 *A tank used for filling helium balloons has a volume of* 0.3 m^3 *and contains* 2.0 *mol of helium gas at* 20.0°*C. Assuming that the helium behaves like an ideal gas.*

(a) What is the total translational kinetic energy of the molecules of the gas?

(b) What is the average kinetic energy per molecule?

Solution (a) Using $(\text{KE})_{\text{Trans}} = \frac{3}{2} nRT$

with $n = 2.0$ mol and $T = 273 + 20 = 293$ K, we find that

$$(\text{KE})_{\text{Trans}} = \frac{3}{2}(2.0)(8.31)(293)$$

$$= 7.3 \times 10^3 \text{ J}$$ **Ans.**

(b) The average kinetic energy per molecule is $\frac{3}{2} kT$.

or

$$\frac{1}{2} m\bar{v}^2 = \frac{1}{2} m\bar{v}^2_{\text{rms}} = \frac{3}{2} kT$$

$$= \frac{3}{2}(1.38 \times 10^{-23})(293)$$

$$= 6.07 \times 10^{-21} \text{ J}$$ **Ans.**

Example 20.20 *Consider an* 1100 *particles gas system with speeds distribution as follows :*

1000 *particles each with speed* 100 *m/s*

2000 *particles each with speed* 200 *m/s*

4000 *particles each with speed* 300 *m/s*

3000 *particles each with speed* 400 *m/s and* 1000 *particles each with speed* 500 *m/s*

Find the average speed, and rms speed.

Solution The average speed is

$$v_{\text{av}} = \frac{(1000)(100) + (2000)(200) + (4000)(300) + (3000)(400) + (1000)(500)}{1100}$$

$$= 309 \text{ m/s}$$ **Ans.**

The rms speed is

$$v_{\text{rms}} = \sqrt{\frac{(1000)(100)^2 + (2000)(200)^2 + (4000)(300)^2 + (3000)(400)^2 + (1000)(500)^2}{1100}}$$

$$= 328 \text{ m/s}$$ **Ans.**

Note *Here,* $\frac{v_{rms}}{v_{av}} \neq \sqrt{\frac{3}{8/\pi}}$ *as values and gas molecules are arbitrarily taken.*

Example 20.21 *Calculate the change in internal energy of 3.0 mol of helium gas when its temperature is increased by 2.0 K.*

Solution Helium is a monatomic gas. Internal energy of n moles of the gas is

$$U = \frac{3}{2} nRT$$

$$\therefore \qquad \Delta U = \frac{3}{2} nR(\Delta T)$$

Substituting the values,

$$\Delta U = \left(\frac{3}{2}\right)(3)(8.31)(2.0)$$

$$= 74.8 \text{ J}$$ **Ans.**

Example 20.22 *In a crude model of a rotating diatomic molecule of chlorine (Cl_2), the two Cl atoms are 2.0×10^{-10} m apart and rotate about their centre of mass with angular speed $\omega = 2.0 \times 10^{12}$ rad/s. What is the rotational kinetic energy of one molecule of Cl_2, which has a molar mass of 70.0 g/mol?*

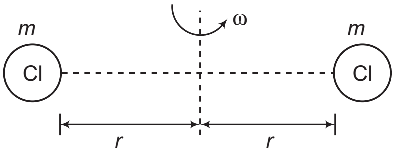


Fig. 20.30

Solution Moment of inertia,

$$I = 2(mr^2) = 2mr^2$$

Here, $$m = \frac{70 \times 10^{-3}}{2 \times 6.02 \times 10^{23}}$$

$$= 5.81 \times 10^{-26} \text{ kg}$$

and $$r = \frac{2.0 \times 10^{-10}}{2}$$

$$= 1.0 \times 10^{-10} \text{ m}$$

$$\therefore \qquad I = 2(5.81 \times 10^{-26})(1.0 \times 10^{-10})^2$$

$$= 1.16 \times 10^{-45} \text{ kg-m}^2$$

$$\therefore \qquad K_R = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} \times (1.16 \times 10^{-45}) \times (2.0 \times 10^{12})^2$$

$$= 2.32 \times 10^{-21} \text{ J}$$ **Ans.**

Example 20.23 *Prove that the pressure of an ideal gas is numerically equal to two third of the mean translational kinetic energy per unit volume of the gas.*

Solution Translational KE per unit volume

$$E = \frac{1}{2}(\text{mass per unit volume})(\overline{v}^2)$$

$$= \frac{1}{2}(\rho)\left(\frac{3p}{\rho}\right) = \frac{3}{2}p$$

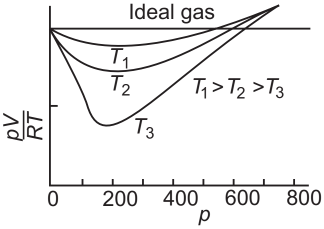
or $$p = \frac{2}{3}E$$ **Hence Proved**.

INTRODUCTORY EXERCISE 20.5

1. Calculate the root mean square speed of hydrogen molecules at 373.15 K.
2. Five gas molecules chosen at random are found to have speed of 500, 600, 700, 800 and 900 m/s. Find the rms speed. Is it the same as the average speed?
3. The average speed of all the molecules in a gas at a given instant is not zero, whereas the average velocity of all the molecules is zero. Explain why?
4. A sample of helium gas is at a temperature of 300 K and a pressure of 0.5 atm. What is the average kinetic energy of a molecule of a gas?
5. A sample of helium and neon gases has a temperature of 300 K and pressure of 1.0 atm. The molar mass of helium is 4.0 g/mol and that of neon is 20.2 g/mol.
   (a) Find the rms speed of the helium atoms and of the neon atoms.
   (b) What is the average kinetic energy per atom of each gas?
6. At what temperature will the particles in a sample of helium gas have an rms speed of 1.0 km/s?
7. For any distribution of speeds $v_{\text{rms}} \geq v_{\text{av}}$. Is this statement true or false?

Final Touch Points

1. **Departures from ideal gas behaviour for a real gas** An ideal gas is a simple theoretical model of a gas. No real gas is truly ideal. Figure shows departures from ideal gas behaviour for a real gas at three different temperatures. Notice that all curves approach the ideal gas behaviour for low pressures and high temperatures.



At low pressures or high temperatures the molecules are far apart and molecular interactions are negligible. Without interactions, the gas behaves like an ideal one .

2. **van der Waal's Equation** Experiments have proved that real gases deviate largely from ideal behaviour. The reason of this deviation is two wrong assumptions in the kinetic theory of gases.
   (i) The size of the molecules is much smaller in comparison to the volume of the gas, hence, it may be neglected.
   (ii) Molecules do not exert intermolecular force on each other.
   van der Waal made corrections for these assumptions and gave a new equation. This equation is known as van der Waal's equation for real gases.
   (a) **Correction for the finite size of molecules:** Molecules occupy some volume. Therefore, the volume in which they perform thermal motion is less than the observed volume of the gas. It is represented by $(V - b)$. Here, b is a constant which depends on the effective size and number of molecules of the gas. Therefore, we should use $(V - b)$ in place of V in gas equation.
   (b) **Correction for intermolecular attraction:** Due to the intermolecular force between gas molecules the molecules which are very near to the wall experiences a net inward force. Due to this inward force there is a decrease in momentum of the particles of a gas. Thus, the pressure exerted by real gas molecules is less than the pressure exerted by the molecules of an ideal gas. So, we use $\left(p + \frac{a}{V^2}\right)$ in place of p in gas equation. Here, again a is a constant.

   van der Waal's equation of state for real gases thus becomes,

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

3. **Critical Temperature, Pressure and Volume :** Gases can't be liquified above a temperature called critical temperature (T_C) however large the pressure may be. The pressure required to liquify the gas at critical temperature is called critical pressure (p_C) and the volume of the gas at critical temperature and pressure is called critical volume (V_C). Value of critical constants in terms of van der Waal's constants a and b are as under :

$$V_C = 3b, \qquad p_C = \frac{a}{27b^2} \qquad \text{and} \qquad T_C = \frac{8a}{27Rb}$$

Further, $\frac{RT_C}{p_C V_C} = \frac{8}{3}$ is called critical coefficient and is same for all gases.

4. Detailed Discussion on Molar Heat Capacity

For monatomic gases value of C_V is $\frac{3}{2}R$. No variation is observed in this. So, value of $C_V, C_p, C_p - C_V$ and γ comes out to be same for different monatomic gases (Table 20.5).

Table 20.5

Gas	C_p	C_V	$C_p - C_V$	$\gamma = C_p/C_V$
Monoatomic Gases				
He	20.8	12.5	8.33	1.67
Ar	20.8	12.5	8.33	1.67
Diatomic Gases				
H_2	28.8	20.4	8.33	1.41
N_2	29.1	20.8	8.33	1.40
O_2	29.4	21.1	8.33	1.40
CO_2	29.3	21.0	8.33	1.40
Cl_2	34.7	25.7	8.96	1.35
Polyatomic Gases				
CO_2	37.0	28.5	8.50	1.30
SO_2	40.4	31.4	9.00	1.29
H_2O	35.4	27.0	8.37	1.30
CH_2	35.5	27.1	8.41	1.31

*All values except that for water were obtained at 300 K. SI units are used for C_p and C_V.

For dia and polyatomic gases these values are not equal for different gases. These values vary from gas to gas. Even for one gas values are different at different temperatures.

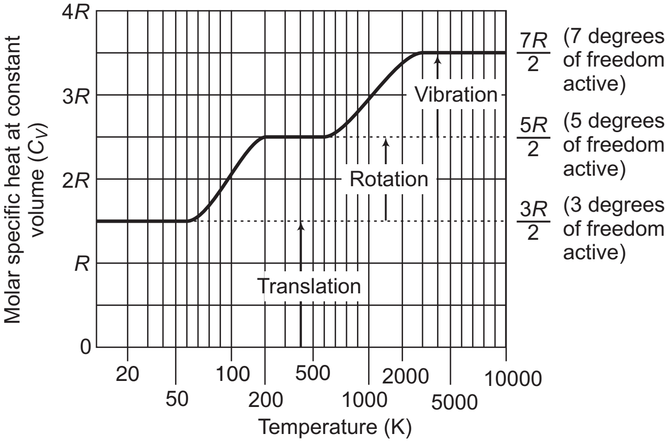
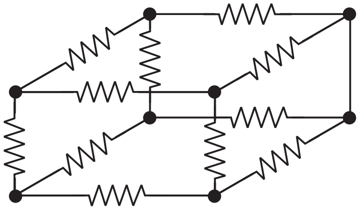


Figure illustrates the variation in the molar specific heat (at constant volume) for H_2 over a wide range in temperatures. (Note that T is drawn on a logarithmic scale). Below about 100 K, C_V is $\frac{3R}{2}$ which is characteristic of three translational degrees of freedom. At room temperature (300 K) it is $\frac{5R}{2}$ which includes the two rotational degrees of freedom. It seems, therefore, that at low temperatures, rotation is not allowed. At high temperatures, C_V starts to rise toward the value $\frac{7R}{2}$. Thus, the vibrational degrees of freedom contribute only at these high temperatures. In Table 20.4 the large values of C_V for some polyatomic molecules show the contributions of vibrational energy. In addition, a molecule with three or more atoms that are not in a straight line has three, not two, rotational degrees of freedom.

From this discussion, we expect heat capacities to be temperature-dependent, generally increasing with increasing temperature.

5. **Solids** In crystalline solids (monoatomic), the atoms are arranged in a three dimensional array, called a lattice. Each atom in a lattice can vibrate along three mutually perpendicular directions, each of which has two degrees of freedom. One corresponding to vibrations KE and the other vibrational PE. Thus, each atom has a total of six degrees of freedom. The volume of a solid does not change significantly with temperature, and so there is little difference between C_V and C_p for a solid. The molar heat capacity is expected to be

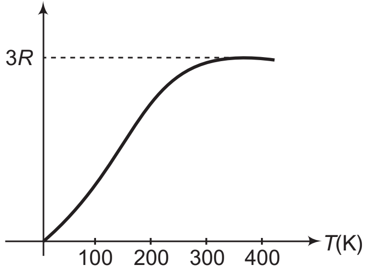


$$C = \frac{f}{2}R = \frac{6}{2}R$$

or $$C = 3R \qquad \text{(ideal monatomic solid)}$$

Its numerical value is $C \approx 25$ J/mol-K ≈ 6 cal/mol-K. This result was first found experimentally by **Dulong and Petit**.

Figure shows that the Dulong and Petit's law is obeyed quite well at high (> 250 K) temperatures. At low temperatures, the heat capacities decreases.



Solved Examples

TYPED PROBLEMS

Type 1. *Conversion of graph*

Concept

(i) In heat, we normally find the following graphs:

S.No	Equation type	Nature of graph
1.	$y \propto x$	Straight line passing through origin
2.	$y \propto \frac{1}{x}$	Rectangular hyperbola
3.	$y =$ constant	Straight line parallel to x-axis
4.	$x =$ constant	Straight line parallel to y-axis
5.	$x =$ constant and $y =$ constant	Dot

(ii) Density of any substance is given by

$$\rho = \frac{m}{V}$$

For given mass of any substance, density only depends on V. It varies with V as

$$\rho \propto \frac{1}{V}$$

If V increases, ρ decreases and *vice-versa*.

(iii) For an ideal gas, density is also given as

$$\rho = \frac{pM}{RT}$$

For a given gas,

$$\rho \propto \frac{p}{T}$$

(iv) For an ideal gas,

$$pV = nRT$$

and

$$U = \frac{nf}{2}RT$$

For given moles of the gas,

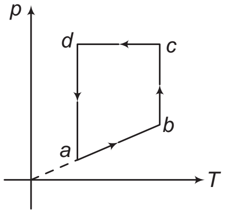
$$U \propto T \quad \text{and} \quad T \propto pV$$

If product of p and V is increasing, it means temperature is increasing. So, U is increasing.

For example, in isothermal process, T is constant.

Therefore, U is constant. Hence, U *versus* T graph is a dot.

Example 1



Corresponding to p-T graph as shown in figure, draw

(a) p-V graph *(b) V-T graph*

(c) ρ-T graph and *(d) U-T graph*

Solution ***ab* process** From the given graph, we can see that

$$p \propto T$$

$$\Rightarrow \qquad V = \text{constant} \qquad \text{(isochoric)}$$

$$\text{or} \qquad \rho = \text{constant} \qquad \left(\text{as } \rho \propto \frac{1}{V}\right)$$

p and T both are increasing.

Therefore, U is also increasing (as $U \propto T$)

Now,

(i) p-V graph is straight line parallel to p-axis as V is constant.

(ii) V-T graph is a straight line parallel to T-axis as V is constant.

(iii) ρ-T graph is a straight line parallel to T-axis as ρ is constant.

(iv) U-T graph is a straight line passing through origin as $U \propto T$.

***bc* process** From the given graph, we can see that

$$T = \text{constant} \qquad \text{(isothermal)}$$

$$\Rightarrow \qquad U = \text{constant}$$

$$\Rightarrow \qquad pV = \text{constant}$$

$$\text{or} \qquad p \propto \frac{1}{V}$$

p is increasing. Therefore, V will decrease.

Hence, ρ will increase.

Now,

(i) p-V graph is a rectangular hyperbola $\left(\text{as } P \propto \frac{1}{V}\right)$.

(ii) V-T graph is a straight line parallel to V-axis, as T = constant

(iii) ρ-T graph is a straight line parallel to ρ-axis, as T = constant

(iv) U-T graph is a dot, as U and T both are constants.

***cd* process** From the given graph, we can see that

$$p = \text{constant} \qquad \text{(isobaric)}$$

$$\Rightarrow \qquad V \propto T$$

Temperature is decreasing. So, volume will also decrease. But density will increase.

Now,

(i) p-V graph is a straight line parallel to V-axis because p is constant.

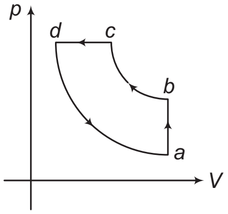
(ii) V-T graph is a straight line passing through origin, as $V \propto T$.

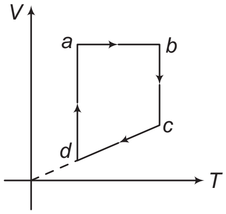
(iii) $\rho = \dfrac{pM}{RT} \Rightarrow \rho \propto \dfrac{1}{T}$ as p, M and R all are constants. Hence, ρ-T graph is rectangular hyperbola.

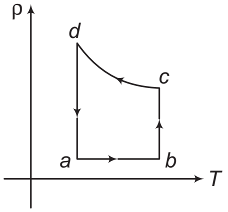
(iv) U-T graph is a straight line passing through origin as $U \propto T$.

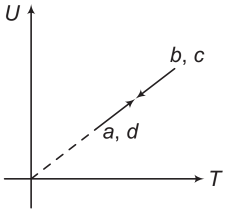
***da* process** This process is just inverse of *bc* process. So, this process will complete the cycle following the steps discussed in process *bc*.

The four graphs are as shown below.









Type 2. *When ordered motion of a gas converts into disordered motion*

Concept

In article 20.7, we have seen that if a container filled with a gas is in motion, then the gas molecules have some ordered motion also. When the container is suddenly stopped, this ordered motion of gas molecules converts into disordered motion. Therefore, internal energy (and hence the temperature) of the gas will increase. In such situation, we can apply the equation,

$$\frac{1}{2}mv^2 = \Delta U$$

where, $$U = \frac{nf}{2}RT$$

$\therefore$ $$\Delta U = \frac{nf}{2}R\Delta T$$

where, $$n = \frac{m}{M}$$

Example 2 *An insulated box containing a monoatomic gas of molar mass M moving with a speed v_0 is suddenly stopped. Find the increment in gas temperature as a result of stopping the box.* **(JEE 2003)**

Solution Decrease in kinetic energy = increase in internal energy of the gas

$$\therefore \qquad \frac{1}{2}mv_0^2 = \frac{nf}{2}R\Delta T = \left(\frac{m}{M}\right)\left(\frac{3}{2}R\right)\Delta T$$

$$\therefore \qquad \Delta T = \frac{Mv_0^2}{3R}$$

Type 3. *Based on the concept of pressure exerted by gas molecules striking elastically with walls of a container.*

Concept

(i) Gas molecules are assumed to be moving with v_{rms}.

(ii) They strike with the wall elastically. So, in each collision change in linear momentum is $2mv_{\text{rms}}$.

(iii) Time interval between two successive collisions is $\frac{2L}{v_{\text{rms}}}$ if we have a cubical box of side L and,

(iv) $F = \frac{\Delta p}{\Delta t}$, Pressure $= \frac{F}{A}$

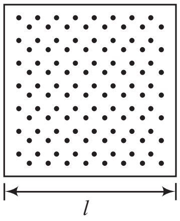
Example 3 *A cubical box of side 1 m contains helium gas (atomic weight 4) at a pressure of $100\ N/m^2$. During an observation time of 1 second, an atom travelling with the root-mean-square speed parallel to one of the edges of the cube, was found to make 500 hits with a particular wall, without any collision with other atoms. Take $R = \frac{25}{3}$ J/mol-K and $k = 1.38 \times 10^{-23}$ J/K.*

(a) Evaluate the temperature of the gas.

(b) Evaluate the average kinetic energy per atom.

(c) Evaluate the total mass of helium gas in the box.

Solution Volume of the box = 1 m^3, pressure of the gas = 100 N/m^2. Let T be the temperature of the gas.



(a) Time between two consecutive collisions with one wall $= \frac{1}{500}$ s.

This time should be equal to $\frac{2l}{v_{\text{rms}}}$, where l is the side of the cube.

$$\frac{2l}{v_{\text{rms}}} = \frac{1}{500}$$

$$\text{or} \qquad v_{\text{rms}} = 1000 \text{ m/s} \qquad (\text{as } l = 1 \text{ m})$$

$$\therefore \qquad \sqrt{\frac{3RT}{M}} = 1000$$

or
$$T = \frac{(1000)^2 M}{3R} = \frac{(10^6)(4\times10^{-3})}{3\left(\frac{25}{3}\right)}$$
$$= 160 \text{ K}$$ **Ans.**

(b) Average kinetic energy per atom $= \frac{3}{2}kT$
$$= \frac{3}{2}(1.38\times10^{-23})(160) \text{ J}$$
$$= 3.312\times10^{-21} \text{ J}$$ **Ans.**

(c) From $pV = nRT = \frac{m}{M}RT$

We get mass of helium gas in the box,
$$m = \frac{pVM}{RT}$$

Substituting the values, we get
$$m = \frac{(100)(1)(4\times10^{-3})}{\left(\frac{25}{3}\right)(160)}$$
$$= 3.0\times10^{-4} \text{ kg}$$ **Ans.**

Example 4 *1 g mole of oxygen at 27°C and 1 atmospheric pressure is enclosed in a vessel.*

(a) Assuming the molecules to be moving with v_{rms}, find the number of collisions per second which the molecules make with one square metre area of the vessel wall.

(b) The vessel is next thermally insulated and moved with a constant speed v_0. It is then suddenly stopped. The process results in a rise of temperature of the gas by 1°C. Calculate the speed v_0. [$k = 1.38\times10^{-23}$ J/K and $N_A = 6.02\times10^{23}$/mol] **(JEE 2002)**

Solution (a) Mass of one oxygen molecule,
$$m = \frac{M}{N_A}$$
$$= \frac{32}{6.02\times10^{23}} \text{ g}$$
$$= 5.316\times10^{-23} \text{ g}$$
$$= 5.316\times10^{-26} \text{ kg}$$
$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$$
$$= \sqrt{\frac{3\times1.38\times10^{-23}\times300}{5.316\times10^{-26}}}$$
$$= 483.35 \text{ m/s}$$

Change in momentum per collision,
$$\Delta p = mv_{\text{rms}} - (-mv_{\text{rms}}) = 2mv_{\text{rms}}$$
$$= (2)(5.316\times10^{-26})(483.35)$$
$$= 5.14\times10^{-23} \text{ kg-m/s}$$

Now, suppose n particles strike per second

$$F = n\Delta p = (n)\,(5.14 \times 10^{-23})\ \text{N} \qquad \left(F_{\text{ext}} = \frac{dp}{dt}\right)$$

Now, as $p = \dfrac{F}{A}$, for unit area $F = p$

$\therefore$ $(n)\,(5.14 \times 10^{-23}) = 1.01 \times 10^5$

or $n = 1.965 \times 10^{27}$ per second **Ans.**

(b) When the vessel is stopped, the ordered motion of the vessel converts into disordered motion and temperature of the gas is increased.

$$\therefore \qquad \frac{1}{2} m v_0^2 = \Delta U \qquad \text{...(i)}$$

$$U = \frac{5}{2} RT \qquad \text{(for 1 g mole of O}_2\text{)}$$

$$\therefore \qquad \Delta U = \frac{5}{2} R \Delta T$$

Here, m is not the mass of one gas molecule but it is the mass of the whole gas.

$$m = \text{ mass of 1 mol} = 32 \times 10^{-3}\ \text{kg}$$

Substituting these values in Eq. (i), we get

$$v_0 = \sqrt{\frac{5R\Delta T}{m}} = \sqrt{\frac{5 \times 8.31 \times 1}{32 \times 10^{-3}}} = 36\ \text{m/s}$$ **Ans.**

Miscellaneous Examples

Example 5 *An ideal diatomic gas with* $C_V = \dfrac{5R}{2}$ *occupies a volume* V_i *at a pressure* p_i. *The gas undergoes a process in which the pressure is proportional to the volume. At the end of the process, it is found that the rms speed of the gas molecules has doubled from its initial value. Determine the amount of energy transferred to the gas by heat.*

Solution Given that, $p \propto V$ or $pV^{-1} = \text{constant}$

As we know, molar heat capacity in the process $pV^x = \text{constant}$ is

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - x} = C_V + \frac{R}{1 - x}$$

In the given problem,

$$C_V = \frac{5R}{2} \quad \text{and} \quad x = -1$$

$$\therefore \qquad C = \frac{5R}{2} + \frac{R}{2} = 3R \qquad \text{...(i)}$$

At the end of the process rms speed is doubled, i.e. temperature has become four times ($v_{\text{rms}} \propto \sqrt{T}$).

Now, $$\Delta Q = nC\Delta T$$
$$= nC\,(T_f - T_i)$$
$$= nC\,(4T_i - T_i)$$
$$= 3T_i nC$$
$$= (3T_i)\,(n)\,(3R)$$
$$= 9\,(nRT_i)$$

or $$\Delta Q = 9P_iV_i$$ **Ans.**

Example 6 *Given, Avogadro's number $N = 6.02 \times 10^{23}$ and Boltzmann's constant $k = 1.38 \times 10^{-23}$ J/K.*

(a) Calculate the average kinetic energy of translation of the molecules of an ideal gas at $0°C$ and at $100°C$.

(b) Also calculate the corresponding energies per mole of the gas.

Solution (a) According to the kinetic theory, the average kinetic energy of translation per molecule of an ideal gas at kelvin temperature T is $\left(\frac{3}{2}\right)kT$, where k is Boltzmann's constant.

At $0°$ C ($T = 273$ K), the kinetic energy of translation $= \frac{3}{2}\,kT$

$$= \frac{3}{2} \times (1.38 \times 10^{-23}) \times 273 = 5.65 \times 10^{-21} \text{ J/mole}$$

At $100°$ C ($T = 373$ K), the energy is

$$\frac{3}{2} \times (1.38 \times 10^{-23}) \times 373 = 7.72\ \times 10^{-21} \text{ J/mole}$$

(b) 1 mole of gas contains $N\ (= 6.02 \times 10^{23})$ molecules. Therefore, at $0°$ C, the kinetic energy of translation of 1 mole of the gas is

$$= (5.65 \times 10^{-21})\,(6.02 \times 10^{23}) \approx 3401 \text{ J/mol}$$

and at $100°$ C

kinetic energy of translation of 1 mole of gas is

$$= (7.72 \times 10^{-21})\,(6.02 \times 10^{23}) \approx 4647 \text{ J/mol}$$

Example 7 *An air bubble starts rising from the bottom of a lake. Its diameter is 3.6 mm at the bottom and 4 mm at the surface. The depth of the lake is 250 cm and the temperature at the surface is $40°C$. What is the temperature at the bottom of the lake? Given atmospheric pressure = 76 cm of Hg and $g = 980\ cm/s^2$.*

Solution At the bottom of the lake, volume of the bubble

$$V_1 = \frac{4}{3}\pi r_1^3 = \frac{4}{3}\pi (0.18)^3 \text{ cm}^3$$

Pressure on the bubble p_1 = Atmospheric pressure + Pressure due to a column of 250 cm of water

$$= 76 \times 13.6 \times 980 + 250 \times 1 \times 980$$
$$= (76 \times 13.6 + 250)\ 980 \text{ dyne/cm}^2$$

At the surface of the lake, volume of the bubble

$$V_2 = \frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi\ (0.2)^3\ \text{cm}^3$$

Pressure on the bubble, $p_2 =$ atmospheric pressure

$$= (76 \times 13.6 \times 980)\ \text{dyne/cm}^2$$

$$T_2 = 273 + 40°\text{C}$$

$$= 313°\ \text{K}$$

Now

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

or

$$\frac{(76 \times 13.6 + 250)\,980 \times \left(\frac{4}{3}\right)\pi\,(0.18)^3}{T_1} = \frac{(76 \times 13.6) \times 980\left(\frac{4}{3}\right)\pi\,(0.2)^3}{313}$$

or

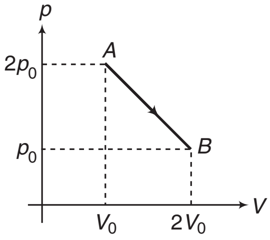
$$T_1 = 283.37\ \text{K}$$

∴

$$T_1 = 283.37 - 273$$

$$= 10.37°\text{C}$$

Example 8 *p-V diagram of n moles of an ideal gas is as shown in figure. Find the maximum temperature between A and B.*



How to Proceed For given number of moles of a gas,

$$T \propto pV \qquad (pV = nRT)$$

Although $(pV)_A = (pV)_B = 2p_0V_0$ or $T_A = T_B$, yet it is not an isothermal process. Because in isothermal process p-V graph is a rectangular hyperbola while it is a straight line. So, to see the behaviour of temperature, first we will find either T-V equation or T-p equation and from that equation we can judge how the temperature varies. From the graph, first we will write p-V equation, then we will convert it either in T-V equation or in T-p equation with the help of equation, $pV = nRT$.

Solution From the graph the p-V equation can be written as

$$p = -\left(\frac{p_0}{V_0}\right)V + 3p_0 \qquad (y = -mx + c)$$

or

$$pV = -\left(\frac{p_0}{V_0}\right)V^2 + 3p_0V$$

or

$$nRT = 3p_0V - \left(\frac{p_0}{V_0}\right)V^2 \qquad (\text{as } pV = nRT)$$

or

$$T = \frac{1}{nR}\left[3p_0V - \left(\frac{p_0}{V_0}\right)V^2\right]$$

This is the required T-V equation. This is quadratic in V. Hence, T-V graph is a parabola. Now, to find maximum or minimum value of T, we can substitute.

$$\frac{dT}{dV}=0$$

or
$$3p_0-\left(\frac{2p_0}{V_0}\right)V=0 \quad \text{or} \quad V=\frac{3}{2}V_0$$

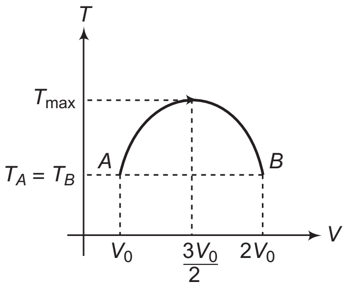
Further $\frac{d^2T}{dV^2}$ is negative at $V=\frac{3}{2}V_0$.

Hence, T is maximum at $V=\frac{3}{2}V_0$ and this maximum value is

$$T_{\text{max}}=\frac{1}{nR}\left[(3p_0)\left(\frac{3V_0}{2}\right)-\left(\frac{p_0}{V_0}\right)\left(\frac{3V_0}{2}\right)^2\right]$$

or
$$T_{\text{max}}=\frac{9p_0V_0}{4nR}$$ **Ans.**

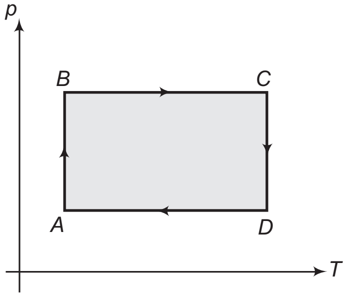
Thus, T-V graph is as shown in figure.



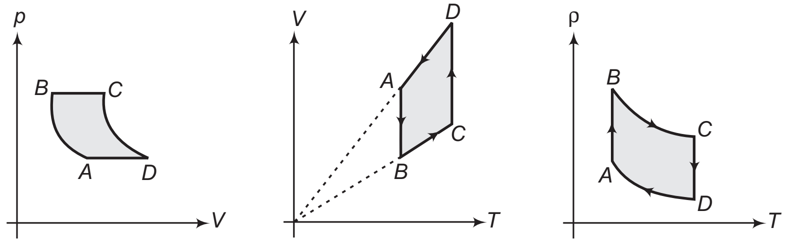
$$T_A=T_B=\frac{2p_0V_0}{nR} \quad \text{and} \quad T_{\text{max}}=\frac{9p_0V_0}{4nR}$$
$$=2.25\,\frac{p_0V_0}{nR}$$

Note *Most of the problems of T_{max}, p_{max} and V_{max} are solved by differentiation. Sometimes graph will be given and sometimes direct equation will be given. For p_{max} you will require either p-V or p-T equation.*

Example 9 *Plot p-V, V-T and ρ-T graph corresponding to the p-T graph for an ideal gas shown in figure.*



Solution ***Process AB*** *is an isothermal process with* T = constant *and* $p_B > p_A$.



p-V graph : $p \propto \dfrac{1}{V}$ i.e. p-V graph is a rectangular hyperbola with $p_B > p_A$

and $$V_B < V_A.$$

V-T graph : T = constant. Therefore, V-T graph is a straight line parallel to V-axis with $V_B < V_A$.

ρ-T graph : $\rho = \dfrac{pM}{RT}$

or $$\rho \propto p$$

As T is constant. Therefore, ρ-T graph is a straight line parallel to ρ-axis with $\rho_B > \rho_A$ as $p_B > p_A$.

Process BC is an isobaric process with P = constant

and $$T_C > T_B.$$

p-V graph : As p is constant. Therefore, p-V graph is a straight line parallel to V-axis with $V_C > V_B$ (because $V \propto T$ in an isobaric process)

V-T graph : In isobaric process $V \propto T$, i.e. V-T graph is a straight line passing through origin, with $T_C > T_B$

and $$V_C > V_B.$$

ρ-T graph : $\rho \propto \dfrac{1}{T}$ (when P = constant), i.e. ρ-T graph is a hyperbola with $T_C > T_B$

and $$\rho_C < \rho_B$$

There is no need of discussing C-D and D-A processes. As they are opposite to AB and BC respectively. The corresponding three graphs are shown above.

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Straight line on p-T graph for an ideal gas represents isochoric process.

Reason : If $p \propto T, V =$ constant.

2. Assertion : Vibrational kinetic energy is insignificant at low temperatures.

Reason : Interatomic forces are responsible for vibrational kinetic energy.

3. Assertion : In the formula $p = \frac{2}{3} E$, the term E represents translational kinetic energy per unit volume of gas.

Reason : In case of monoatomic gas, translational kinetic energy and total kinetic energy are equal.

4. Assertion : If a gas container is placed in a moving train, the temperature of gas will increase.

Reason : Kinetic energy of gas molecules will increase.

5. Assertion : According to the law of equipartition of energy, internal energy of an ideal gas at a given temperature, is equally distributed in translational and rotational kinetic energies.

Reason : Rotational kinetic energy of a monoatomic gas is zero

6. Assertion : Real gases behave as ideal gases most closely at low pressure and high temperature.

Reason : Intermolecular force between ideal gas molecules is assumed to be zero.

7. Assertion : A glass of water is filled at 4°C. Water will overflow, if temperature is increased or decreased. (Ignore expansion of glass).

Reason : Density of water is minimum at 4°C.

8. Assertion : If pressure of an ideal gas is doubled and volume is halved, then its internal energy will remain unchanged.

Reason : Internal energy of an ideal gas is a function of only temperature.

9. Assertion : In equation $p = \frac{1}{3} \alpha\, v_{\text{rms}}^2$, the term α represents density of gas.

Reason : $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$.

10. **Assertion :** In isobaric process, V-T graph is a straight line passing through origin. Slope of this line is directly proportional to mass of the gas. V is taken on y-axis.

Reason : $V = \left(\frac{nR}{p}\right)T$

$\therefore$ slope $\propto n$

or slope $\propto m$

Objective Questions

1. The average velocity of molecules of a gas of molecular weight M at temperature T is

(a) $\sqrt{\frac{3RT}{M}}$ (b) $\sqrt{\frac{8RT}{\pi M}}$

(c) $\sqrt{\frac{2RT}{M}}$ (d) zero

2. Four particles have velocities 1, 0, 2 and 3 m/s. The root mean square velocity of the particles (definition wise) is

(a) 3.5 m/s (b) $\sqrt{3.5}$ m/s

(c) 1.5 m/s (d) $\sqrt{\frac{14}{3}}$ m/s

3. The temperature of an ideal gas is increased from 27°C to 927°C. The rms speed of its molecules becomes

(a) twice (b) half

(c) four times (d) one-fourth

4. In case of hydrogen and oxygen at NTP, which of the following is the same for both?

(a) Average linear momentum per molecule (b) Average KE per molecule

(c) KE per unit volume (d) KE per unit mass

5. The average kinetic energy of the molecules of an ideal gas at 10°C has the value E. The temperature at which the kinetic energy of the same gas becomes $2E$ is

(a) 5°C (b) 10°C

(c) 40°C (d) None of these

6. A polyatomic gas with n degrees of freedom has a mean energy per molecule given by

(a) $\frac{n}{2}RT$ (b) $\frac{1}{2}RT$

(c) $\frac{n}{2}kT$ (d) $\frac{1}{2}kT$

7. In a process, the pressure of a gas remains constant. If the temperature is doubled, then the change in the volume will be

(a) 100% (b) 200%

(c) 50% (d) 25%

8. A steel rod of length 1 m is heated from 25° to 75°C keeping its length constant. The longitudinal strain developed in the rod is (Given, coefficient of linear expansion of steel $= 12 \times 10^{-6}/{}^\circ\mathrm{C}$)

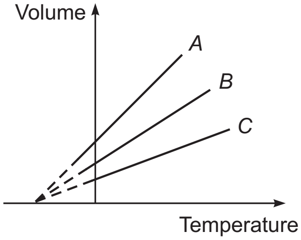
(a) 6×10^{-4} (b) -6×10^{-5}

(c) -6×10^{-4} (d) zero

9. The coefficient of linear expansion of steel and brass are $11 \times 10^{-6}/°C$ and $19 \times 10^{-6}/°C$, respectively. If their difference in lengths at all temperatures has to kept constant at 30 cm, their lengths at 0°C should be

(a) 71.25 cm and 41.25 cm
(b) 82 cm and 52 cm
(c) 92 cm and 62 cm
(d) 62.25 cm and 32.25 cm

10. The expansion of an ideal gas of mass m at a constant pressure p is given by the straight line B. Then, the expansion of the same ideal gas of mass $2m$ at a pressure $2p$ is given by the straight line



(a) C
(b) A
(c) B
(d) data insufficient

Subjective Questions

1. Change each of the given temperatures to the Celsius and Kelvin scales: 68° F, 5° F and 176° F.

2. Change each of the given temperatures to the Fahrenheit and Reaumur scales: 30° C, 5° C and –20° C.

3. At what temperature do the Celsius and Fahrenheit readings have the same numerical value?

4. You work in a materials testing lab and your boss tells you to increase the temperature of a sample by 40.0° C. The only thermometer you can find at your workbench reads in °F. If the initial temperature of the sample is 68.2 °F. What is its temperature in °F, when the desired temperature increase has been achieved?

5. The steam point and the ice point of a mercury thermometer are marked as 80° and 20°. What will be the temperature in centigrade mercury scale when this thermometer reads 32°?

6. A platinum resistance thermometer reads 0° C when its resistance is $80\,\Omega$ and 100° C when its resistance is $90\,\Omega$. Find the temperature at which the resistance is $86\,\Omega$.

7. The steam point and the ice point of a mercury thermometer are marked as 80° and 10°. At what temperature on centigrade scale the reading of this thermometer will be 59°?

8. Find the temperature at which oxygen molecules would have the same rms speed as of hydrogen molecules at 300 K.

9. Find the mass (in kilogram) of an ammonia molecule NH_3.

10. Three moles of an ideal gas having $\gamma = 1.67$ are mixed with 2 moles of another ideal gas having $\gamma = 1.4$. Find the equivalent value of γ for the mixture.

11. How many degrees of freedom have the gas molecules, if under standard conditions the gas density is $\rho = 1.3\ \text{kg/m}^3$ and velocity of sound propagation on it is $v = 330$ m/s?

12. 4 g hydrogen is mixed with 11.2 litre of He at STP in a container of volume 20 litre. If the final temperature is 300 K, find the pressure.

13. One mole of an ideal monoatomic gas is taken at a temperature of 300 K. Its volume is doubled keeping its pressure constant. Find the change in internal energy.

14. Two perfect monoatomic gases at absolute temperatures T_1 and T_2 are mixed. There is no loss of energy. Find the temperature of the mixture if the number of moles in the gases are n_1 and n_2.

15. If the water molecules in 1.0 g of water were distributed uniformly over the surface of earth, how many such molecules would there be in 1.0 cm^2 of earth's surface?

16. If the kinetic energy of the molecules in 5 litres of helium at 2 atm is E. What is the kinetic energy of molecules in 15 litres of oxygen at 3 atm in terms of E ?

17. At what temperature is the "effective" speed of gaseous hydrogen molecules (molecular weight = 2) equal to that of oxygen molecules (molecular weight = 32) at 47° C ?

18. At what temperature is v_{rms} of H_2 molecules equal to the escape speed from earth's surface. What is the corresponding temperature for escape of hydrogen from moon's surface? Given $g_m = 1.6\ \text{m/s}^2$, $R_e = 6367$ km and $R_m = 1750$ km.

19. The pressure of the gas in a constant volume gas thermometer is 80 cm of mercury in melting ice. When the bulb is placed in a liquid, the pressure becomes 160 cm of mercury. Find the temperature of the liquid.

20. The resistances of a platinum resistance thermometer at the ice point, the steam point and the boiling point of sulphur are 2.50, 3.50 and 6.50 Ω respectively. Find the boiling point of sulphur on the platinum scale. The ice point and the steam point measure 0° and 100°, respectively.

21. In a constant volume gas thermometer, the pressure of the working gas is measured by the difference in the levels of mercury in the two arms of a U-tube connected to the gas at one end . When the bulb is placed at the room temperature 27.0 °C, the mercury column in the arm open to atmosphere stands 5.00 cm above the level of mercury in the other arm. When the bulb is placed in a hot liquid, the difference of mercury levels becomes 45.0 cm. Calculate the temperature of the liquid.(Atmospheric pressure = 75.0 cm of mercury.)

22. A steel wire of 2.0 mm^2cross-section is held straight (but under no tension) by attaching it firmly to two points a distance 1.50 m apart at 30 ° C. If the temperature now decreases to –10° C and if the two points remain fixed, what will be the tension in the wire? For steel, $Y = 20{,}0000$ MPa.

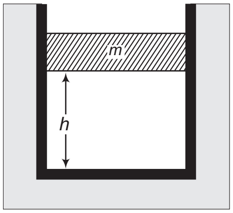
23. A metallic bob weighs 50 g in air. If it is immersed in a liquid at a temperature of 25° C, it weighs 45 g. When the temperature of the liquid is raised to 100° C, it weighs 45.1 g. Calculate the coefficient of cubical expansion of the liquid. Given that coefficient of cubical expansion of the metal is $12 \times 10^{-6}\ {}^\circ\text{C}^{-1}$.

24. An ideal gas exerts a pressure of 1.52 MPa when its temperature is 298.15 K and its volume is 10^{-2}m^3. (a) How many moles of gas are there? (b) What is the mass density if the gas is molecular hydrogen? (c) What is the mass density if the gas is oxygen?

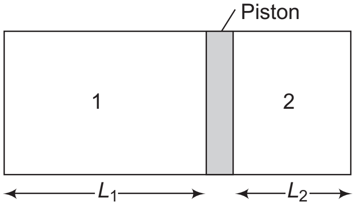
25. A compressor pumps 70 L of air into a 6 L tank with the temperature remaining unchanged. If all the air is originally at 1 atm. What is the final absolute pressure of the air in the tank?

26. A partially inflated balloon contains 500 m^3 of helium at 27° C and 1 atm pressure. What is the volume of the helium at an altitude of 18000 ft, where the pressure is 0.5 atm and the temperature is –3° C?

27. A cylinder whose inside diameter is 4.00 cm contains air compressed by a piston of mass $m = 13.0$ kg which can slide freely in the cylinder. The entire arrangement is immersed in a water bath whose temperature can be controlled. The system is initially in equilibrium at temperature $t_i = 20°$ C. The initial height of the piston above the bottom of the cylinder is $h_i = 4.00$ cm. The temperature of the water bath is gradually increased to a final temperature $t_f = 100°$ C. Calculate the final height h_f of the piston.



28. The closed cylinder shown in figure has a freely moving piston separating chambers 1 and 2. Chamber 1 contains 25 mg of N_2 gas and chamber 2 contains 40 mg of helium gas. When equilibrium is established what will be the ratio L_1 / L_2? What is the ratio of the number of moles of N_2 to the number of moles of He? (Molecular weights of N_2 and He are 28 and 4).



29. Two gases occupy two containers *A* and *B*. The gas in *A* of volume 0.11 m^3 exerts a pressure of 1.38 MPa. The gas in *B* of volume 0.16 m^3 exerts a pressure of 0.69 MPa. Two containers are united by a tube of negligible volume and the gases are allowed to intermingle. What is the final pressure in the container if the temperature remains constant ?

30. A glass bulb of volume 400 cm^3 is connected to another of volume 20 cm^3 by means of a tube of negligible volume. The bulbs contain dry air and are both at a common temperature and pressure of 20°C and 1.00 atm. The larger bulb is immersed in steam at 100°C and the smaller in melting ice at 0°C. Find the final common pressure.

31. The condition called standard temperature and pressure (STP) for a gas is defined as temperature of $0°$ C = 273.15 K and a pressure of 1 atm = 1.013×10^5 Pa. If you want to keep a mole of an ideal gas in your room at STP, how big a container do you need?

32. A large cylindrical tank contains 0.750 m^3 of nitrogen gas at 27° C and 1.50×10^5 Pa (absolute pressure). The tank has a tightfitting piston that allows the volume to be changed. What will be the pressure if the volume is decreased to 0.480 m^3 and the temperature is increased to 157°C.

33. A vessel of volume 5 litres contains 1.4 g of N_2 and 0.4 g of He at 1500 K. If 30% of the nitrogen molecules are dissociated into atoms then find the gas pressure.

34. Temperature of diatomic gas is 300 K. If moment of inertia of its molecules is 8.28×10^{-38} g-cm^2. Calculate their root mean square angular velocity.

35. Find the number of degrees of freedom of molecules in a gas. Whose molar heat capacity
(a) at constant pressure $C_p = 29 \text{ J mol}^{-1}\text{K}^{-1}$
(b) $C = 29 \text{ J mol}^{-1}\text{K}^{-1}$ in the process $pT =$ constant.

36. In a certain gas $\frac{2}{5}$th of the energy of molecules is associated with the rotation of molecules and the rest of it is associated with the motion of the centre of mass.

(a) What is the average translational energy of one such molecule when the temperature is 27°C?

(b) How much energy must be supplied to one mole of this gas at constant volume to raise the temperature by 1°C?

37. A mixture contains 1 mole of helium ($C_p = 2.5\,R, C_V = 1.5\,R$) and 1 mole of hydrogen ($C_p = 3.5R, C_V = 2.5\,R$). Calculate the values of C_p, C_V and γ for the mixture.

38. An ideal gas ($C_p / C_V = \gamma$) is taken through a process in which the pressure and the volume vary as $P = aV^b$. Find the value of b for which the specific heat capacity in the process is zero.

39. An ideal gas is taken through a process in which the pressure and the volume are changed according to the equation $p = kV$. Show that the molar heat capacity of the gas for the process is given by $C = C_V + \frac{R}{2}$.

40. The pressure of a gas in a 100 mL container is 200 kPa and the average translational kinetic energy of each gas particle is 6×10^{-21} J. Find the number of gas particles in the container. How many moles are there in the container?

41. One gram mole NO_2 at 57° C and 2 atm pressure is kept in a vessel. Assuming the molecules to be moving with rms velocity. Find the number of collisions per second which the molecules make with one square metre area of the vessel wall.

42. A 2.00 mL volume container contains 50 mg of gas at a pressure of 100 kPa. The mass of each gas particle is 8.0×10^{-26} kg. Find the average translational kinetic energy of each particle.

43. Call the rms speed of the molecules in an ideal gas v_0 at temperature T_0 and pressure p_0. Find the speed if (a) the temperature is raised from $T_0 = 293$ K to 573 K (b) the pressure is doubled and $T = T_0$ (c) the molecular weight of each of the gas molecules is tripled.

44. (a) What is the average translational kinetic energy of a molecule of an ideal gas at temperature of 27° C?

(b) What is the total random translational kinetic energy of the molecules in one mole of this gas?

(c) What is the rms speed of oxygen molecules at this temperature?

45. At 0°C and 1.0 atm ($= 1.01 \times 10^5$ N/m^2) pressure the densities of air, oxygen and nitrogen are 1.284 kg/m^3, 1.429 kg/m^3 and 1.251 kg/m^3 respectively. Calculate the percentage of nitrogen in the air from these data, assuming only these two gases to be present.

46. An air bubble of 20 cm^3 volume is at the bottom of a lake 40 m deep where the temperature is 4°C. The bubble rises to the surface which is at a temperature of 20°C. Take the temperature to be the same as that of the surrounding water and find its volume just before it reaches the surface.

47. For a certain gas the heat capacity at constant pressure is greater than that at constant volume by 29.1 J/K.

(a) How many moles of the gas are there?

(b) If the gas is monatomic, what are heat capacities at constant volume and pressure?

(c) If the gas molecules are diatomic which rotate but do not vibrate, what are heat capacities at constant volume and at constant pressure.

48. The heat capacity at constant volume of a sample of a monatomic gas is 35 J/K. Find
(a) the number of moles
(b) the internal energy at 0°C
(c) the molar heat capacity at constant pressure.

LEVEL 2

Single Correct Option

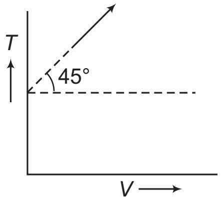
1. Two thermally insulated vessels 1 and 2 are filled with air at temperatures (T_1, T_2), volumes (V_1, V_2) and pressures (p_1, p_2) respectively. If the valve joining the two vessels is opened, the temperature inside the vessel at equilibrium will be (P = common pressure)

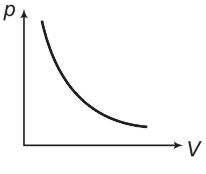
(a) $T_1 + T_2$
(b) $(T_1 + T_2)/2$
(c) $\dfrac{T_1T_2p(V_1 + V_2)}{p_1V_1T_2 + p_2V_2T_1}$
(d) $\dfrac{T_1T_2(p_1V_1 + p_2V_2)}{p_1V_1T_2 + p_2V_2T_1}$

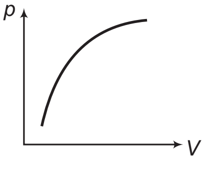
2. Two marks on a glass rod 10 cm apart are found to increase their distance by 0.08 mm when the rod is heated from 0°C to 100°C. A flask made of the same glass as that of rod measures a volume of 100 cc at 0°C. The volume it measures at 100°C in cc is

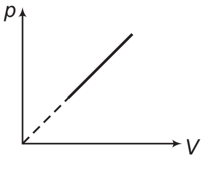
(a) 100.24
(b) 100.12
(c) 100.36
(d) 100.48

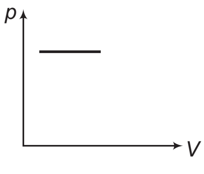
3. The given curve represents the variation of temperature as a function of volume for one mole of an ideal gas. Which of the following curves best represents the variation of pressure as a function of volume?



(a)


(b)


(c)


(d)


4. A gas is found to be obeyed the law p^2V = constant. The initial temperature and volume are T_0 and V_0. If the gas expands to a volume $3V_0$, then the final temperature becomes

(a) $\sqrt{3}\,T_0$
(b) $\sqrt{2}\,T_0$
(c) $\dfrac{T_0}{\sqrt{3}}$
(d) $\dfrac{T_0}{\sqrt{2}}$

5. Air fills a room in winter at 7°C and in summer at 37°C. If the pressure is the same in winter and summer, the ratio of the weight of the air filled in winter and that in summer is

(a) 2.2
(b) 1.75
(c) 1.1
(d) 3.3

6. Three closed vessels A, B and C are at the same temperature T and contain gases which obey Maxwell distribution law of velocities. Vessel A contains O_2, B only N_2 and C mixture of equal quantities of O_2 and N_2. If the average speed of the O_2 molecules in vessel A is v_1 that of N_2 molecules in vessel B is v_2, then the average speed of the O_2 molecules in vessel C is

(a) $\frac{(v_1 + v_2)}{2}$ (b) v_1

(c) $\sqrt{v_1 v_2}$ (d) None of these

7. In a very good vacuum system in the laboratory, the vacuum attained was 10^{-13} atm. If the temperature of the system was 300 K, the number of molecules present in a volume of 1 cm^3 is

(a) 2.4×10^6 (b) 24

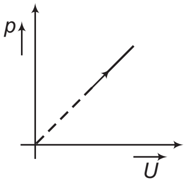
(c) 2.4×10^9 (d) zero

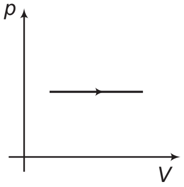
8. If nitrogen gas molecule goes straight up with its rms speed at 0°C from the surface of the earth and there are no collisions with other molecules, then it will rise to an approximate height of

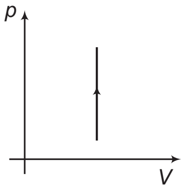
(a) 8 km (b) 12 km

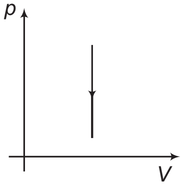
(c) 12 m (d) 8 m

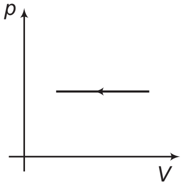
9. The given p-U graph shows the variation of internal energy of an ideal gas with increase in pressure. Which of the following pressure-volume graph is equivalent to this graph?



(a)


(b)


(c)


(d)


10. 28 g of N_2 gas is contained in a flask at a pressure of 10 atm and at a temperature of 57°C. It is found that due to leakage in the flask, the pressure is reduced to half and the temperature to 27°C. The quantity of N_2 gas that leaked out is

(a) 11/20 g (b) 20/11 g

(c) 5/63 g (d) 63/5 g

11. A mixture of 4 g of hydrogen and 8 g of helium at NTP has a density about

(a) 0.22 kg/m^3 (b) 0.62 kg/m^3

(c) 1.12 kg/m^3 (d) 0.13 kg/m^3

12. The pressure (p) and the density (ρ) of given mass of a gas expressed by Boyle's law, $p = K\rho$ holds true

(a) for any gas under any condition

(b) for same gas under any condition

(c) only if the temperature is kept constant

(d) None of the above

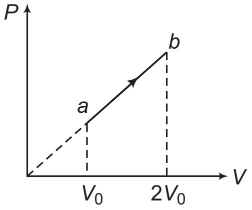
More than One Correct Options

1. During an experiment, an ideal gas is found to obey a condition $\frac{p^2}{\rho}$ = constant. (ρ = density of the gas). The gas is initially at temperature T, pressure p and density ρ. The gas expands such that density changes to ρ/2.
(a) The pressure of the gas changes to $\sqrt{2}\ p$
(b) The temperature of the gas changes to $\sqrt{2}\ T$
(c) The graph of the above process on p-T diagram is parabola
(d) The graph of the above process on p-T diagram is hyperbola

2. During an experiment, an ideal gas is found to obey a condition Vp^2 = constant. The gas is initially at a temperature T, pressure p and volume V. The gas expands to volume $4V$.
(a) The pressure of gas changes to $\frac{p}{2}$
(b) The temperature of gas changes to $4T$
(c) The graph of the above process on p-T diagram is parabola
(d) The graph of the above process on p-T diagram is hyperbola

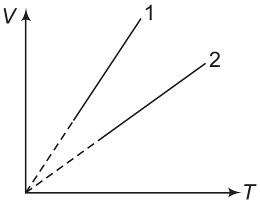
3. Find the correct options.
(a) Ice point in Fahrenheit scale is 32°F
(b) Ice point in Fahrenheit scale is 98.8°F
(c) Steam point in Fahrenheit scale is 212°F
(d) Steam point in Fahrenheit scale is 252°F

4. In the P-V diagram shown in figure, choose the correct options for the process a-b :
(a) density of gas has reduced to half
(b) temperature of gas has increased to two times
(c) internal energy of gas has increased to four times
(d) T-V graph is a parabola passing through origin



5. Choose the **wrong** options
(a) Translational kinetic energy of all ideal gases at same temperature is same
(b) In one degree of freedom all ideal gases has internal energy = $\frac{1}{2}RT$
(c) Translational degree of freedom of all ideal gases is three
(d) Translational kinetic energy of one mole of all ideal gases is $\frac{3}{2}RT$

6. Along the line-1, mass of gas is m_1 and pressure is p_1. Along the line-2 mass of same gas is m_2 and pressure is p_2. Choose the correct options.



(a) m_1 may be less than m_2
(b) m_2 may be less than m_1
(c) p_1 may be less than p_2
(d) p_2 may be less than p_1

7. Choose the correct options.

(a) In $p = \frac{m}{M} RT$, m is mass of gas per unit volume

(b) In $pV = \frac{m}{M} RT$, m is mass of one molecule of gas

(c) In $p = \frac{1}{3} \frac{mN}{V} v_{\text{rms}}^2$, m is total mass of gas.

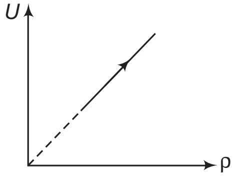
(d) In $v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$, m is mass of one molecule of gas

Match the Columns

1. Match the following two columns for 2 moles of an ideal diatomic gas at room temperature T.

Column I	Column II
(a) Translational kinetic energy	(p) $2RT$
(b) Rotational kinetic energy	(q) $4RT$
(c) Potential energy	(r) $3RT$
(d) Total internal energy	(s) None of these

2. In the graph shown, U is the internal energy of gas and ρ the density. Corresponding to given graph, match the following two columns.



Column I	Column II
(a) Pressure	(p) is constant
(b) Volume	(q) is increasing
(c) Temperature	(r) is decreasing
(d) Ratio T/V	(s) data insufficient

3. At a given temperature T,

$v_1 = \sqrt{\frac{x_1 RT}{M}}$ = rms speed of gas molecules, $v_2 = \sqrt{\frac{x_2 RT}{M}}$ = average speed of gas molecules

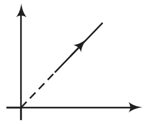
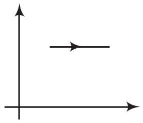
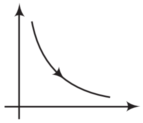
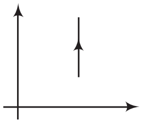
$v_3 = \sqrt{\frac{x_3 RT}{M}}$ = most probable speed of gas molecules, $v_4 = \sqrt{\frac{x_4 RT}{M}}$ = speed of sound

Column I	Column II
(a) x_1	(p) 1.5
(b) x_2	(q) 2.0
(c) x_3	(r) 3.0
(d) x_4	(s) data insufficient

4. With increase in temperature, match the following two columns.

Column I	Column II
(a) Density of water	(p) will increase
(b) Fraction of a solid floating in a liquid	(q) will decrease
(c) Apparent weight of a solid immersed in water	(r) will remain unchanged
(d) Time period of pendulum	(s) may increase or decrease

5. Corresponding to isobaric process, match the following two columns.

Column I	Column II
(a) p-T graph	(p) 
(b) U-ρ graph	(q) 
(c) T-V graph	(r) 
(d) T-ρ graph	(s) 

Note *First physical quantity is along y-axis.*

Subjective Questions

1. Show that the volume thermal expansion coefficient for an ideal gas at constant pressure is $\frac{1}{T}$.

2. The volume of a diatomic gas $(\gamma = 7/5)$ is increased two times in a polytropic process with molar heat capacity $C = R$. How many times will the rate of collision of molecules against the wall of the vessel be reduced as a result of this process?

3. A perfectly conducting vessel of volume $V = 0.4\ \text{m}^3$ contains an ideal gas at constant temperature $T = 273\ \text{K}$. A portion of the gas is let out and the pressure of the gas falls by $\Delta p = 0.24$ atm. (Density of the gas at STP is $\rho = 1.2\ \text{kg/m}^3$). Find the mass of the gas which escapes from the vessel.

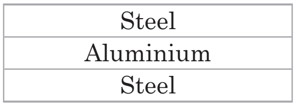
4. A thin-walled cylinder of mass m, height h and cross-sectional area A is filled with a gas and floats on the surface of water. As a result of leakage from the lower part of the cylinder, the depth of its submergence has increased by Δh. Find the initial pressure p_1 of the gas in the cylinder if the atmospheric pressure is p_0 and the temperature remains constant.

5. Find the minimum attainable pressure of an ideal gas in the process $T = T_0 + \alpha V^2$, where T_0 and α are positive constants and V is the volume of one mole of gas.

6. A solid body floats in a liquid at a temperature $t = 50°\text{C}$ being completely submerged in it. What percentage of the volume of the body is submerged in the liquid after it is cooled to $t_0 = 0°\text{C}$, if the coefficient of cubic expansion for the solid is $\gamma_s = 0.3 \times 10^{-5}\ °\text{C}^{-1}$ and of the liquid $\gamma_l = 8 \times 10^{-5}\ °\text{C}^{-1}$.

7. Two vessels connected by a pipe with a sliding plug contain mercury. In one vessel, the height of mercury column is 39.2 cm and its temperature is 0° C, while in the other, the height of mercury column is 40 cm and its temperature is 100° C. Find the coefficient of cubical expansion for mercury. The volume of the connecting pipe should be neglected.

8. Two steel rods and an aluminium rod of equal length l_0 and equal cross-section are joined rigidly at their ends as shown in the figure below. All the rods are in a state of zero tension at 0° C.



Find the length of the system when the temperature is raised to θ. Coefficient of linear expansion of aluminium and steel are α_a and α_s, respectively. Young's modulus of aluminium is Y_a and of steel is Y_s.

9. A metal rod A of 25 cm length expands by 0.050 cm when its temperature is raised from 0° C to 100° C. Another rod B of a different metal of length 40 cm expands by 0.040 cm for the same rise in temperature. A third rod C of 50 cm length is made up of pieces of rods A and B placed end to end expands by 0.03 cm on heating from 0° C to 50° C. Find the lengths of each portion of the composite rod.

Answers

Introductory Exercise 20.1

1. (a) -17.8°C (b) -459.67°F **2.** (a) 160°C (b) -24.6°C **3.** 122°F
4. 574.25 **5.** -40°C

Introductory Exercise 20.2

1. Gains, 15.55 s **2.** It will first increase and then decrease **3.** $(\gamma_2 - \gamma_1)\,\Delta T$
4. Cool the system **5.** 50.8°C **6.** (a) 0.064 cm (b) 88.48 cm
7. -0.042%

Introductory Exercise 20.3

1. $m_1 > m_2$ **2.** 12 atm **3.** 1.5 kg/m^3 **4.** 8×10^{15} **5.** $p_1 > p_2$
6. Straight line passing through origin **7.** A dot

Introductory Exercise 20.4

1. (d) **2.** (c) **3.** $\frac{3}{2}K_0,\ \frac{5}{2}K_0$

Introductory Exercise 20.5

1. 2.15 km/s **2.** $v_{\text{rms}} = 714$ m/s, $v_{\text{av}} = 700$ m/s
3. Speed is a scalar quantity while velocity is a vector quantity.
4. 6.21×10^{-21} J **5.** (a) 1368 m/s, 609 m/s (b) 6.21×10^{-21} J **6.** 160 K
7. True

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (b) **3.** (b) **4.** (d) **5.** (d) **6.** (b) **7.** (c) **8.** (b) **9.** (b) **10.** (a)

Objective Questions

1. (b) **2.** (b) **3.** (a) **4.** (b) **5.** (d) **6.** (c) **7.** (a) **8.** (c) **9.** (a) **10.** (c)

Subjective Questions

1. 20°C, -15°C, 80°C, 293K, 258K, 353K **2.** 86°F, 41°F, -4°F, 546°R, 501°R, 456°R
3. $-40^\circ\text{F} = -40^\circ\text{C}$ **4.** 140.2°F **5.** 20°C **6.** 60°C **7.** 70°C
8. 4800 K **9.** 2.82×10^{-26} kg **10.** 1.53 **11.** 5
12. 3.12×10^5 N/m^2 **13.** 450 R **14.** $T = \dfrac{n_1T_1 + n_2T_2}{n_1 + n_2}$ **15.** 6.5×10^3
16. 7.5 E **17.** -253°C **18.** $T_E = 10059$ K, $T_M = 449$ K **19.** 546.30 K **20.** 400°
21. 177.07°C **22.** 192 N **23.** 3.1×10^{-4} per $^\circ$C
24. (a) 6.135 mol (b) 1.24 kg/m^3 (c) 19.6 kg/m^3 **25.** 11.7 atm absolute pressure
26. 900 m^3 **27.** 5.09 cm **28.** 0.089, 0.089 **29.** 0.97 MPa

30. 1.13 atm **31.** 22.4 L **32.** 3.36×10^5 Pa **33.** 4.1×10^5 N/m^2 **34.** 10^{13} rad/s

35. (a) 5 (b) 3 **36.** (a) 6.21×10^{-21} J (b) 20.8 J **37.** $3R$, $2R$, 1.5

38. $-\gamma$ **40.** 5×10^{21}, 8.3×10^{-3} mol **41.** 3.1×10^{27} **42.** 4.8×10^{-22} J

43. (a) 1.40 v_0 (b) v_0 (c) 0.58 v_0 or $v_0/\sqrt{3}$ **44.** (a) 6.21×10^{-21} J (b) 3740 J (c) 484 m/s

45. 76.5% by mass **46.** 105 cm^3

47. (a) 3.5 mol (b) 43.65 J/K, 72.75 J/K (c) 72.75 J/K, 101.85 J/K

48. (a) 2.81 mol (b) 9.56 kJ (c) 20.8 J/mol·K

LEVEL 2

Single Correct Option

1.(c) **2.**(a) **3.**(a) **4.**(a) **5.**(c) **6.**(b) **7.**(a) **8.**(b) **9.**(b) **10.**(d)

11.(d) **12.**(c)

More than One Correct Options

1. (b,d) **2.** (a,d) **3.** (a,c) **4.** (a,c,d) **5.** (a,b) **6.** (a,b,c,d) **7.** (a,d)

Match the Columns

1. (a) → r (b) → p (c) → s (d) → s

2. (a) → q (b) → r (c) → q (d) → q

3. (a) → r (b) → s (c) → q (d) → s

4. (a) → s (b) → s (c) → s (d) → p

5. (a) → q (b) → r (c) → p (d) → r

Subjective Questions

2. $(2)^{4/3}$ times **3.** 115.2 g **4.** $p_1 = \left(p_0 + \frac{mg}{A}\right)\left(1 - \frac{\Delta h}{h}\right)$

5. $2R\sqrt{\alpha T_0}$ **6.** 99.99% **7.** 2.0×10^{-4} per °C

8. $l_0\left[1 + \left(\frac{\alpha_a Y_a + 2\alpha_s Y_s}{Y_a + 2Y_s}\right)\theta\right]$ **9.** 10 cm, 40 cm

21

Laws of Thermodynamics

Chapter Contents

21.1 The First Law of Thermodynamics

21.2 Further Explanation of Three Terms Used in First Law

21.3 Different Thermodynamic Processes

21.4 Heat Engine and its Efficiency

21.5 Refrigerator

21.6 Zeroth Law of Thermodynamics

21.7 Second Law of Thermodynamics

21.1 The First Law of Thermodynamics

The first law of thermodynamics is basically law of conservation of energy. This law can be applied for any type of system like solid, liquid and gas. But, in most of the cases the system will be an ideal gas. A process in which there are changes in the state of a thermodynamic system (like p, V, T, U and ρ etc.) is called a **thermodynamic process.**

We now come to the first law

Suppose Q heat is given to a system, then part of it is used by the system in doing work W against the surroundings (like atmosphere) and part is used by the system in increasing its internal energy ΔU. Thus,

$$\boxed{Q = W + \Delta U} \qquad \ldots\text{(i)}$$

Let us take a real life situation similar to first law.

Consider a person X. Suppose his monthly income is Rs. 50,000 (Q). He spends Rs. 30,000 (W) as his monthly expenditure. Then, obviously the remaining Rs. 20,000 goes to his savings (ΔU). In some month it is also possible that he spends more than his income. In that case he will withdraw it from his bank and his savings will get reduced ($\Delta U < 0$). In the similar manner, other combinations can be made.

Sign Convention

(i) ***Q*** If heat is given to the system, then Q is positive and if heat is taken from the system, then it is negative.

(ii) ***W*** Work done used in Eq. (i) is the work done by the system (not work done on the system). This work done is positive if volume of the system increases. Sign of work done in different situations is given in tabular form as below

Table 21.1

S.No	Volume of the system	Work done by the system used in Eq. (i)	Work done on the system
1.	is increasing	positive	negative
2.	is decreasing	negative	positive
3.	is constant	zero	zero

(iii) **ΔU** Internal energy of a system is due to disordered motion of its constituent particles. It mainly depends on state (solid, liquid and gas) and temperature. For example, two different states at same temperature will have different internal energy. Ice at 0°C and water at 0°C have different energies. Similarly, same state at different temperatures will have different energies. For example, internal energy of an ideal gas is given by

$$U = \frac{nf}{2} RT$$

or

$$U \propto T$$

If temperature increases, internal energy also increases and $\Delta U = U_f - U_i$ will be positive.

Extra Points to Remember

- If a thermodynamic system changes from an initial equilibrium state A to final equilibrium state B through three different paths 1, 2 and 3, then Q and W will be different along these three paths. But $Q - W$ or ΔU will be same along all three paths. This is because, Q and W are path functions. But U is a state function. Thus,

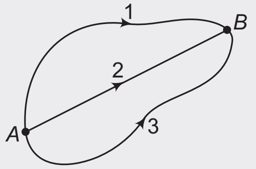


Fig. 21.1

$$\boxed{Q_1 - W_1 = Q_2 - W_2 = Q_3 - W_3 = \Delta U}$$

In a closed path, $\Delta U = 0$

Example 21.1 *When a system goes from state A to state B, it is supplied with 400 J of heat and it does 100 J of work.*

(a) For this transition, what is the system's change in internal energy?

(b) If the system moves from B to A, what is the change in internal energy?

(c) If in moving from A to B along a different path in which $W'_{AB} = 400$ J of work is done on the system, how much heat does it absorb?

Solution (a) From the first law,

$$\Delta U_{AB} = Q_{AB} - W_{AB} = (400 - 100)\text{ J} = 300\text{ J}$$

(b) Consider a closed path that passes through the state A and B. Internal energy is a state function so ΔU is zero for a closed path.

Thus, $\quad \Delta U = \Delta U_{AB} + \Delta U_{BA} = 0 \quad$ or $\quad \Delta U_{BA} = -\Delta U_{AB} = -300$ J

(c) The change in internal energy is the same for any path, so

$$\Delta U_{AB} = \Delta U'_{AB} = Q'_{AB} - W'_{AB}$$

$$300\text{ J} = Q'_{AB} - (-400\text{ J})$$

and the heat exchanged is $\quad Q'_{AB} = -100$ J

The negative sign indicates that the system loses heat in this transition.

INTRODUCTORY EXERCISE 21.1

1. The quantities in the following table represent four different paths for the same initial and final states. Find *a, b, c, d, e, f* and *g.*

Table 21.2

Q(J)	W(J)	ΔU(J)
–80	–120	*d*
90	*c*	*e*
a	40	*f*
b	–40	*g*

2. In a certain chemical process, a lab technician supplies 254 J of heat to a system. At the same time, 73 J of work are done on the system by its surroundings. What is the increase in the internal energy of the system?

21.2 Further Explanation of Three Terms Used in First Law

First law of thermodynamics basically revolves round the three terms Q, ΔU and W. If you substitute these three terms correctly with proper signs in the equation $Q = \Delta U + W$, then you are able to solve most of the problems of first law. Let us take each term one by one. Here, we are taking the system an ideal gas.

(i) **Heat Transfer (Q or ΔQ)** There are two methods of finding Q or ΔQ.

***Method* 1.**

$$\boxed{Q = nC\Delta T}$$

or

$$\boxed{\Delta Q = nC\Delta T}$$

where, C is the molar heat capacity of the gas and n is the number of moles of the gas. Always take,

$$\boxed{\Delta T = T_f - T_i}$$

where, T_f is the final temperature and T_i the initial temperature of the gas. Further, we have discussed in chapter 20, that molar heat capacity of an ideal gas in the process $pV^x =$ constant is

$$\boxed{C = \frac{R}{\gamma - 1} + \frac{R}{1 - x} = C_V + \frac{R}{1 - x}}$$

$C = C_V = \dfrac{R}{\gamma - 1}$ in isochoric process and

$C = C_p = C_V + R$ in isobaric process

Mostly C_p and C_V are used.

Note *For finding C, nature of gas and process should be known. Nature of gas will give us C_V and process the value of x.*

***Method* 2.** We can also find Q by finding ΔU and W by the equation

$$Q = W + \Delta U$$

(ii) **Change in internal energy (ΔU)** There are two methods of finding ΔU.

***Method* 1.** For change in internal energy of the gas

$$\Delta U = nC_V \Delta T$$

Students are often confused that the result $\Delta U = nC_V \Delta T$ can be applied only in case of an isochoric process (as C_V is here used). However, it is not so. It can be applied in any process, whether it is isobaric, isothermal, adiabatic or else.

Note *In the above expression, value of C_V depends on the nature of gas.*

***Method* 2.** We can also find ΔU from the basic equation of first law of thermodynamics,

$$\Delta U = Q - W$$

(iii) **Work done (W)** This is the most important of the three.

Work Done During Volume Changes

A gas in a cylinder with a movable piston is a simple example of a thermodynamic system.

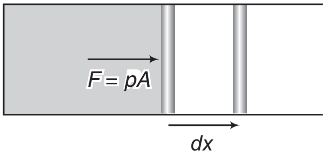


Fig. 21.2

Figure shows a gas confined to a cylinder that has a movable piston at one end. If the gas expands against the piston, it exerts a force and does work on the piston. If the piston compresses the gas as it is moved inward, work is done on the gas. The work associated with such volume changes can be determined as follows.

Let the gas pressure on the piston face be p. Then, the force on the piston due to the gas is pA, where A is the area of the face.

When the piston is pushed outward an infinitesimal distance dx, the work done by the gas is

$$dW = Fdx = pA\,dx$$

which, since the change in volume of the gas is $dV = Adx$, becomes

$$\boxed{dW = pdV}$$

For a finite change in volume from V_i to V_f, this equation is then integrated between V_i to V_f to find the net work

$$\boxed{W = \int dW = \int_{V_i}^{V_f} pdV}$$

Now, there are five methods of finding work done by a gas.

***Method* 1.** This is used when p - V equation is known to us. Suppose p as a function of V is known to us.

$$p = f(V)$$

then work done can be found by

$$W = \int_{V_i}^{V_f} f(V)\,dV$$

***Method* 2.** The work done by a gas is also equal to the area under p - V graph. Following different cases are possible :

Case 1 When volume is constant

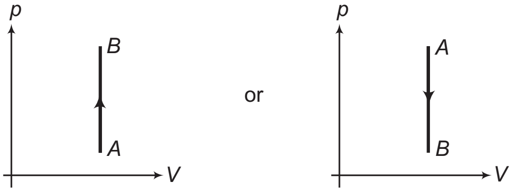


Fig. 21.3

$$V = \text{constant}$$

$$\therefore \qquad W_{AB} = 0$$

Case 2 When volume is increasing

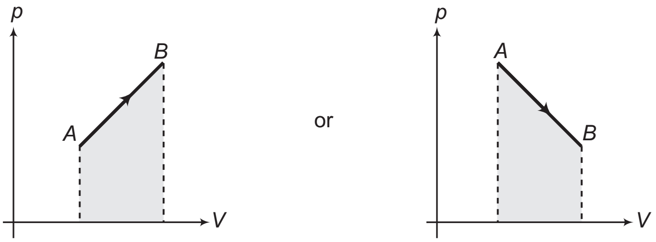


Fig. 21.4

$$V \text{ is increasing}$$

$$\therefore \qquad W_{AB} > 0$$

$$W_{AB} = \text{Shaded area}$$

Case 3 When volume is decreasing

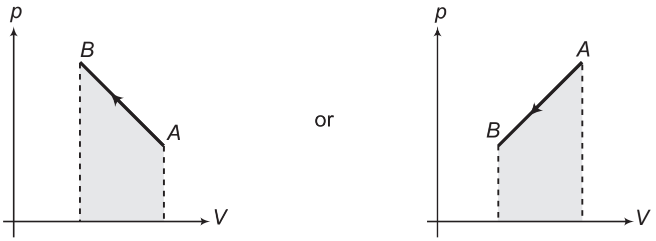


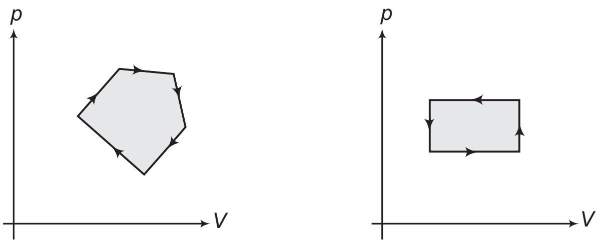
Fig. 21.5

$$V \text{ is decreasing}$$

$$\therefore \qquad W_{AB} < 0$$

$$W_{AB} = -\text{ Shaded area}$$

Case 4 Cyclic process



(a) (b)

Fig. 21.6

$$W_{\text{clockwise cycle}} = +\text{ Shaded area} \qquad \text{[in figure (a)]}$$

$$W_{\text{anticlockwise cycle}} = -\text{ Shaded area} \qquad \text{[in figure (b)]}$$

Case 5 Incomplete cycle

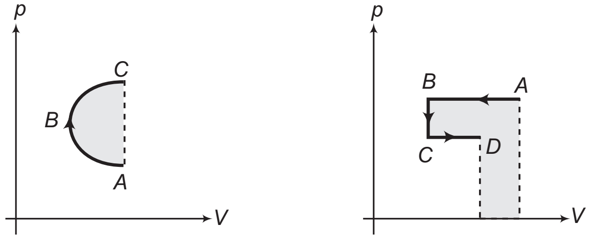


Fig. 21.7

$$W_{ABC} = +\text{ Shaded area}$$
$$W_{ABCD} = -\text{ Shaded area.}$$

***Method* 3.** Sometimes work done by the gas is also obtained by finding the forces against which work is done by the gas.

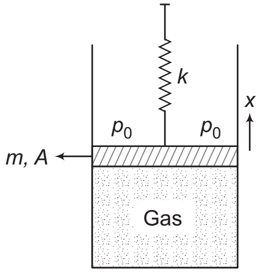


Fig. 21.8

For example, in the figure shown, work is done by the gas against the following forces (when the piston is displaced upwards)

(i) Against gravity force mg. Since, mg is a constant force.

∴ $W_1 = \text{Force} \times \text{displacement} = mgx$

(ii) Against the force $p_0 A$. Further, this is also a constant force.

∴ $W_2 = \text{Force} \times \text{displacement} = p_0 Ax$

But, $Ax = \Delta V$

∴ $W_2 = p_0 \Delta V$

(iii) Against spring force kx. This is a variable force.

Hence,

$$W_3 = \int_0^x (kx)\, dx = \frac{1}{2} kx^2$$

Note *While calculating W_3, we have assumed that initially spring is in its natural length.*

***Method* 4.** In some cases, one process and limits of temperature (or temperature change) is given. In those cases, with the help of given process and the standard ideal gas equation $pV = nRT$, first we convert pdV into $f(T)dT$ and then integrate this expression with the limits of temperature. Thus,

$$W = \int pdV = \int_{T_i}^{T_f} f(T)\, dT$$

***Method* 5.** The last method of finding work done is from the fundamental equation of first law.

or
$$W = Q - \Delta U$$

Now, let us take example of each one of them.

Example 21.2 ***Method* 1 *of Q***

Temperature of two moles of a monoatomic gas is increased by 300 K in the process $p \propto V$.

(a) Find molar heat capacity of the gas in the given process.

(b) Find heat given to the gas in that.

Solution (a) $p \propto V \quad \Rightarrow \quad pV^{-1} = \text{constant}$

If we compare with pV^x = constant, then

$$x = -1$$

Now,

$$C = C_V + \frac{R}{1-x}$$

$$C_V = \frac{3}{2}R \text{ for a monoatomic gas}$$

$$\therefore \qquad C = \frac{3}{2}R + \frac{R}{1-(-1)} = 2R$$ **Ans.**

(b) $Q = nC\Delta T$

Substituting the values, we get

$$Q = (2)(2R)(300)$$
$$= 1200R$$ **Ans.**

Example 21.3 ***Method* 2 *of Q***

In a given process work done on a gas is 40 J and increase in its internal energy is 10 J. Find heat given or taken to/from the gas in this process.

Solution Given, $\Delta U = +10$ J

Work done on the gas is 40 J. Therefore, work done by the gas used in the equation, $Q = W + \Delta U$ will be –40 J. Now, putting the values in the equation,

$$Q = W + \Delta U$$

We have,

$$Q = -40 + 10$$
$$= -30 \text{ J}$$ **Ans.**

Here, negative sign indicates that heat is taken out from the gas.

Example 21.4 ***Method* 1 *of* ΔU**

Temperature of two moles of a monoatomic gas is increased by 600 K in a given process. Find change in internal energy of the gas.

Solution Using the equation,

$$\Delta U = nC_V \Delta T \qquad \text{for change in internal energy}$$

$$C_V = \frac{3}{2}R \qquad \text{for monoatomic gas}$$

$$\therefore \qquad \Delta U = (2)\left(\frac{3}{2}R\right)(600)$$

$$= 1800\,R$$ **Ans.**

Example 21.5 Method 2 of ΔU

Work done by a gas in a given process is –20 J. Heat given to the gas is 60 J. Find change in internal energy of the gas.

Solution $\Delta U = Q - W$

Substituting the values we have,

$$\Delta U = 60 - (-20)$$

$$= 80\text{ J}$$ **Ans.**

ΔU is positive. Hence, internal energy of the gas is increasing.

Example 21.6 Method 1 of W

By integration, make expressions of work done by gas in

(a) Isobaric process (p = constant)

(b) Isothermal process (pV = constant)

(c) Adiabatic process (pV^{γ} = constant)

Solution (a) **Isobaric process**

$$W = \int_{V_i}^{V_f} p\,dV = p\int_{V_i}^{V_f} dV \qquad (\text{as } p = \text{constant})$$

$$= p\,[V]_{V_i}^{V_f} = p\,(V_f - V_i)$$

$$= p\Delta V$$ **Ans.**

Note *Any process freely taking place in atmosphere is considered isobaric. For example, melting of ice, boiling of water etc. Here, the constant pressure is p_0. Therefore,*

$$W = p_0 \Delta V = p_0 (V_f - V_i)$$

(b) **Isothermal process**

$$W = \int_{V_i}^{V_f} p\,dV = \int_{V_i}^{V_f} \left(\frac{nRT}{V}\right) dV \qquad \left(\text{as } p = \frac{nRT}{V}\right)$$

$$= nRT \int_{V_i}^{V_f} \frac{dV}{V} \qquad (\text{as } T = \text{constant})$$

$$= nRT \ln\left(\frac{V_f}{V_i}\right) = nRT \ln\left(\frac{p_i}{p_f}\right) \qquad \left(\text{as } p_i V_i = p_f V_f \text{ So, } \frac{V_f}{V_i} = \frac{p_i}{p_f}\right)$$

(c) **Adiabatic process**

$$pV^{\gamma} = \text{constant} = k\ (\text{say}) = p_i V_i^{\gamma} = p_f V_f^{\gamma}$$

Further,
$$p = \frac{k}{V^{\gamma}} = kV^{-\gamma}$$

$$W = \int_{V_i}^{V_f} pdV = \int_{V_i}^{V_f} kV^{-\gamma}\, dV = \left[\frac{kV^{-\gamma+1}}{-\gamma+1}\right]_{V_i}^{V_f}$$

$$= \frac{kV_f^{-\gamma+1} - kV_i^{-\gamma+1}}{-\gamma+1} = \frac{p_f V_f^{\gamma} V_f^{-\gamma+1} - p_i V_i^{\gamma} V_i^{-\gamma+1}}{1-\gamma}$$

$$= \frac{p_f V_f - p_i V_i}{1-\gamma} = \frac{nRT_f - nRT_i}{1-\gamma} = \frac{nR\Delta T}{1-\gamma}$$

Example 21.7 Method 2 of W

In the given p-V diagram, find

(a) pressures at c and d

(b) work done in different processes separately

(c) work done in complete cycle abcd.

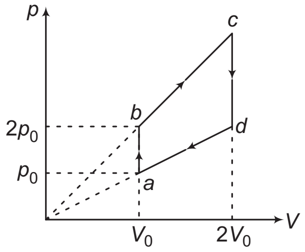


Fig. 21.9

Solution (a) Line *bc* is passing through origin. Hence,

$$p \propto V$$

From *b* to *c*, volume is doubled. Hence, pressure is also doubled.

$$\therefore \qquad p_c = 2p_b = 4p_0 \qquad \textbf{Ans.}$$

Similarly, line *ad* is also passing through origin.

$$\therefore \qquad p_d = 2p_a = 2p_0 \qquad \textbf{Ans.}$$

(b) *ab* and *cd* processes are isochoric ($V = \text{constant}$). Hence,

$$W_{ab} = W_{cd} = 0$$

$$W_{bc} = \text{area under the line } bc$$
$$= \text{area of trapezium}$$
$$= \frac{1}{2}(p_b + p_c)(2V_0 - V_0)$$
$$= \frac{1}{2}(2p_0 + 4p_0)(V_0)$$
$$= 3\, p_0 V_0 \qquad \textbf{Ans.}$$

Since, volume is increasing. Therefore, W_{bc} is positive.

$$W_{da} = -\text{area under the line } da$$
$$= -\text{area of trapezium}$$
$$= -\frac{1}{2}(p_a + p_d)(2V_0 - V_0)$$

$$= -\frac{1}{2}(p_0 + 2p_0)(V_0)$$

$$= -1.5\, p_0V_0$$ **Ans.**

Volume is decreasing. Therefore, work done is negative.

(d) Work done in complete cycle

$$W_{\text{net}} = W_{ab} + W_{bc} + W_{cd} + W_{da}$$
$$= 0 + 3\, p_0V_0 + 0 - 1.5\, p_0V_0$$
$$= 1.5\, p_0V_0$$ **Ans.**

Note *Net work done is also equal to the area between the cycle. Since, cycle is clockwise. Hence, net work done is positive.*

Example 21.8 *Method 3 of W*

Mass of a piston shown in Fig. 21.10 is m and area of cross-section is A. Initially spring is in its natural length. Find work done by the gas.

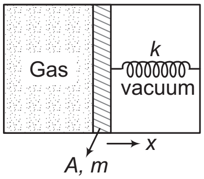


Fig. 21.10

Solution In the given condition, work is done by the gas only against spring force kx. This force is a variable force. Hence,

$$W = \int_0^x (kx)\,dx = \frac{1}{2}kx^2$$

Example 21.9 *Method 4 of W*

The temperature of n-moles of an ideal gas is increased from T_0 to $2T_0$ through a process $p = \frac{\alpha}{T}$. Find work done in this process.

Solution $pV = nRT$ (ideal gas equation) ...(i)

and $p = \frac{\alpha}{T}$ ...(ii)

Dividing Eq. (i) by Eq. (ii), we get

$$V = \frac{nRT^2}{\alpha} \quad \text{or} \quad dV = \frac{2nRT}{\alpha}\,dT$$

$$\therefore \quad W = \int_{V_i}^{V_f} p\,dV = \int_{T_0}^{2T_0} \left(\frac{\alpha}{T}\right)\left(\frac{2nRT}{\alpha}\right) dT$$

$$= 2nRT_0$$ **Ans.**

Example 21.10 *Method 5 of W*

Heat taken from a gas in a process is 80 J and increase in internal energy of the gas is 20 J. Find work done by the gas in the given process.

Solution Heat is taken from the gas.

Therefore, Q is negative. Or, $Q = -80$ J

Internal energy of the gas is increasing.

Therefore, ΔU is positive. Or $\Delta U = +20\text{ J}$

Using the first law equation,

$$Q = W + \Delta U \quad \text{or} \quad W = Q - \Delta U = -80 - 20 = -100\text{ J} \qquad \textbf{Ans.}$$

Here, negative sign indicates that volume of the gas is decreasing and work is done on the gas.

INTRODUCTORY EXERCISE 21.2

1. A gas in a cylinder is held at a constant pressure of 1.7×10^5 Pa and is cooled and compressed from 1.20 m^3 to 0.8 m^3. The internal energy of the gas decreases by 1.1×10^5 J.
   (a) Find the work done by the gas.
   (b) Find the magnitude of the heat flow into or out of the gas and state the direction of heat flow.
   (c) Does it matter whether or not the gas is ideal?
2. A thermodynamic system undergoes a cyclic process as shown in figure.

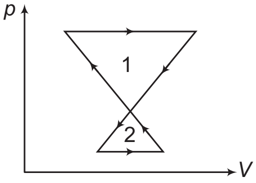
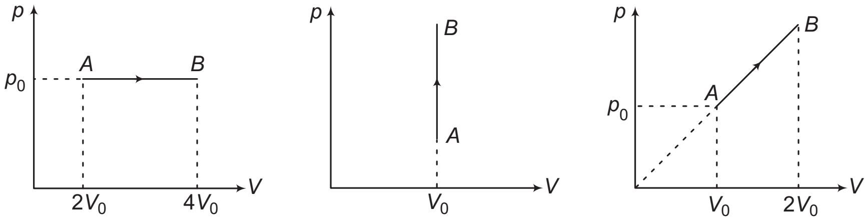


Fig. 21.12

   (a) over one complete cycle, does the system do positive or negative work.
   (b) over one complete cycle, does heat flow into or out of the system.
   (c) In each of the loops 1 and 2, does heat flow into or out of the system.
3. How many moles of helium at temperature 300 K and 1.00 atm pressure are needed to make the internal energy of the gas 100 J?
4. Temperature of four moles of a monoatomic gas is increased by 300 K in isochoric process. Find W, Q and ΔU.
5. Find work done by the gas in the process AB shown in the following figures.



(i) (ii) (iii)

Fig. 21.12

6. Temperature of two moles of an ideal gas is increased by 300 K in a process $V = \dfrac{a}{T}$, where a is positive constant. Find work done by the gas in the given process.
7. Pressure and volume of a gas changes from (p_0, V_0) to $\left(\dfrac{p_0}{4}, 2V_0\right)$ in a process $pV^2 =$ constant. Find work done by the gas in the given process.

21.3 Different Thermodynamic Processes

Different thermodynamic processes and their important points are given below in tabular form.

Table 21.3

S.No	Name of the process	Important points in the process	$Q = nC\Delta T = W + \Delta U$	$\Delta U = nC_V\Delta T$	W
1.	Isothermal	$T, pV, U =$ constant $\Delta T = \Delta(pV) = \Delta U = 0$ $p_1V_1 = p_2V_2$ or $p \propto \frac{1}{V}$	$Q = W$	0	$nRT\ln\left(\frac{V_f}{V_i}\right) = nRT\ln\left(\frac{p_i}{p_f}\right)$
2.	Isochoric	$V, \rho, \frac{p}{T} =$ constant $\Delta V = \Delta\rho = \Delta\left(\frac{p}{T}\right) = 0$ $\frac{p_1}{T_1} = \frac{p_2}{T_2}$ or $p \propto T$	$C = C_V$ $\therefore\ Q = nC_V\Delta T$	$nC_V\Delta T$	0
3.	Isobaric	$p, \frac{V}{T} =$ constant $\Delta p = \Delta\left(\frac{V}{T}\right) = 0$ $\frac{V_1}{T_1} = \frac{V_2}{T_2}$ or $V \propto T$	$C = C_P$ $\therefore\ Q = nC_P\Delta T$	$nC_V\Delta T$	$p\Delta V \rightarrow$ For any system $Q - \Delta U = n(C_P - C_V)\Delta T$ $= nR\Delta T \rightarrow$ for an ideal gas
4.	Adiabatic process	$pV^\gamma =$ constant $TV^{\gamma-1} =$ constant $T^\gamma p^{1-\gamma} =$ constant	0	$nC_V\Delta T$	$W = -\Delta U$ $\therefore\ W = -nC_V\Delta T$ $= -n\left(\frac{R}{\gamma-1}\right)(T_f - T_i)$ $= \frac{p_iV_i - p_fV_f}{\gamma - 1}$
5.	Cyclic process	$(p_i, V_i, T_i) = (p_f, V_f, T_f)$ Since $T_i = T_f$ $\Rightarrow\ U_i = U_f$ or $\Delta T = \Delta U = 0$	$Q_{\text{net}} = W_{\text{net}}$	0	$W_{\text{net}} =$ area between cycle on p-V diagram
6.	Polytropic process $pV^X =$ constant	$C = \frac{R}{\gamma-1} + \frac{R}{1-x}$ $= C_V + \frac{R}{1-x}$	$nC\Delta T$	$nC_V\Delta T$	$Q - \Delta U = \frac{nR\Delta T}{1-x}$ $= \frac{nR(T_i - T_f)}{1-x}$ $= \frac{(p_iV_i - p_fV_f)}{1-x}$
7.	Free expansion in vacuum	$\Delta U = 0$ $\Rightarrow\ U, T$ and $pV =$ constant or $p \propto \frac{1}{V}$	0	0	0

Important Points in the Above Table

In isobaric process In isobaric process, neither of the three terms (Q, ΔU and W) is zero but they have a constant ratio which depends on nature of gas (like monoatomic or diatomic etc.)

$$Q : \Delta U : W = nC_p\Delta T : nC_V\Delta T : nR\Delta T = C_p : C_V : R$$

So, this ratio is $C_p : C_V : R$. For example, $C_p = \frac{5}{2}R$ and $C_V = \frac{3}{2}R$ for a monoatomic gas. Therefore, for a monoatomic gas this ratio is $\frac{5}{2}R : \frac{3}{2}R : R$ or $5 : 3 : 2$. If Q is 50 J, then ΔU will be 30 J and W is 20 J.

In adiabatic process (i) In a thermodynamic process, there are three variables p, V and T. If relation between any two (p-V, V-T or p-T) are known, then other relations can be obtained using the ideal gas equation

$$pV = nRT$$

or $$pV \propto T \qquad \text{(For given value of } n\text{)}$$

or $$p \propto \frac{T}{V}$$

and $$V \propto \frac{T}{p}$$

For example, p-V equation $pV^{\gamma} =$ constant can be converted into p-T or V-T equation.

***p*-*T* equation** Replace V with $\frac{T}{p}$

$$\therefore \qquad p\left(\frac{T}{p}\right)^{\gamma} = \text{constant} \quad \text{or} \quad p^{1-\gamma}T^{\gamma} = \text{constant}$$

***V*-*T* equation** Replace p with $\frac{T}{V}$

$$\therefore \qquad \left(\frac{T}{V}\right)V^{\gamma} = \text{constant} \quad \text{or} \quad TV^{\gamma-1} = \text{constant}$$

(ii) An adiabatic process is defined as one with no heat transfer into or out of a system : $Q = 0$. We can prevent heat flow either by surrounding the system with thermally insulating material or by carrying out the process so **quickly** that there is not enough time for appreciable heat flow. From the first law, we find that for every adiabatic process,

$$W = -\Delta U \qquad \text{(as } Q = 0\text{)}$$

Therefore, if the work done by a gas is positive (i.e. volume of the gas is increasing), then ΔU will be negative. Hence, U and therefore T will decrease. The cooling of air can be experienced practically during bursting of a tyre. The process is so fast that it can be assumed as adiabatic. As the gas expands. Therefore, it cools. On the other hand, the compression stroke in an internal combustion engine is an approximately adiabatic process. The temperature rises as the air fuel mixture in the cylinder is compressed.

Note *Contrary to adiabatic process which is very fast an isothermal process is very slow. Because the system needs sufficient time to interact with surroundings to keep its temperature constant.*

In cyclic process In a cyclic process, initial and final points are same.

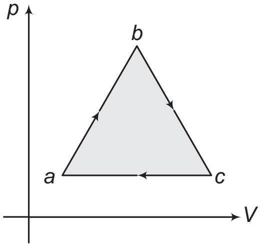


Fig. 21.13

Therefore,

$$(p_i, V_i, T_i) = (p_f, V_f, T_f)$$

Internal energy is a state function which only depends on temperature (in case of an ideal gas).

$$T_i = T_f$$

$$\Rightarrow \quad U_i = U_f$$

$$\text{or} \quad \Delta U_{\text{net}} = 0$$

If there are three processes in a cycle *abc*, then

$$\Delta U_{ab} + \Delta U_{bc} + \Delta U_{ca} = 0$$

From first law of thermodynamics,

$$Q = W + \Delta U, \text{ if } \Delta U_{\text{net}} = 0, \text{ then}$$

$$Q_{\text{net}} = W_{\text{net}}$$

$$\text{or} \quad Q_{ab} + Q_{bc} + Q_{ca} = W_{ab} + W_{bc} + W_{ca}$$

Further, W_{net} = area under p-V diagram. For example, W_{net} = + area of triangle '*abc*' in the shown diagram. Cycle is clockwise. So, work done will be positive.

Free expansion in vacuum A gas in a closed adiabatic chamber is taken to vacuum and then chamber is opened. So, the gas expands. Since, there is no external force which opposes this expansion. So, work done by gas

$$W = 0$$

Further no heat is supplied or taken from the gas. Therefore, heat exchange is also zero. Or

$$Q = 0$$

From first law of thermodynamics, $Q = W + \Delta U$ change in internal energy is also zero. Or

$$\Delta U = 0$$

$$\text{or} \quad U = \text{constant}$$

$$\Rightarrow \quad T \text{ or } pV \text{ is also constant}$$

$$\text{or} \quad p \propto \frac{1}{V}$$

This behaviour is similar to an isothermal process. The only difference is, in this process all three terms Q, ΔU and W are zero. But in isothermal process, only ΔU is zero.

Extra Points to Remember

- **Slope of p-V diagram** In a general polytropic process,

$$pV^x = \text{constant}$$

or differentiating, we get

$$p\,(xV^{x-1})\,dV + V^x(dp) = 0 \quad \Rightarrow \quad \frac{dp}{dV} = -x\frac{p}{V}$$

or slope of p-V graph $= -x\dfrac{p}{V}$

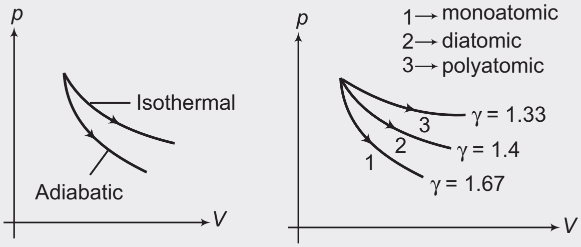
In isobaric process $p =$ constant $\Rightarrow x = 0$, therefore slope $= 0$

In isothermal process $pV =$ constant $\Rightarrow x = 1$, therefore slope $= -\dfrac{p}{V}$

In adiabatic process $pV^\gamma =$ constant $\Rightarrow x = \gamma$, therefore slope $= -\gamma\dfrac{p}{V}$

Thus, slope of an adiabatic graph is γ times slope of isothermal graph at that point.

Because $\gamma > 1$, the isothermal curve is not as steep as that for the adiabatic expansion.



Adiabatic and isothermal expansion of an ideal gas

Adiabatic expansion of mono,dia and polyatomic gases.

Fig. 21.14

p - V diagram of different processes is shown in one graph as below.

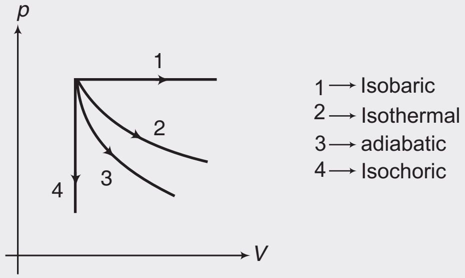


Fig. 21.15

Example 21.11 *p-V plots for two gases during adiabatic processes are shown in the figure. Plots 1 and 2 should correspond respectively to* **(JEE 2001)**

(a) He *and* O_2

(b) O_2 *and* He

(c) He *and* Ar

(d) O_2 *and* N_2

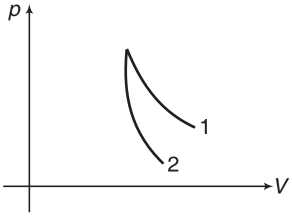


Fig. 21.16

Solution In adiabatic process :

slope of p-V graph, $$\frac{dp}{dV} = -\gamma \frac{p}{V}$$

$$\text{slope} \propto \gamma \qquad \text{(with negative sign)}$$

From the given graph,

$$(\text{slope})_2 > (\text{slope})_1$$

$$\therefore \qquad \gamma_2 > \gamma_1$$

Therefore, 1 should correspond to O_2 ($\gamma = 1.4$) and 2 should correspond to He ($\gamma = 1.67$).

Hence, the correct option is (b).

Example 21.12 *Starting with the same initial conditions, an ideal gas expands from volume V_1 to V_2 in three different ways, the work done by the gas is W_1 if the process is purely isothermal, W_2 if purely isobaric and W_3 if purely adiabatic, then* **(JEE 2000)**

(a) $W_2 > W_1 > W_3$

(b) $W_2 > W_3 > W_1$

(c) $W_1 > W_2 > W_3$

(d) $W_1 > W_3 > W_2$

Solution The corresponding p-V graphs (also called indicator diagram) in three different processes will be as shown below.

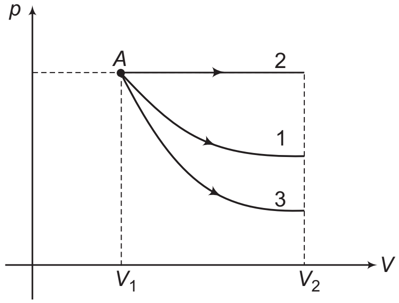


Fig. 21.17

Area under the graph gives the work done by the gas.

$$(\text{Area})_2 > (\text{Area})_1 > (\text{Area})_3 \quad \Rightarrow \quad W_2 > W_1 > W_3$$

Therefore, the correct option is (a).

Example 21.13 *When an ideal diatomic gas is heated at constant pressure, the fraction of the heat energy supplied, which increases the internal energy of the gas, is* **(JEE 1990)**

(a) $\frac{2}{5}$

(b) $\frac{3}{5}$

(c) $\frac{3}{7}$

(d) $\frac{5}{7}$

Solution The desired fraction is

$$f = \frac{\Delta U}{\Delta Q} = \frac{nC_V \Delta T}{nC_p \Delta T} = \frac{C_V}{C_p} = \frac{1}{\gamma} \quad \text{or} \quad f = \frac{5}{7} \qquad \left(\text{as } \gamma = \frac{7}{5}\right) \qquad \textbf{Ans.}$$

Therefore, the correct option is (d).

Example 21.14 *What is the heat input needed to raise the temperature of 2 moles of helium gas from 0°C to 100°C*

(a) at constant volume,

(b) at constant pressure?

(c) What is the work done by the gas in part (b)?

Give your answer in terms of R.

Solution Helium is monoatomic gas. Therefore,

$$C_V = \frac{3R}{2} \quad \text{and} \quad C_p = \frac{5R}{2}$$

(a) At constant volume,

$$\begin{aligned} Q &= nC_V \Delta T \\ &= (2)\left(\frac{3R}{2}\right)(100) \\ &= 300R \end{aligned}$$

(b) At constant pressure,

$$\begin{aligned} Q &= nC_p \Delta T \\ &= (2)\left(\frac{5R}{2}\right)(100) \\ &= 500R \end{aligned}$$

(c) At constant pressure,

$$\begin{aligned} W &= Q - \Delta U \\ &= nC_p \Delta T - nC_V \Delta T \\ &= nR\Delta T = (2)(R)(100) \\ &= 200R \end{aligned}$$

Example 21.15 *An ideal monoatomic gas at 300K expands adiabatically to twice its volume. What is the final temperature?*

Solution For an ideal monoatomic gas,

$$\gamma = \frac{5}{3}$$

In an adiabatic process, $TV^{\gamma - 1} = \text{constant}$

$\therefore$ $T_f V_f^{\gamma - 1} = T_i V_i^{\gamma - 1}$

or

$$\begin{aligned} T_f &= T_i \left(\frac{V_i}{V_f}\right)^{\gamma - 1} \\ &= (300)\left(\frac{1}{2}\right)^{\frac{5}{3} - 1} = 189\ \text{K} \end{aligned}$$

Example 21.16 *p-T graph of 2 moles of an ideal monoatomic gas* $\left(C_V = \frac{3}{2}R \text{ and } C_p = \frac{5}{2}R\right)$ *is as shown below. Find Q, W and ΔU for each of the four processes separately and then show that,*

$$\Delta U_{net} = 0$$

and

$$Q_{net} = W_{net}$$

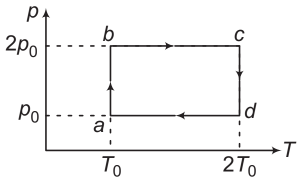


Fig. 21.18

Solution

Table 21.4

Process	Name of process	Q	ΔU	W
ab	Isothermal (as T = constant)	$Q = W = -2RT_0 \ln(2)$	0	$nRT \ln\left(\frac{p_i}{p_f}\right)$ $= 2RT_0 \ln\left(\frac{p_0}{2p_0}\right)$ $= -2RT_0 \ln(2)$
bc	Isobaric (as p = constant)	$Q = nC_p\Delta T$ $= 2\left(\frac{5}{2}R\right)(2T_0 - T_0)$ $= 5RT_0$	$\Delta U = nC_V\Delta T$ $= 2\left(\frac{3}{2}R\right)(2T_0 - T_0)$ $= 3RT_0$	$Q - \Delta U$ or $nR\Delta T$ $= 2RT_0$
cd	Isothermal (as T = constant)	$Q = W = 4RT_0 \ln(2)$	0	$nRT \ln\left(\frac{p_i}{p_f}\right)$ $= 2R(2T_0) \ln\left(\frac{2p_0}{p_0}\right)$ $= 4RT_0 \ln(2)$
da	Isobaric (as p = constant)	$Q = nC_p\Delta T$ $= 2\left(\frac{5}{2}R\right)(T_0 - 2T_0)$ $= -5RT_0$	$\Delta U = nC_V\Delta T$ $= 2\left(\frac{3}{2}R\right)(T_0 - 2T_0)$ $= -3RT_0$	$Q - \Delta U$ or $nR\Delta T$ $= -2RT_0$
	Net values	$2RT_0 \ln(2)$	0	$2RT_0 \ln(2)$

In the given table, we can see that

$$\Delta U_{\text{net}} = 0$$

and

$$Q_{\text{net}} = W_{\text{net}}$$

INTRODUCTORY EXERCISE 21.3

1. One mole of an ideal monoatomic gas is initially at 300 K. Find the final temperature if 200 J of heat are added as follows :

   (a) at constant volume (b) at constant pressure.

2. An ideal gas expands while the pressure is kept constant. During this process, does heat flow into the gas or out of the gas? Justify your answer.

3. Consider the cyclic process depicted in figure. If Q is negative for the process BC and if ΔU is negative for the process CA, what are the signs of Q, W and ΔU that are associated with each process?

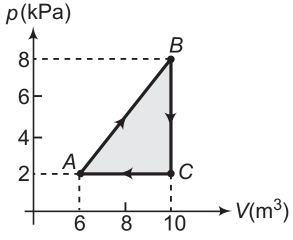


Fig. 21.19

4. A well insulated box contains a partition dividing the box into two equal volumes as shown in figure. Initially, the left hand side contains an ideal monoatomic gas and the other half is a vacuum. The partition is suddenly removed so that the gas expands throughout the entire box.

   (a) Does the temperature of the gas change?

   (b) Does the internal energy of the system change?

   (c) Does the gas work?

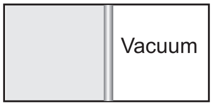


Fig. 21.20

5. Find the ratio of $\frac{\Delta Q}{\Delta U}$ and $\frac{\Delta Q}{\Delta W}$ in an isobaric process. The ratio of molar heat capacities $\frac{C_p}{C_V} = \gamma$.

6. A certain amount of an ideal gas passes from state A to B first by means of process 1, then by means of process 2. In which of the process is the amount of heat absorbed by the gas greater?

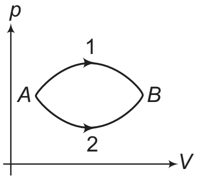


Fig. 21.21

7. A sample of ideal gas is expanded to twice its original volume of 1.00 m^3 in a quasi-static process for which $p = \alpha V^2$, with $\alpha = 5.00$ atm/m^6, as shown in Fig 21.22. How much work is done by the expanding gas?

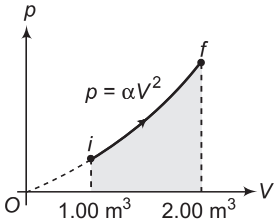


Fig. 21.22

8. As a result of the isobaric heating by $\Delta T = 72$ K, one mole of a certain ideal gas obtains an amount of heat $Q = 1.6$ kJ. Find the work performed by the gas, the increment of its internal energy and γ.

21.4 Heat Engine and its Efficiency

A heat engine is a device which converts heat energy into mechanical energy. In every heat engine, there are the following three components :

(i) Working substance (which is normally a gas in a cylinder)
(ii) Source (at temperature T_1)
(iii) Sink (at temperature T_2), $T_2 < T_1$ and sink is normally atmosphere.

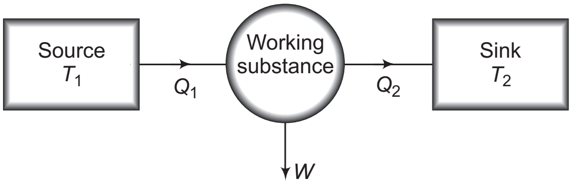


Fig. 21.23

The working substance absorbs some heat (Q_1) from the source, converts a part of it into work (W) and the rest (Q_2) is rejected to the sink.

From conservation of energy,

$$Q_1 = W + Q_2$$

Now, the above work done is used by us for different purposes.

It is just like a shopkeeper. He takes some money from you. (Suppose he takes Rs. 100/- from you). So, you are source. In lieu of this he provides services to you (suppose he provides services of worth Rs. 80/-). This is work done. The remaining Rs. 20/- is his profit which goes to his account and this is basically sink. Then, the efficiency of the shopkeeper is 80%. There can't be a shopkeeper whose efficiency is 100%.

Similarly, efficiency of a heat engine is defined as the ratio of net work done per cycle by the engine to the total amount of heat absorbed per cycle by the working substance from the source. It is denoted by η. Thus,

$$\boxed{\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}}$$

Work is done by the working substance in a cyclic process. In the above expression of efficiency, W is net work done in the complete cycle which should be positive. So, cycle should be clockwise on p-V diagram. Q_1 is the total heat given to the working substance or the total positive heat. Q_2 is total heat rejected by the working substance or it is the magnitude of total negative heat.

Thus, efficiency (η) of a cycle can also be defined as

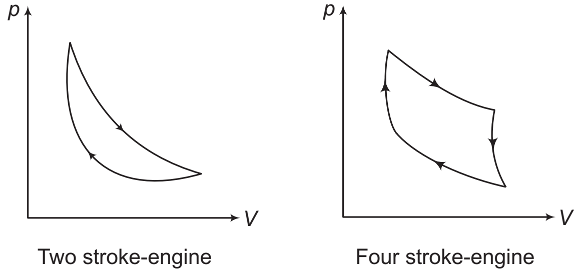
$$\eta = \left(\frac{\begin{array}{c}\text{Work done by the working substance} \\ \text{(an ideal gas in our case) during a cycle}\end{array}}{\text{Heat supplied to the gas during the cycle}} \right) \times 100$$

$$= \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100$$

$$= \frac{|Q_{+\text{ve}}| - |Q_{-\text{ve}}|}{|Q_{+\text{ve}}|} \times 100 = \left\{1 - \left|\frac{Q_{-\text{ve}}}{Q_{+\text{ve}}}\right|\right\} \times 100$$

Thus, $$\eta = \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100 = \left\{1 - \left|\frac{Q_{-\text{ve}}}{Q_{+\text{ve}}}\right|\right\} \times 100$$

Depending on the number of processes in a cycle, it may be called a two-stroke engine or four-stroke engine.



Two stroke-engine Four stroke-engine

Fig. 21.24

Note *There cannot be a heat engine whose efficiency is 100%. It is always less than 100%. Thus,*

$$\eta \neq 100\%$$

or $$W \neq Q_1 \quad \text{or} \quad W_{net} \neq Q_{+ve}$$

or $$Q_2 \neq 0 \quad \text{or} \quad |Q_{-ve}| \neq 0$$

Efficiency of a Cycle

By the similar method discussed above, we can also find efficiency of a cycle provided net work done in the whole cycle comes out to be positive or it is clockwise cycle on p-V diagram. But always remember that, in a cyclic process,

$$\Delta U_{\text{net}} = 0$$

and $$Q_{\text{net}} = W_{\text{net}}$$

If there are four processes in the cycle, then

$$\Delta U_1 + \Delta U_2 + \Delta U_3 + \Delta U_4 = 0 \text{ and}$$

$$Q_1 + Q_2 + Q_3 + Q_4 = W_1 + W_2 + W_3 + W_4$$

Carnot Engine

Carnot cycle consists of the following four processes :

(i) Isothermal expansion (process AB)
(ii) Adiabatic expansion (process BC)
(iii) Isothermal compression (process CD) and
(iv) Adiabatic compression (process DA)

The p-V diagram of the cycle is shown in the figure.

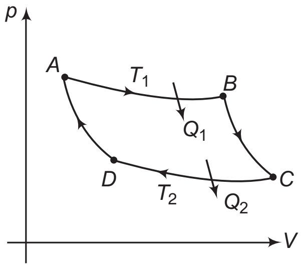


Fig. 21.25

Table 21.5

Process	Name of the process	Q	ΔU	W
AB	Isothermal expansion $T_1 =$ constant	$Q = W =$ positive $= Q_1$	0	positive
BC	Adiabatic expansion	0	negative	positive
CD	Isothermal compression $T_2 =$ constant	$Q = W =$ negative $= Q_2$	0	negative
DA	Adiabatic compression	0	positive	negative

In process AB, heat Q_1 is taken by the working substance at constant temperature T_1 and in process CD heat Q_2 is rejected from the working substance at constant temperature T_2. The net work done is area of graph $ABCD$.

Note (i) *In the whole cycle only Q_1 is the positive heat and Q_2 the negative heat. Thus,*

$$Q_{+ve} = Q_1$$

and $$|Q_{-ve}| = Q_2$$

$\therefore$ $$\eta = \left(1 - \frac{Q_2}{Q_1}\right) \times 100$$

Specially for Carnot cycle, $\frac{Q_2}{Q_1}$ also comes out to be $\frac{T_2}{T_1}$.

$\therefore$ $$\eta = \left(1 - \frac{T_2}{T_1}\right) \times 100$$

(ii) *Efficiency of Carnot engine is maximum (not 100%) for given temperatures T_1 and T_2. But still Carnot engine is not a practical engine because many ideal situations have been assumed while designing this engine which can practically not be obtained.*

21.5 Refrigerator

Refrigerator is an apparatus which takes heat from a cold body, work is done on it and the work done together with the heat absorbed is rejected to the source.

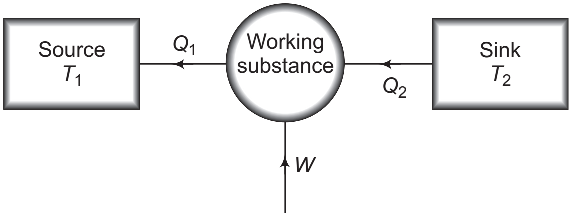


Fig. 21.26

An ideal refrigerator can be regarded as Carnot ideal heat engine working in the reverse direction.

Coefficient of Performance

Coefficient of performance (β) of a refrigerator is defined as the ratio of quantity of heat removed per cycle (Q_2) to the work done on the working substance per cycle to remove this heat. Thus,

$$\beta = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2}$$

We can also show that

$$\beta = \frac{T_2}{T_1 - T_2} = \frac{1-\eta}{\eta}$$

Here, η is the efficiency of Carnot cycle.

Example 21.17 *An ideal gas is taken through a cyclic thermodynamic process through four steps. The amounts of heat involved in these steps are $Q_1 = 5960\,J$, $Q_2 = -5585\,J$, $Q_3 = -2980\,J$ and $Q_4 = 3645\,J$ respectively. The corresponding quantities of work involved are $W_1 = 2200\,J$, $W_2 = -825\,J$, $W_3 = -1100\,J$ and W_4 respectively.* **(JEE 1994)**

(a) Find the value of W_4.

(b) What is the efficiency of the cycle?

Solution (a) In a cyclic process, $\Delta U = 0$

Therefore, $Q_{\text{net}} = W_{\text{net}}$

or $Q_1 + Q_2 + Q_3 + Q_4 = W_1 + W_2 + W_3 + W_4$

Hence, $W_4 = (Q_1 + Q_2 + Q_3 + Q_4) - (W_1 + W_2 + W_3)$

$= \{(5960 - 5585 - 2980 + 3645) - (2200 - 825 - 1100)\}$

or $W_4 = 765\text{ J}$ **Ans.**

(b) Efficiency,

$$\eta = \frac{\text{Total work done in the cycle}}{\text{Heat absorbed (positive heat) by the gas during the cycle}} \times 100$$

$$= \left(\frac{W_1 + W_2 + W_3 + W_4}{Q_1 + Q_4}\right) \times 100$$

$$= \left\{\frac{(2200 - 825 - 1100 + 765)}{5960 + 3645}\right\} \times 100$$

$$= \frac{1040}{9605} \times 100$$

$$\eta = 10.82\%$$ **Ans.**

Note *From energy conservation,*

$$W_{\text{net}} = Q_{+\text{ve}} - Q_{-\text{ve}} \qquad \textit{(in a cycle)}$$

$$\therefore \quad \eta = \frac{W_{\text{net}}}{Q_{+\text{ve}}} \times 100 = \frac{(Q_{+\text{ve}} - Q_{-\text{ve}})}{Q_{+\text{ve}}} \times 100$$

$$= \left(1 - \frac{Q_{-\text{ve}}}{Q_{+\text{ve}}}\right) \times 100$$

In the above question,

$$Q_{-\text{ve}} = |Q_2| + |Q_3| = (5585 + 2980)\text{ J} = 8565\text{ J}$$

and $$Q_{+\text{ve}} = Q_1 + Q_4 = (5960 + 3645)\ \text{J} = 9605\ \text{J}$$

$\therefore$ $$\eta = \left(1 - \frac{8565}{9605}\right) \times 100$$

$$\eta = 10.82\%$$

Example 21.18 *The density versus pressure graph of one mole of an ideal monatomic gas undergoing a cyclic process is shown in figure. The molecular mass of the gas is M.*

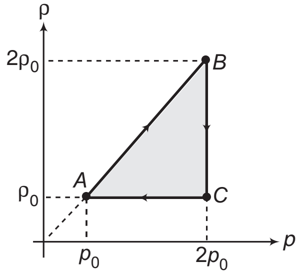


Fig. 21.27

(a) Find the work done in each process.
(b) Find heat rejected by gas in one complete cycle.
(c) Find the efficiency of the cycle.

Solution (a) Given, $n = 1$ $\therefore$ $m = M$

Process *AB* $\rho \propto p$, i.e. It is an isothermal process (T = constant), because $\rho = \dfrac{pM}{RT}$.

$\therefore$ $$W_{AB} = RT_A \ln\left(\frac{p_A}{p_B}\right) = RT_A \ln\left(\frac{1}{2}\right)$$

$$= -\frac{p_0 M}{\rho_0} \ln (2)$$

$$\Delta U_{AB} = 0$$

and $$Q_{AB} = W_{AB} = -\frac{p_0 M}{\rho_0} \ln (2)$$

Process *BC* is an isobaric process (p = constant)

$$W_{BC} = p_B (V_C - V_B) = 2p_0 \left(\frac{M}{\rho_C} - \frac{M}{\rho_B}\right) = 2p_0\left(\frac{M}{\rho_0} - \frac{M}{2\rho_0}\right) = \frac{p_0 M}{\rho_0}$$

$$\Delta U_{BC} = C_V \Delta T$$

$$= \left(\frac{3}{2}R\right)\left[\frac{2p_0 M}{\rho_0 R} - \frac{2p_0 M}{2\rho_0 R}\right] = \frac{3p_0 M}{2\rho_0}$$

$$Q_{BC} = W_{BC} + \Delta U_{BC} = \frac{5p_0 M}{2\rho_0}$$

Process *CA* As $\rho =$ constant

$$\therefore \qquad V = \text{constant}$$

So, it is an isochoric process.

$$W_{CA} = 0$$

$$\begin{aligned}\Delta U_{CA} &= C_V \Delta T \\ &= \left(\frac{3}{2}R\right)(T_A - T_C) \\ &= \left(\frac{3}{2}R\right)\left[\frac{p_0 M}{\rho_0 R} - \frac{2p_0 M}{\rho_0 R}\right] \\ &= -\frac{3p_0 M}{2\rho_0}\end{aligned}$$

$$Q_{CA} = \Delta U_{CA} = -\frac{3p_0 M}{2\rho_0}$$

(b) Heat rejected by gas $= |Q_{AB}| + |Q_{CA}|$

$$= \frac{p_0 M}{\rho_0}\left[\frac{3}{2} + \ln(2)\right]$$ **Ans.**

(c) Efficiency of the cycle (in fraction)

$$\eta = \frac{\text{Total work done}}{\text{Heat supplied}} = \frac{W_{\text{Total}}}{Q_{+\text{ve}}}$$

$$= \frac{\frac{p_0 M}{\rho_0}[1 - \ln(2)]}{\frac{5}{2}\left(\frac{p_0 M}{\rho_0}\right)}$$

$$= \frac{2}{5}[1 - \ln(2)]$$ **Ans.**

Example 21.19 *Carnot engine takes one thousand kilo calories of heat from a reservoir at 827°C and exhausts it to a sink at 27°C. How, much work does it perform? What is the efficiency of the engine?*

Solution Given, $Q_1 = 10^6$ cal

$$T_1 = (827 + 273) = 1100\text{ K}$$

and $$T_2 = (27 + 273) = 300\text{ K}$$

as, $$\frac{Q_2}{Q_1} = \frac{T_2}{T_1} \Rightarrow Q_2 = \frac{T_2}{T_1}\cdot Q_1 = \left(\frac{300}{1100}\right)(10^6)$$

$$= 2.72 \times 10^5 \text{ cal}$$

$$W = Q_1 - Q_2 = 7.28 \times 10^5 \text{ cal}$$ **Ans.**

Efficiency of the cycle,

$$\eta = \left(1 - \frac{T_2}{T_1}\right) \times 100 \quad \text{or} \quad \eta = \left(1 - \frac{300}{1100}\right) \times 100$$

$$= 72.72\% \qquad \textbf{Ans.}$$

Example 21.20 *In a refrigerator, heat from inside at 277 K is transferred to a room at 300 K. How many joules of heat shall be delivered to the room for each joule of electrical energy consumed ideally?*

Solution Coefficient of performance of a refrigerator,

$$\beta = \frac{Q_2}{W} = \frac{T_2}{T_1 - T_2}$$

$$\therefore \qquad Q_2 = W \frac{T_2}{T_1 - T_2}$$

But W = Energy consumed by the refrigerator = 1 J, $T_1 = 300$ K, $T_2 = 277$ K

$$\therefore \qquad Q_2 = 1 \times \frac{277}{300 - 277} = \frac{277}{23} = 12 \text{ J}$$

Heat rejected by the refrigerator,

$$Q_1 = W + Q_2 = 1 + 12$$

$$= 13 \text{ J} \qquad \textbf{Ans.}$$

Example 21.21 *Calculate the least amount of work that must be done to freeze one gram of water at 0°C by means of a refrigerator. Temperature of surroundings is 27°C. How much heat is passed on the surroundings in this process? Latent heat of fusion L = 80 cal/g.*

Solution $Q_2 = mL = 1 \times 80 = 80 \text{ cal}$

$$T_2 = 0° \text{ C} = 273 \text{ K}$$

and $$T_1 = 27° \text{ C} = 300 \text{ K}$$

Least amount of work will be needed for carnot's type of cycle.

$$\frac{Q_2}{W} = \frac{T_2}{T_1 - T_2}$$

$$\therefore \qquad W = \frac{Q_2 (T_1 - T_2)}{T_2}$$

$$= \frac{80\,(300 - 273)}{273}$$

$$= 7.91 \text{ cal} \qquad \textbf{Ans.}$$

$$Q_1 = Q_2 + W$$

$$= (80 + 7.91)$$

$$= 87.91 \text{ cal} \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 21.4

1. Carnot engine takes 1000 K cal of heat from a reservoir at 827°*C* and exhausts it to a sink at 27°*C*. How much heat is rejected to the sink? What is the efficiency of the engine?
2. One of the most efficient engines ever developed operated between 2100 K and 700 K. Its actual efficiency is 40%. What percentage of its maximum possible efficiency is this?
3. In a heat engine, the temperature of the source and sink are 500 K and 375 K. If the engine consumes 25×10^5 J per cycle, find (a) the efficiency of the engine, (b) work done per cycle, and (c) heat rejected to the sink per cycle.
4. A Carnot engine takes 3×10^6 cal of heat from a reservoir at 627 °C and gives it to a sink at 27 °C. Find the work done by the engine.
5. The efficiency of a Carnot cycle is 1/6. If on reducing the temperature of the sink by 65 °C, the efficiency becomes 1/3, find the source and sink temperatures between which the cycle is working.
6. Refrigerator *A* works between –10 °C and 27 °C, while refrigerator *B* works between –27 °C and 17 °C, both removing heat equal to 2000 J from the freezer. Which of the two is the better refrigerator?
7. A refrigerator has to transfer an average of 263 J of heat per second from temperature –10 °C to 25 °C. Calculate the average power consumed, assuming no energy losses in the process.
8. *n* moles of a monoatomic gas are taken around in a cyclic process consisting of four processes along *ABCDA* as shown. All the lines on the *p*-*V* diagram have slope of magnitude p_0/V_0. The pressure at *A* and *C* is p_0 and the volumes at *A* and *C* are $V_0/2$ and $3V_0/2$, respectively. Calculate the percentage efficiency of the cycle.

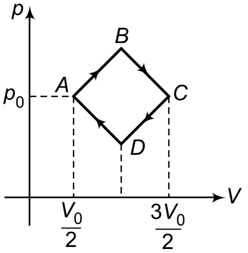


Fig. 21.28

21.6 Zeroth Law of Thermodynamics

The zeroth law of thermodynamics states that if two system *A* and *B* are in thermal equilibrium with a third system *C*, then *A* and *B* are also in thermal equilibrium with each other. It is analogous to the transitive property in math (if $A = C$ and $B = C$, then $A = B$). Another way of stating the zeroth law is that every object has a certain temperature, and when two objects are in thermal equilibrium, their temperatures are equal. It is called the zeroth law because it came to light after the first and second laws of thermodynamics had already been established and named, but was considered more fundamental and thus was given a lower number zero.

21.7 Second Law of Thermodynamics

The first law of thermodynamics is the principle of conservation of energy. Common experience shows that there are many conceivable processes that are perfectly allowed by the first law and yet are never observed. For example, nobody has ever seen a book lying on a table jumping to a height by itself. But such a thing would be possible if the principle of conservation of energy were the only restriction.

Thus, the second law of thermodynamics is a general principle which places constraints upon the direction of and the attainable efficiency of or coefficient of performance. It also imposes restrictions on entropy.

Second law of thermodynamics can be better understood by the following three statements

In Terms of Entropy

The second law can be expressed in several ways, the simplest being that heat will naturally flow from a hotter to a colder body. As its heat is a property of thermodynamic systems called entropy represented by "S" — in loose terms, a measure of the amount of disorder within a system. This can be represented in many ways, for example in the arrangement of the molecules — water molecules in an ice cube are more ordered than the same molecules after they have been heated into a gas. The entropy of the ice cube is, therefore, lower than that of the gas. Similarly, the entropy of a plate is higher when it is in pieces on the floor compared with when it is in one piece in the sink.

The second equation is a way to express the second law of thermodynamics in terms of entropy. The formula says that the entropy of an isolated natural system will always tend to stay the same or increase — in other words, the energy in the universe is gradually moving towards disorder.

Kelvin Planck's Statement

Kelvin-Planck's statement is based on the fact that the efficiency of the heat engine cycle is never 100%. This means that in the heat engine cycle some heat is always rejected to the low temperature reservoir. The heat engine cycle always operates between two heat reservoirs and produces work.

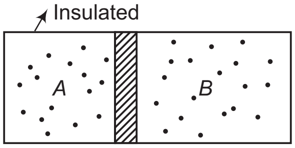
Clausius Statement

As we know from the previous statement, the natural tendency of the heat is to flow from high temperature reservoir to the low temperature reservoir.

The Clausius statement says that, to transfer the heat from low temperature to high temperature reservoir some external work should be done on the cycle. This statement has been the basis for the working for all refrigerators, heat pumps and air-conditioners. This work cannot be zero or coefficient of performance of a refrigerator cannot be infinite.

Final Touch Points

1. Adiabatic and Diathermic Wall



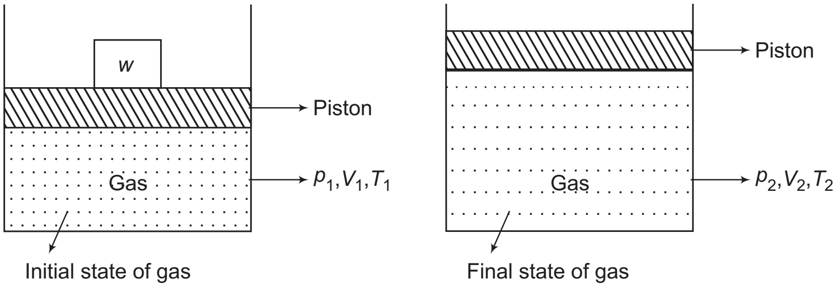
Adiabatic wall An insulating wall (can be movable also) that does not allow flow of energy (heat) from one chamber to another is called an adiabatic wall. If two thermodynamic systems *A* and *B* are separated by an adiabatic wall then the thermodynamic state of *A* will be independent of the state of *B* and *vice-versa* if wall is fixed otherwise if wall is movable, only pressure will be same on both sides.

Diathermic wall A conducting wall that allows energy flow (heat) from one chamber to another is called a diathermic wall. If two thermodynamic systems *A* and *B* are separated by a diathermic wall then thermal equilibrium is attained in due course of time (if wall is fixed). If wall is movable, then temperature and pressure on both sides will become same.

Note *In the above two cases, thermodynamic systems A and B are insulated from the external surroundings.*

2. Quasi-Static Process

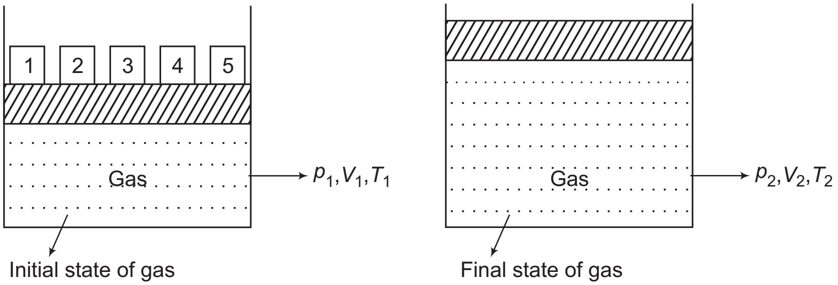
Quasi means almost or near to.Quasi-static process means very nearly static process. Let us consider a system of gas contained in cylinder. The gas is held by a moving piston. A weight *w* is placed over the piston. Due to the weight , the gas in cylinder is compressed. After the gas reaches equilibrium, the properties of gas are denoted by p_1, V_1 and T_1. The weight placed over the piston is balanced by upward force exerted by the gas. If the weight is suddenly removed, then there will be an unbalanced force between the system and the surroundings. The gas under pressure will expand and push the piston upwards. The properties at this state after reaching equilibrium are P_2, V_2 and T_2. But the intermediate states passed through, by the system are non-equilibrium states which cannot be described by thermodynamic coordinates. In this case, we only have initial and final states and do not have a path connecting them.



Initial state of gas

Final state of gas

Sudden expansion of gas

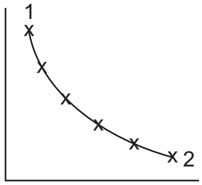


Initial state of gas

Final state of gas

Quasi-static expansion of gas

Suppose, the weight is made of large numbers of small weights. And one by one each of these small weights are removed and allowed the system to reach an equilibrium state. Then, we have intermediate equilibrium states and the path described by these states will not deviate much from the thermodynamic equilibrium state. Such a process, which is the locus of all the intermediate points passed by the system is known as quasi-static process. It means, this process is almost near to the thermodynamically equilibrium process. Infinite slowness is the characteristic feature of quasi-static process.

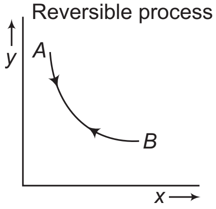


A quasi-static process

A quasi-static process is obviously a hypothetical concept. In practice, processes that are sufficiently slow and do not involve accelerated motion of the piston are reasonably approximation to an ideal quasi-static process. We shall from now onwards deal with quasi-static processes only, except when stated otherwise.

3. Reversible and Irreversible Process

Reversible process The process in which the system and surroundings can be restored to the initial state from the final state without producing any changes in the thermodynamic properties of the universe is called a reversible process. In the figure below, let us suppose that the system has undergone a change from state *A* to state *B*. If the system can be restored from state *B* to state *A*, and there is no change in the universe, then the process is said to be a reversible process. The reversible process can be reversed completely and there is no trace left to show that the system had undergone thermodynamic change.



For the system to undergo reversible change, it should occur infinitely slowly or it should be quasi-static process. During reversible process, all the changes in state that occur in the system are in thermodynamic equilibrium with each other.

Thus, there are two important conditions for the reversible process to occur. Firstly, the process should occur very slowly and secondly all of the initial and final states of the system should be in equilibrium with each other.

For example, a quasi-static isothermal expansion of an ideal gas in a cylinder fitted with a frictionless movable piston is a reversible process.

In actual practice, the reversible process never occurs, thus it is an ideal or hypothetical process.

Irreversible process The process is said to be an irreversible process if it cannot return the system and the surroundings to their original conditions when the process is reversed. The irreversible process is not at equilibrium throughout the process. Several examples can be cited. For examples,

(i) When we are driving the car uphill, it consumes a lot of fuel and this fuel is not returned when we are driving down the hill.

(ii) The base of a vessel on an oven is hotter than its other parts. When the vessel is removed, heat is transferred from the base to the other parts, bringing the vessel to a uniform temperature (which in due course cools to the temperature of the surroundings). The process cannot be reversed; a part of the vessel will not get cooler spontaneously and warm up the base. It will violate the second law of thermodynamics, if it did.

(iii) The free expansion of a gas is irreversible.

(iv) Cooking gas leaking from a gas cylinder in the kitchen diffuses to the entire room. The diffusion process will not spontaneously reverse and bring the gas back to the cylinder.

Many factors contribute in making any process irreversible. The most common of these are: friction, viscosity and other dissipative effects.

The irreversible process is also called the natural process because all the processes occurring in nature are irreversible processes.

4. $\boldsymbol{C_V = \frac{dU}{dT}}$ Let us derive the relation $C_V = \frac{dU}{dT}$, where U = internal energy of 1 mole of the gas.

Consider 1 mole ($n = 1$) of an ideal monoatomic gas which undergoes an isochoric process (V = constant).

From the first law of thermodynamics.

$$dQ = dW + dU \qquad \ldots\text{(i)}$$

Here,

$$dW = 0 \quad \text{as} \quad V = \text{constant}$$

$$dQ = CdT = C_V dT \qquad (\text{In } dQ = nC_V dT,\ n = 1 \text{ and } C = C_V)$$

Substituting in Eq. (i), we have

$$C_V dT = dU$$

or

$$C_V = \frac{dU}{dT}$$

Hence proved.

5. $\boldsymbol{C_p - C_V = R}$ To prove this relation (also known as **Mayor's formula**) let us consider 1 mole of an ideal gas which undergoes an isobaric (p = constant) process.

From first law of thermodynamics,

$$dQ = dW + dU \qquad \ldots\text{(ii)}$$

Here,

$$dQ = C_p dT \qquad (\text{as } n = 1 \text{ and } C = C_p)$$

$$dU = C_V dT$$

and

$$dW = pdV = pd\left(\frac{RT}{p}\right) \qquad \left(\text{as } V = \frac{RT}{P}\right)$$

$$= d(RT) \qquad (\text{as } P = \text{constant})$$

$$= RdT$$

Substituting these values in Eq. (ii)

We have

$$C_p dT = RdT + C_V dT \quad \text{or} \quad C_p - C_V = R$$

Hence proved.

6. $\boldsymbol{C_V = \frac{R}{\gamma - 1}}$ We have already derived,

$$C_p - C_V = R$$

Dividing this equation by C_V, we have

$$\frac{C_p}{C_V} - 1 = \frac{R}{C_V} \quad \text{or} \quad \gamma - 1 = \frac{R}{C_V} \qquad \left(\text{as } \frac{C_P}{C_V} = \gamma\right)$$

$\therefore$

$$C_V = \frac{R}{\gamma - 1}$$

Hence proved.

7. **Polytropic process** When p and V bear the relation pV^x = constant, where $x \neq 1$ or γ the process is called a polytropic one. In this process the molar heat capacity is,

$$C = C_V + \frac{R}{1-x} = \frac{R}{\gamma-1} + \frac{R}{1-x}$$

Let us now derive this relation. The molar heat capacity is defined as

$$C = \frac{\Delta Q}{\Delta T} \quad \text{for 1 mole}$$

$$= \frac{\Delta U + \Delta W}{\Delta T}$$

$$= \frac{\Delta U}{\Delta T} + \frac{\Delta W}{\Delta T} \qquad \left(\frac{\Delta U}{\Delta T} = C_V\right)$$

$$\therefore \quad C = C_V + \frac{\Delta W}{\Delta T} \qquad \text{...(iii)}$$

Here,

$$\Delta W = \int_{V_i}^{V_f} p\,dV = \int_{V_i}^{V_f} kV^{-x}\,dV = \left[\frac{kV^{-x+1}}{-x+1}\right]_{V_i}^{V_f}$$

$$= \frac{kV_f^{-x+1} - kV_i^{-x+1}}{-x+1}$$

$$= \frac{p_f V_f^x V_f^{-x+1} - p_i V_i^x V_i^{-x+1}}{1-x}$$

$$= \frac{p_f V_f - p_i V_i}{1-x} = \frac{RT_f - RT_i}{1-x}$$

$$= \frac{R\Delta T}{1-x}$$

$$\therefore \quad \frac{\Delta W}{\Delta T} = \frac{R}{1-x}$$

Substituting in Eq. (iii), we get the result i.e.

$$\therefore \quad C = C_V + \frac{R}{1-x} = \frac{R}{\gamma-1} + \frac{R}{1-x}$$

Solved Examples

TYPED PROBLEMS

Type 1. *Based on first law of thermodynamics applied to a general system*

Concept

First law of thermodynamics is simply law of conservation of energy which can be applied for any system.

Example 1 ***Boiling water :*** *Suppose* 1.0 *g of water vaporizes isobarically at atmospheric pressure* $(1.01 \times 10^5 \ Pa)$. *Its volume in the liquid state is* $V_i = V_{liquid} = 1.0 \ cm^3$ *and its volume in vapour state is* $V_f = V_{vapour} = 1671 \ cm^3$. *Find the work done in the expansion and the change in internal energy of the system. Ignore any mixing of the steam and the surrounding air. Take latent heat of vaporization* $L_v = 2.26 \times 10^6 \ J/kg$.

Solution Because the expansion takes place at constant pressure, the work done is

$$W = \int_{V_i}^{V_f} p_0 dV = p_0 \int_{V_i}^{V_f} dV = p_0 \ (V_f - V_i)$$

$$= (1.01 \times 10^5)\ (1671 \times 10^{-6} - 1.0 \times 10^{-6})$$

$$= 169 \text{ J}$$ **Ans.**

$$Q = mL_v = (1.0 \times 10^{-3})\ (2.26 \times 10^6)$$

$$= 2260 \text{ J}$$ **Ans.**

Hence, from the first law, the change in internal energy

$$\Delta U = Q - W = 2260 - 169$$

$$= 2091 \text{ J}$$ **Ans.**

Note *The positive value of ΔU indicates that the internal energy of the system increases. We see that most $\left(\frac{2091 \ J}{2260 \ J} = 93\%\right)$ of the energy transferred to the liquid goes into increasing the internal energy of the system only $\frac{169 \ J}{2260 \ J} = 7\%$ leaves the system by work done by the steam on the surrounding atmosphere.*

Example 2 *A metal of mass* 1 *kg at constant atmospheric pressure and at initial temperature* 20°*C is given a heat of* 20000 *J. Find the following* **(JEE 2005)**

(a) change in temperature,

(b) work done and

(c) change in internal energy.

(Given, specific heat = 400 *J/kg-°C, coefficient of cubical expansion,* $\gamma = 9 \times 10^{-5}/°C$, *density* $\rho = 9000 \ kg/m^3$, *atmospheric pressure* $= 10^5 \ N/m^2$*)*

Solution (a) From $\Delta Q = ms\Delta T$

$$\Delta T = \frac{\Delta Q}{ms} = \frac{20000}{1 \times 400} = 50°\text{C}$$

(b) $$\Delta V = V\gamma\,\Delta T = \left(\frac{1}{9000}\right)(9 \times 10^{-5})\,(50)$$

$$= 5 \times 10^{-7}\ \text{m}^3$$

$\therefore$ $$W = p_0 \cdot \Delta V = (10^5)\,(5 \times 10^{-7}) = 0.05\ \text{J}$$

(c) $$\Delta U = \Delta Q - W = (20000 - 0.05)\ \text{J}$$

$$= 19999.95\ \text{J}$$

Type 2. *To make p-V, V-T or p-T equation corresponding to a given process*

Concept

Suppose we wish to make p-V equation for a given process then with the help of equation of first law of thermodynamics and $pV = nRT$, first make an equation of type

$$f(p)\,dp + f(V)\,dV = 0$$

Now, integrating this equation we will get the desired p-V equation.

Example 3 *Make p-V equation for an adiabatic process.*

Solution In adiabatic process, $dQ = 0$

and $$dW = -\,dU$$

$\therefore$ $$pdV = -\,C_V dT \qquad \text{(for } n = 1\text{)}$$

$\therefore$ $$dT = -\frac{pdV}{C_V} \qquad \ldots\text{(i)}$$

Also, for 1 mole of an ideal gas,

$$d\,(pV) = d\,(RT)$$

or $$pdV + Vdp = RdT$$

or $$dT = \frac{pdV + Vdp}{R} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$C_V Vdp + (C_V + R)\,pdV = 0$$

or $$C_V Vdp + C_p pdV = 0$$

Dividing this equation by PV, we are left with

$$C_V \frac{dp}{p} + C_p \frac{dV}{V} = 0$$

or $$\frac{dp}{p} + \gamma \frac{dV}{V} = 0$$

or $$\int \frac{dp}{p} + \gamma \int \frac{dV}{V} = 0$$

or $$\ln\,(p) + \gamma \ln\,(V) = \text{constant}$$

We can write this in the form

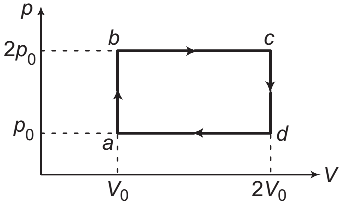
$$\boxed{pV^{\gamma} = \text{constant}}$$

Type 3. *To find values of all three terms of first law of thermodynamics from a p-V diagram provided nature of gas is given.*

Concept

Use the equation $nRT = pV$

Example 4 *A cyclic process abcd is given for a monoatomic gas* $\left(C_V = \frac{3}{2}R \text{ and } C_p = \frac{5}{2}R\right)$ *as shown in figure. Find Q, W and ΔU in each of the four processes separately. Also find the efficiency of cycle.*



Solution **Process *ab***

$$V = \text{constant} \qquad (\therefore \text{ Isochoric process})$$

$$W_{ab} = 0$$

$$\therefore \quad Q_{ab} = \Delta U_{ab} = nC_V\Delta T$$

$$= n\left(\frac{3}{2}R\right)(T_b - T_a)$$

$$= \frac{3}{2}(nRT_b - nRT_a)$$

$$= \frac{3}{2}(p_bV_b - p_aV_a)$$

$$= \frac{3}{2}(2p_0V_0 - p_0V_0)$$

$$= 1.5\ p_0V_0$$

Process *bc*

$$p = \text{constant} \qquad (\therefore \text{ Isobaric process})$$

$$Q_{bc} = nC_p\Delta T$$

$$= n\left(\frac{5}{2}R\right)(T_c - T_b)$$

$$= \frac{5}{2}(nRT_c - nRT_b)$$

$$= \frac{5}{2}(p_cV_c - p_bV_b)$$

$$= \frac{5}{2}(4p_0V_0 - 2p_0V_0)$$

$$= 5\ p_0V_0$$

$$\begin{aligned}\Delta U_{bc} &= nC_V\Delta T\\ &= n\left(\frac{3}{2}R\right)(T_c - T_b)\\ &= \frac{3}{2}(nRT_c - nRT_b)\\ &= \frac{3}{2}(p_cV_c - p_bV_b)\\ &= \frac{3}{2}(4p_0V_0 - 2p_0V_0)\\ &= 3p_0V_0\\ W_{bc} &= Q_{bc} - \Delta U_{bc} = 2p_0V_0\end{aligned}$$

Process *cd*

Again an isochoric process.

$\therefore$

$$\begin{aligned}W_{cd} &= 0\\ Q_{cd} &= \Delta U_{cd} = nC_V\Delta T\\ &= n\left(\frac{3}{2}R\right)(T_d - T_c)\\ &= \frac{3}{2}(nRT_d - nRT_c)\\ &= \frac{3}{2}(p_dV_d - p_cV_c)\\ &= \frac{3}{2}(2p_0V_0 - 4p_0V_0)\\ &= -3p_0V_0\end{aligned}$$

Process *da*

This is an isobaric process.

$\therefore$

$$\begin{aligned}Q_{da} &= nC_p\Delta T\\ &= n\left(\frac{5}{2}R\right)(T_a - T_d)\\ &= \frac{5}{2}(nRT_a - nRT_d)\\ &= \frac{5}{2}(p_aV_a - p_dV_d)\\ &= \frac{5}{2}(p_0V_0 - 2p_0V_0)\\ &= -2.5\ p_0V_0\\ \Delta U_{da} &= nC_V\Delta T\\ &= n\left(\frac{3}{2}R\right)(T_a - T_d)\\ &= \frac{3}{2}(nRT_a - nRT_d)\\ &= \frac{3}{2}(p_aV_a - p_dV_d)\end{aligned}$$

$$= \frac{3}{2}(p_0V_0 - 2p_0V_0)$$

$$= -1.5\ p_0V_0$$

$$W_{da} = Q_{da} - \Delta U_{da}$$

$$= -\ p_0V_0$$

Efficiency of cycle

In the complete cycle,

$$W_{\text{net}} = W_{ab} + W_{bc} + W_{cd} + W_{da}$$

$$= 0 + 2p_0V_0 + 0 - p_0V_0$$

$$= p_0V_0$$

Note *This W_{net} is also equal to area under the cycle.*

$$\Sigma Q_{+\text{ve}} = Q_{ab} + Q_{bc}$$

$$= 1.5p_0V_0 + 5p_0V_0$$

$$= 6.5p_0V_0$$

$\therefore$

$$\eta = \frac{W_{\text{net}}}{\Sigma Q_{+\text{ve}}} \times 100$$

$$= \left(\frac{p_0V_0}{6.5p_0V_0}\right) \times 100$$

$$= 15.38\ \%$$

Ans.

Miscellaneous Examples

Example 5 *For a Carnot cycle (or engine) discussed in article 21.4, prove that efficiency of cycle is given by*

$$\eta = \left(1 - \frac{T_2}{T_1}\right)$$

Solution Efficiency $= \dfrac{\text{net work done by gas}}{\text{heat absorbed by gas}} = \dfrac{|W_1| + |W_2| - |W_3| - |W_4|}{|Q_1|}$ ...(i)

Process 1 On this isothermal expansion process, the constant temperature is T_1 so work done by the gas

$$W_1 = nRT_1 \ln\left(\frac{V_b}{V_a}\right) \quad \text{...(ii)}$$

Remember that $V_b > V_a$, so this quantity is positive, as expected. (In process 1, the gas does work by lifting something)

In isothermal process,

$$Q = W$$

$\therefore$

$$|Q_1| = |W_1| \quad \text{...(iii)}$$

Process 2 On this adiabatic expansion process, the temperature and volume are related through

$$TV^{\gamma-1} = \text{constant}$$

∴ $$T_1 V_b^{\gamma-1} = T_2 V_c^{\gamma-1}$$

or $$\frac{T_1}{T_2} = \left(\frac{V_c}{V_b}\right)^{\gamma-1} \qquad \text{...(iv)}$$

Work done by the gas in this adiabatic process is

$$W_2 = \frac{p_c V_c - p_b V_b}{1-\gamma} = \frac{nRT_c - nRT_b}{1-\gamma}$$

$$= \left(\frac{T_2 - T_1}{1-\gamma}\right) nR = \left(\frac{T_1 - T_2}{\gamma - 1}\right) nR \qquad \text{...(v)}$$

Once again, as expected, this quantity is positive.

Process 3 In the isothermal compression process, the work done by the gas is

$$W_3 = nRT_2 \ln \frac{V_d}{V_c} \qquad \text{...(vi)}$$

Because $V_d < V_c$, the work done by the gas is negative. The work done on the gas is

$$|W_3| = -nRT_2 \ln \frac{V_d}{V_c} = nRT_2 \ln \frac{V_c}{V_d} \qquad \text{...(vii)}$$

Furthermore, just as in process 1,

$$|Q_3| = |W_3| \qquad \text{...(viii)}$$

Process 4 In the adiabatic compression process, the calculations are exactly the same as they were in process 2 but of course with different variables. Therefore,

$$\frac{T_2}{T_1} = \left(\frac{V_a}{V_d}\right)^{\gamma-1} \qquad \text{...(ix)}$$

and $$W_4 = \frac{nR}{\gamma - 1}(T_2 - T_1) \qquad \text{...(x)}$$

As expected, this quantity is negative and

$$|W_4| = -W_4 = \frac{nR}{\gamma - 1}(T_1 - T_2) \qquad \text{...(xi)}$$

We can now calculate the efficiency.

$$\text{Efficiency} = \frac{|W_1| + |W_2| - |W_3| - |W_4|}{|Q_1|}$$

$$= 1 + \frac{|W_2| - |W_3| - |W_4|}{|Q_1|} \qquad (\text{as } |W_1| = |Q_1|)$$

But our calculations show that $|W_2| = |W_4|$.

$$\text{Efficiency} = 1 - \frac{|W_3|}{|Q_1|} = 1 - \frac{nRT_2 \ln (V_c/V_d)}{nRT_1 \ln (V_b/V_a)} = 1 - \frac{T_2 \ln (V_c/V_d)}{T_1 \ln (V_b/V_a)} \qquad \text{...(xii)}$$

We have seen that

$$\frac{T_1}{T_2} = \left(\frac{V_c}{V_b}\right)^{\gamma-1} = \left(\frac{V_d}{V_a}\right)^{\gamma-1} \quad \text{or} \quad \frac{V_c}{V_b} = \frac{V_d}{V_a} \quad \text{or} \quad \frac{V_c}{V_d} = \frac{V_b}{V_a}$$

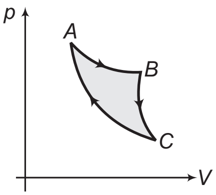
Substituting in Eq. (xii), we get

$$\boxed{\text{Efficiency} = 1 - \frac{T_2}{T_1}}$$

Example 6 *An ideal gas expands isothermally along AB and does 700 J of work.*

(a) How much heat does the gas exchange along AB?

(b) The gas then expands adiabatically along BC and does 400 J of work. When the gas returns to A along CA, it exhausts 100 J of heat to its surroundings. How much work is done on the gas along this path?



Solution (a) AB is an isothermal process. Hence,

$$\Delta U_{AB} = 0$$

and $$Q_{AB} = W_{AB} = 700 \text{ J}$$ **Ans.**

(b) BC is an adiabatic process. Hence,

$$Q_{BC} = 0$$

$$W_{BC} = 400 \text{ J}$$

∴ $$\Delta U_{BC} = -W_{BC} = -400 \text{ J}$$

ABC is a cyclic process and internal energy is a state function. Therefore,

$$(\Delta U)_{\text{whole cycle}} = 0 = \Delta U_{AB} + \Delta U_{BC} + \Delta U_{CA}$$

and from first law of thermodynamics,

$$Q_{AB} + Q_{BC} + Q_{CA} = W_{AB} + W_{BC} + W_{CA}$$

Substituting the values,

$$700 + 0 - 100 = 700 + 400 + \Delta W_{CA}$$

∴ $$\Delta W_{CA} = -500 \text{ J}$$ **Ans.**

Negative sign implies that work is done on the gas.

Table below shows different values in different processes.

Table 21.6

Process	Q (J)	W (J)	ΔU (J)
AB	700	700	0
BC	0	400	–400
CA	–100	–500	400
For complete cycle	600	600	0

Note *Total work done is 600 J, which implies that area of the closed curve is also 600 J.*

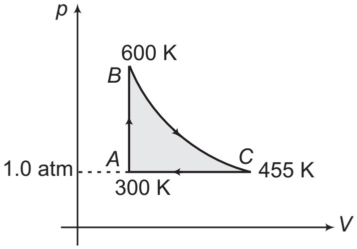
Example 7 *The p-V diagram of 0.2 mol of a diatomic ideal gas is shown in figure. Process BC is adiabatic. The value of γ for this gas is 1.4.*

(a) Find the pressure and volume at points A, B and C.

(b) Calculate $\Delta Q, \Delta W$ and ΔU for each of the three processes.

(c) Find the thermal efficiency of the cycle.

Take 1 atm $= 1.0 \times 10^5 \ N/m^2$.



Solution (a) $p_A = p_C = 1 \text{ atm} = 1.01 \times 10^5 \text{ N/m}^2$

Process AB is an isochoric process.

$$\therefore \qquad p \propto T \quad \text{or} \quad \frac{p_B}{p_A} = \frac{T_B}{T_A}$$

$$\therefore \qquad p_B = \left(\frac{T_B}{T_A}\right) p_A = \left(\frac{600}{300}\right) (1 \text{ atm}) = 2 \text{ atm}$$

$$= 2.02 \times 10^5 \text{ N/m}^2$$

From ideal gas equation $\qquad V = \dfrac{nRT}{p}$

$$\therefore \qquad V_A = V_B = \frac{nRT_A}{p_A}$$

$$= \frac{(0.2)\,(8.31)\,(300)}{(1.01 \times 10^5)} \approx 5.0 \times 10^{-3} \text{ m}^3$$

$$= 5 \text{ L}$$

and $\qquad V_C = \dfrac{nRT_C}{p_C} = \dfrac{(0.2)\,(8.31)\,(455)}{(1.01 \times 10^5)}$

$$= 7.5 \times 10^{-3} \text{ m}^3$$

$$\approx 7.5 \text{ L}$$

Table 21.7

State	p	V
A	1 atm	5 L
B	2 atm	5 L
C	1 atm	7.5 L

(b) **Process AB** is an isochoric process. Hence,

$$\Delta W_{AB} = 0$$

$$\Delta Q_{AB} = \Delta U_{AB} = nC_V \Delta T = n\left(\frac{5}{2}R\right)(T_B - T_A)$$

$$= (0.2)\left(\frac{5}{2}\right)(8.31)\,(600 - 300)$$

$$\approx 1246 \text{ J}$$

Process BC is an adiabatic process. Hence,

$$\Delta Q_{BC} = 0$$

$$\therefore \qquad \Delta W_{BC} = -\,\Delta U_{BC}$$

$$\Delta U_{BC} = nC_V \Delta T = nC_V\,(T_C - T_B)$$

$$= (0.2)\left(\frac{5}{2}R\right)(455 - 600)$$

$$= (0.2)\left(\frac{5}{2}\right)(8.31)\,(-\,145) \text{ J}$$

$$\approx -\,602 \text{ J}$$

$$\therefore \qquad \Delta W_{BC} = -\,\Delta U_{BC} = 602 \text{ J}$$

Process *CA* is an isobaric process. Hence,

$$\Delta Q_{CA} = nC_p\Delta T = n\left(\frac{7}{2}R\right)(T_A - T_C)$$

$$= (0.2)\left(\frac{7}{2}\right)(8.31)(300 - 455)$$

$$\approx -902 \text{ J}$$

$$\Delta U_{CA} = nC_V\Delta T$$

$$= \frac{\Delta Q_{CA}}{\gamma} \qquad \left(\text{as } \gamma = \frac{C_p}{C_V}\right)$$

$$= -\frac{902}{1.4} \approx -644 \text{ J}$$

$$\therefore \quad \Delta W_{CA} = \Delta Q_{CA} - \Delta U_{CA}$$

$$= -258 \text{ J}$$

Table 21.8

Process	ΔQ (in J)	ΔW (in J)	ΔU (in J)
AB	1246	0	1246
BC	0	602	– 602
CA	– 902	– 258	– 644
Total	344	344	0

(c) Efficiency of the cycle

$$\eta = \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100 = \frac{344}{1246} \times 100$$

$$= 27.6\%$$

Example 8 *Find the molar specific heat of the process* $p = \frac{a}{T}$ *for a monoatomic gas, a being constant.*

Solution We know that

$$dQ = dU + dW$$

Specific heat,

$$C = \frac{dQ}{dT} = \frac{dU}{dT} + \frac{dW}{dT} \qquad \text{...(i)}$$

Since,

$$dU = C_V dT \qquad \text{...(ii)}$$

$$C = C_V + \frac{dW}{dT}$$

$$= C_V + \frac{pdV}{dT}$$

$$pV = RT$$

∴ For the given process,

$$V = \frac{RT}{p} = \frac{RT^2}{a}$$

$$\frac{dV}{dT} = \frac{2RT}{a}$$

$$\therefore \qquad C = C_V + p\left(\frac{2RT}{a}\right)$$

$$= C_V + 2R$$

$$= \frac{3}{2}R + 2R = \frac{7}{2}R$$

Example 9 *At 27° C two moles of an ideal monatomic gas occupy a volume V. The gas expands adiabatically to a volume $2V$. Calculate*

(a) final temperature of the gas

(b) change in its internal energy and

(c) the work done by the gas during the process.

$[R = 8.31\ J/mol\text{-}K]$

Solution (a) In case of adiabatic change

$$TV^{\gamma-1} = \text{constant}$$

So that $$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \text{ with } \gamma = \left(\frac{5}{3}\right)$$

i.e. $$300 \times V^{2/3} = T(2V)^{2/3}$$

or $$T = \frac{300}{(2)^{2/3}} = 189 \text{ K}$$

(b) As $$\Delta U = nC_V\Delta T = n\left(\frac{3}{2}R\right)\Delta T$$

So, $$\Delta U = 2 \times \left(\frac{3}{2}\right) \times 8.31\ (189 - 300)$$

$$= -2767.23 \text{ J}$$

Negative sign means internal energy will decrease.

(c) According to first law of thermodynamics

$$Q = \Delta U + \Delta W$$

And as for adiabatic change $\Delta Q = 0$,

$$\Delta W = -\Delta U = 2767.23 \text{ J}$$

Example 10 *Two moles of a diatomic ideal gas is taken through pT = constant. Its temperature is increased from T to $2T$. Find the work done by the system?*

Solution pT = constant

$$\therefore \qquad p(pV) = \text{constant} \qquad (\text{as } T \propto pV)$$

or $$pV^{1/2} = \text{constant}$$

Comparing with $$pV^x = \text{constant}$$

We have, $$x = \frac{1}{2}$$

$$W = \frac{nR\Delta T}{1-x} = \frac{2R(2T-T)}{1-\frac{1}{2}}$$

$$= 4RT$$

Example 11 *An ideal monatomic gas at temperature 27°C and pressure $10^6\ N/m^2$ occupies 10 L volume. 10,000 cal of heat is added to the system without changing the volume. Calculate the change in temperature of the gas. Given : $R = 8.31$ J/mol-K and $J = 4.18$ J/cal.*

Solution For n moles of gas, we have $pV = nRT$

Here, $p = 10^6\ \text{N/m}^2$, $V = 10\ L = 10^{-2}\ \text{m}^3$ and $T = 27°\text{C} = 300\ \text{K}$

$$\therefore \qquad n = \frac{pV}{RT} = \frac{10^6 \times 10^{-2}}{8.31 \times 300} = 4.0$$

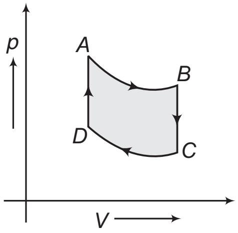
For monatomic gas, $C_V = \frac{3}{2}R$

Thus,
$$C_V = \frac{3}{2} \times 8.31\ \text{J/mol-K}$$
$$= \frac{3}{2} \times \frac{8.31}{4.18} \approx 3\ \text{cal/mol-K}$$

Let ΔT be the rise in temperature when n moles of the gas is given Q cal of heat at constant volume. Then,

$$Q = nC_V\Delta T \quad \text{or} \quad \Delta T = \frac{Q}{nC_V}$$
$$= \frac{10000\ \text{cal}}{4.0\ \text{mole} \times 3\ \text{cal/mol-K}}$$
$$= 833\ \text{K}$$

Example 12 *One mole of a monoatomic ideal gas is taken through the cycle shown in figure.*



$A \to B$ Adiabatic expansion
$B \to C$ Cooling at constant volume
$C \to D$ Adiabatic compression.
$D \to A$ Heating at constant volume

The pressure and temperature at A, B etc., are denoted by p_A, T_A; p_B, T_B etc. respectively.

Given, $T_A = 1000K$, $p_B = \left(\frac{2}{3}\right)p_A$ and $p_C = \left(\frac{1}{3}\right)p_A$. Calculate

(a) the work done by the gas in the process $A \to B$
(b) the heat lost by the gas in the process $B \to C$

Given, $\left(\frac{2}{3}\right)^{0.4} = 0.85$ and $R = 8.31$ J/mol-K

Solution (a) As for adiabatic change $pV^\gamma = \text{constant}$

$$\text{i.e.} \qquad p\left(\frac{nRT}{p}\right)^\gamma = \text{constant} \qquad (\text{as } pV = nRT)$$

$$\text{i.e.} \qquad \frac{T^\gamma}{p^{\gamma-1}} = \text{constant so} \left(\frac{T_B}{T_A}\right)^\gamma = \left(\frac{p_B}{p_A}\right)^{\gamma-1}, \text{ where } \gamma = \frac{5}{3}$$

i.e. $$T_B = T_A\left(\frac{2}{3}\right)^{1-\frac{1}{\gamma}} = 1000\left(\frac{2}{3}\right)^{2/5} = 850\text{ K}$$

So, $$W_{AB} = \frac{nR[T_F - T_I]}{[1-\gamma]} = \frac{1\times 8.31\,[1000-850]}{\left[\left(\frac{5}{3}\right)-1\right]}$$

i.e. $$W_{AB} = 1869.75\text{ J}$$

(b) For $B \to C$, $V =$ constant so $\Delta W = 0$

So, from first law of thermodynamics

$$\Delta Q = \Delta U + \Delta W = nC_V\Delta T + 0$$

or $$\Delta Q = 1\times\left(\frac{3}{2}R\right)(T_C - 850) \qquad \left(\text{as } C_V = \frac{3}{2}R\right)$$

Now, along path BC, $V =$ constant; $p \propto T$

i.e. $$\frac{p_C}{p_B} = \frac{T_C}{T_B}$$

$$T_C = \frac{\left(\frac{1}{3}\right)p_A}{\left(\frac{2}{3}\right)p_A}\times T_B = \frac{T_B}{2} = \frac{850}{2} = 425\text{ K} \qquad \text{...(ii)}$$

So, $$\Delta Q = 1\times\frac{3}{2}\times 8.31\,(425-850) = -5297.625\text{ J}$$

[Negative heat means, heat is lost by the system]

Example 13 *A gas undergoes a process such that $p \propto \frac{1}{T}$. If the molar heat capacity for this process is $C = 33.24$ J/mol-K, find the degree of freedom of the molecules of the gas.*

Solution As $$p \propto \frac{1}{T}$$

or $$pT = \text{constant} \qquad \text{...(i)}$$

We have for one mole of an ideal gas

$$pV = RT \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$p^2V = \text{constant}$$

or $$pV^{1/2} = K\ \text{(say)} = p_iV_i^{1/2} = p_fV_f^{1/2} \qquad \text{...(iii)}$$

From first law of thermodynamics,

$$\Delta Q = \Delta U + \Delta W$$

or $$C\Delta T = C_V\Delta T + \Delta W$$

or $$C = C_V + \frac{\Delta W}{\Delta T} \qquad \text{...(iv)}$$

Here, $$\Delta W = \int p\,dV = K\int_{V_i}^{V_f} V^{-1/2}dV$$

$$= 2K\,[V_f^{1/2} - V_i^{1/2}] = 2\,[p_fV_f^{1/2}V_f^{1/2} - p_iV_i^{1/2}V_i^{1/2}]$$

$$= 2\,[p_fV_f - p_iV_i] = 2R\,[T_f - T_i]$$

$$= \frac{R\Delta T}{1/2} \quad \Rightarrow \quad \frac{\Delta W}{\Delta T} = 2R$$

Substituting in Eq. (iv), we have

$$C = C_V + 2R = \frac{R}{\gamma - 1} + 2R$$

Substituting the values, $33.24 = R\left(\frac{1}{\gamma - 1} + 2\right) = 8.31\left(\frac{1}{\gamma - 1} + 2\right)$

Solving this we get $\gamma = 1.5$

Now, $\gamma = 1 + \frac{2}{F}$

or degree of freedom $F = \frac{2}{\gamma - 1} = \frac{2}{1.5 - 1} = 4$

Alternate Solution In the process pV^x = constant, molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$$

The given process is $pV^{1/2}$ = constant or $x = \frac{1}{2}$

$\therefore$ $$C = \frac{R}{\gamma - 1} + \frac{R}{1 - \frac{1}{2}} = \frac{R}{\gamma - 1} + 2R$$

Now, we may proceed in the similar manner.

Example 14 *A gaseous mixture enclosed in a vessel consists of one gram mole of a gas A with* $\gamma = \left(\frac{5}{3}\right)$ *and some amount of gas B with* $\gamma = \frac{7}{5}$ *at a temperature T. The gases A and B do not react with each other and are assumed to be ideal. Find the number of gram moles of the gas B if* γ *for the gaseous mixture is* $\left(\frac{19}{13}\right)$.

Solution As for an ideal gas, $C_p - C_V = R$ and $\gamma = \left(\frac{C_p}{C_V}\right)$

So, $$C_V = \frac{R}{(\gamma - 1)}$$

$\therefore$ $$(C_V)_1 = \frac{R}{\left(\frac{5}{3}\right) - 1} = \frac{3}{2}R;$$

$$(C_V)_2 = \frac{R}{\left(\frac{7}{5}\right) - 1} = \frac{5}{2}R$$

and $$(C_V)_{\text{mix}} = \frac{R}{\left(\frac{19}{13}\right) - 1} = \frac{13}{6}R$$

Now, from conservation of energy,

i.e $$\Delta U = \Delta U_1 + \Delta U_2$$

$$(n_1 + n_2)\,(C_V)_{\text{mix}}\,\Delta T = [n_1 (C_V)_1 + n_2\,(C_V)_2]\,\Delta T$$

i.e.
$$(C_V)_{\text{mix}} = \frac{n_1\,(C_V)_1 + n_2\,(C_V)_2}{n_1 + n_2}$$

We have
$$\frac{13}{6}R = \frac{1 \times \frac{3}{2}R + n_2 \frac{5}{2}R}{1 + n_2}$$

$$= \frac{(3 + 5n_2)R}{2\,(1 + n_2)}$$

or
$$13 + 13n_2 = 9 + 15n_2,$$

i.e.
$$n_2 = 2$$ **Ans.**

Example 15 *An ideal gas having initial pressure p, volume V and temperature T is allowed to expand adiabatically until its volume becomes $5.66\,V$, while its temperature falls to $T/2$.*

(a) How many degrees of freedom do the gas molecules have?

(b) Obtain the work done by the gas during the expansion as a function of the initial pressure p and volume V.

Given that $(5.66)^{0.4} = 2$

Solution (a) For adiabatic expansion,

$$TV^{\gamma - 1} = \text{constant}$$

i.e.
$$TV^{\gamma - 1} = T'\,V'^{\gamma - 1}$$

$$= \frac{T}{2}\,(5.66\,V)^{\gamma - 1}$$

i.e.
$$(5.66)^{\gamma - 1} = 2$$

i.e.
$$\gamma = 1.4$$

Using $\gamma = 1 + \dfrac{2}{F}$

We get degree of freedom,

$$F = 5$$ **Ans.**

(b) Work done during adiabatic process for one mole gas is

$$W = \frac{1}{1 - \gamma}\,[p'\,V' - pV]$$

From relation,
$$\frac{pV}{T} = \frac{p'\,V'}{T'}$$

We get
$$p' = \frac{T'}{T} \cdot \frac{pV}{V'} = \frac{1}{2} \times \frac{1}{5.66}\,p = \frac{p}{11.32}$$

∴
$$W = \frac{1}{1 - 1.4}\left[\frac{p}{11.32} \times \frac{V}{5.66} - pV\right]$$

$$= \frac{1}{0.4}\left[1 - \frac{1}{11.32 \times 5.66}\right] pV$$

$$= 2.461\,pV$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***
(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***
(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*
(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. Assertion : In adiabatic expansion, temperature of gas always decreases.
Reason : In adiabatic process exchange of heat is zero.

2. Assertion : In a thermodynamic process, initial volume of gas is equal to final volume of gas. Work done by gas in this process should be zero.
Reason : Work done by gas in isochoric process is zero.

3. Assertion : First law of thermodynamics can be applied for ideal gases only.
Reason : First law is simply, law of conservation of energy.

4. Assertion : When ice melts, work is done by atmosphere on (ice + water) system.
Reason : On melting of ice volume of (ice + water) system decreases.

5. Assertion : Between two thermodynamic states, the value of $(Q - W)$ is constant for any process.
Reason : Q and W are path functions.

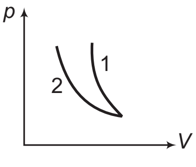
6. Assertion : Efficiency of a heat engine can't be greater than efficiency of Carnot engine.
Reason : Efficiency of any engine is never 100%.

7. Assertion : In the process $pT =$ constant, if temperature of gas is increased work done by the gas is positive.
Reason : For the given process, $V \propto T$.

8. Assertion : In free expansion of a gas inside an adiabatic chamber Q, W and ΔU all are zero.
Reason : In such an expansion $p \propto \frac{1}{V}$.

9. Assertion : For an ideal gas in a cyclic process and in an isothermal process change in internal energy is zero.
Reason : In both processes there is no change in temperature.

10. Assertion : Isothermal and adiabatic, two processes are shown on p-V diagram. Process-1 is adiabatic and process-2 is isothermal.
Reason : At a given point, slope of adiabatic process $= \gamma \times$ slope of isothermal process.

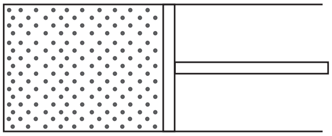


Objective Questions

1. In a process, the pressure of an ideal gas is proportional to square of the volume of the gas. If the temperature of the gas increases in this process, then work done by this gas

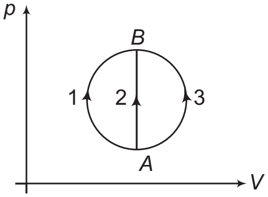
(a) is positive
(b) is negative
(c) is zero
(d) may be positive or negative

2. n moles of a gas are filled in a container at temperature T. If the gas is slowly and isothermally compressed to half its initial volume, the work done by the atmosphere on the gas is



(a) $\frac{nRT}{2}$
(b) $-\frac{nRT}{2}$
(c) $nRT \ln 2$
(d) $-nRT \ln 2$

3. A gas undergoes A to B through three different processes 1, 2 and 3 as shown in the figure. The heat supplied to the gas is Q_1, Q_2 and Q_3 respectively, then

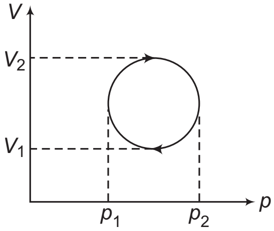


(a) $Q_1 = Q_2 = Q_3$
(b) $Q_1 < Q_2 < Q_3$
(c) $Q_1 > Q_2 > Q_3$
(d) $Q_1 = Q_3 > Q_2$

4. For an adiabatic compression the quantity pV

(a) increases
(b) decreases
(c) remains constant
(d) depends on γ

5. The cyclic process form a circle on a pV diagram as shown in figure. The work done by the gas is

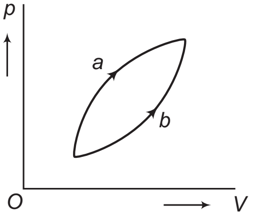


(a) $\frac{\pi}{4}(p_2 - p_1)^2$
(b) $\frac{\pi}{4}(V_2 - V_1)^2$
(c) $\frac{\pi}{2}(p_2 - p_1)(V_2 - V_1)$
(d) $\frac{\pi}{4}(p_2 - p_1)(V_1 - V_2)$

6. An ideal gas has initial volume V and pressure p. In doubling its volume the minimum work done will be in the process (of the given processes)

(a) isobaric process
(b) isothermal process
(c) adiabatic process
(d) same in all given processes

7. Figure shows two processes a and b for a given sample of a gas. If $\Delta Q_1, \Delta Q_2$ are the amounts of heat absorbed by the system in the two cases and $\Delta U_1, \Delta U_2$ are changes in internal energies respectively, then



(a) $\Delta Q_1 = \Delta Q_2; \Delta U_1 = \Delta U_2$
(b) $\Delta Q_1 > \Delta Q_2; \Delta U_1 > \Delta U_2$
(c) $\Delta Q_1 < \Delta Q_2; \Delta U_1 < \Delta U_2$
(d) $\Delta Q_1 > \Delta Q_2; \Delta U_1 = \Delta U_2$

8. A Carnot engine works between 600 K and 300 K. The efficiency of the engine is
(a) 50% (b) 70%
(c) 20% (d) 80%

9. Air in a cylinder is suddenly compressed by a piston which is then maintained at the same position. As the time passes pressure of the gas
(a) increases
(b) decreases
(c) remains the same
(d) may increase or decrease depending on the nature of the gas

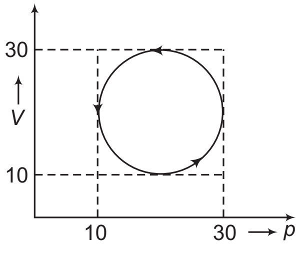
10. A cycle pump becomes hot near the nozzle after a few quick strokes even if they are smooth because
(a) the volume of air decreases (b) the number of air molecules increases
(c) the compression is adiabatic (d) collision between air particles increases

11. In an adiabatic change, the pressure p and temperature T of a diatomic gas are related by the relation $p \propto T^{\alpha}$, where α equals
(a) 1.67 (b) 0.4
(c) 0.6 (d) 3.5

12. A diatomic gas obeys the law pV^x = constant. For what value of x, it has negative molar specific heat ?
(a) $x > 1.4$ (b) $x < 1.4$
(c) $1 < x < 1.4$ (d) $0 < x < 1$

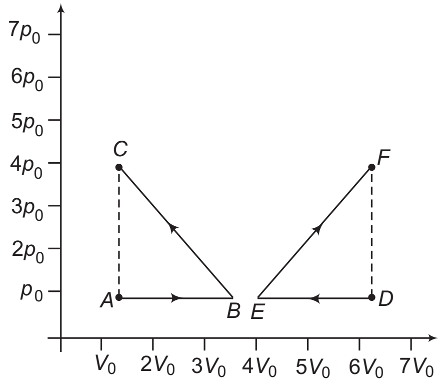
13. The molar specific heat at constant volume of gas mixture is $\frac{13R}{6}$. The gas mixture consists of
(a) 2 moles of O_2 and 4 moles of H_2 (b) 2 moles of O_2 and 4 moles of argon
(c) 2 moles of argon and 4 moles of O_2 (d) 2 moles of CO_2 and 4 moles of argon

14. Heat energy absorbed by a system in going through a cyclic process as shown in the figure [V in litres and p in kPa] is



(a) $10^7 \pi$J (b) $10^4 \pi$J (c) $10^2 \pi$J (d) $10^3 \pi$J

15. If W_{ABC} is the work done in process $A \to B \to C$ and W_{DEF} is work done in process $D \to E \to F$ as shown in the figure, then



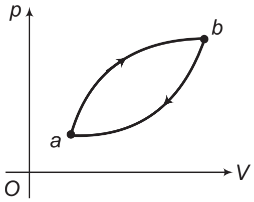
(a) $|W_{DEF}| > |W_{ABC}|$
(b) $|W_{DEF}| < |W_{ABC}|$
(c) $W_{DEF} = W_{ABC}$
(d) $W_{DEF} = -W_{ABC}$

Subjective Questions

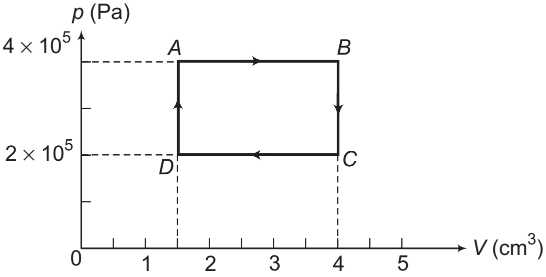
1. How many moles of helium at temperature 300 K and 1.00 atm pressure are needed to make the internal energy of the gas 100 J?

2. Show how internal energy U varies with T in isochoric, isobaric and adiabatic process?

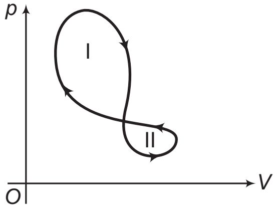
3. A system is taken around the cycle shown in figure from state a to state b and then back to state a. The absolute value of the heat transfer during one cycle is 7200 J. (a) Does the system absorb or liberate heat when it goes around the cycle in the direction shown in the figure? (b) What is the work W done by the system in one cycle? (c) If the system goes around the cycle in a counter-clock wise direction, does it absorb or liberate heat in one cycle? What is the magnitude of the heat absorbed or liberated in one counter-clockwise cycle?



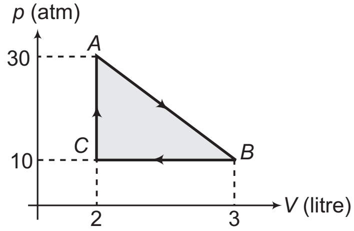
4. For the thermodynamic cycle shown in figure find (a) net output work of the gas during the cycle, (b) net heat flow into the gas per cycle.



5. A thermodynamic system undergoes a cyclic process as shown in figure. The cycle consists of two closed loops, loop I and loop II. (a) Over one complete cycle, does the system do positive or negative work? (b) In each of loops I and II, is the net work done by the system positive or negative ? (c) Over one complete cycle, does heat flow into or out of the system? (d) In each of loops I and II, does heat flow into or out of the system?



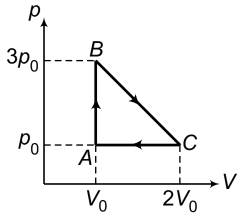
6. A gas undergoes the cycle shown in figure. The cycle is repeated 100 times per minute. Determine the power generated.



7. One mole of an ideal monoatomic gas is initially at 300 K. Find the final temperature if 200 J of heat is added (a) at constant volume (b) at constant pressure.

8. A closed vessel 10 L in volume contains a diatomic gas under a pressure of 10^5 N/m^2. What amount of heat should be imparted to the gas to increase the pressure in the vessel five times?

9. One mole of an ideal monatomic gas is taken round the cyclic process *ABCA* as shown in figure. Calculate



(a) the work done by the gas.
(b) the heat rejected by the gas in the path *CA* and heat absorbed in the path *AB*.
(c) the net heat absorbed by the gas in the path *BC*.
(d) the maximum temperature attained by the gas during the cycle.

10. A diatomic ideal gas is heated at constant volume until its pressure becomes three times. It is again heated at constant pressure until its volume is doubled. Find the molar heat capacity for the whole process.

11. Two moles of a certain gas at a temperature $T_0 = 300\,\text{K}$ were cooled isochorically so that the pressure of the gas got reduced 2 times. Then as a result of isobaric process, the gas is allowed to expand till its temperature got back to the initial value. Find the total amount of heat absorbed by gas in this process.

12. Five moles of an ideal monoatomic gas with an initial temperature of 127°C expand and in the process absorb 1200 J of heat and do 2100 J of work. What is the final temperature of the gas?

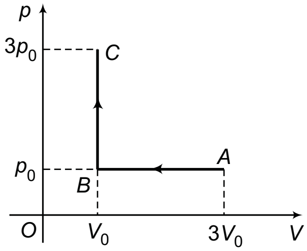
13. Find the change in the internal energy of 2 kg of water as it is heated from 0°C to 4°C. The specific heat capacity of water is 4200 J/kg-K and its densities at 0°C and 4°C are 999.9 kg/m^3 and 1000 kg/m^3, respectively. Atmospheric pressure = 10^5 Pa.

14. Calculate the increase in the internal energy of 10 g of water when it is heated from 0°C to 100°C and converted into steam at 100 kPa. The density of steam = 0.6 kg/m^3. Specific heat capacity of water = 4200 J/kg-°C and the latent heat of vaporisation of water = 2.5×10^6 J/kg.

15. One gram of water (1 cm^3) becomes 1671 cm^3 of steam when boiled at a constant pressure of 1 atm (1.013×10^5 Pa). The heat of vaporization at this pressure is $L_v = 2.256 \times 10^6$ J/kg. Compute (a) the work done by the water when it vaporizes and (b) its increase in internal energy.

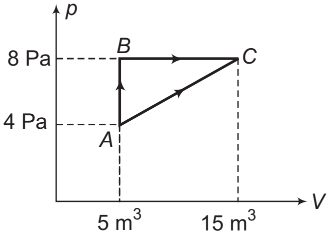
16. A gas in a cylinder is held at a constant pressure of 2.30×10^5 Pa and is cooled and compressed from 1.70 m^3 to 1.20 m^3. The internal energy of the gas decreases by 1.40×10^5 J. (a) Find the work done by the gas. (b) Find the absolute value $|Q|$ of the heat flow into or out of the gas and state the direction of the heat flow. (c) Does it matter whether or not the gas is ideal? Why or why not?

17. p-V diagram of an ideal gas for a process ABC is as shown in the figure.



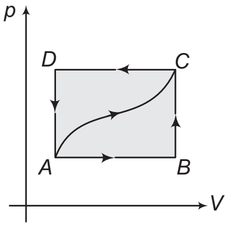
(a) Find total heat absorbed or released by the gas during the process ABC.
(b) Change in internal energy of the gas during the process ABC.
(c) Plot pressure *versus* density graph of the gas for the process ABC.

18. In the given graph, an ideal gas changes its state from A to C by two paths ABC and AC.



(a) Find the path along which work done is less.
(b) The internal energy of gas at A is 10 J and the amount of heat supplied in path AC is 200 J. Calculate the internal energy of gas at C.
(c) The internal energy of gas at state B is 20 J. Find the amount of heat supplied to the gas to go from A to B.

19. When a gas expands along *AB*, it does 500 J of work and absorbs 250 J of heat. When the gas expands along *AC*, it does 700 J of work and absorbs 300 J of heat.



(a) How much heat does the gas exchange along *BC*?

(b) When the gas makes the transition from *C* to *A* along *CDA*, 800 J of work are done on it from *C* to *D*. How much heat does it exchange along *CDA*?

20. A 1.0 kg bar of copper is heated at atmospheric pressure ($1.01 \times 10^5\ \text{N/m}^2$). If its temperature increases from 20°C to 50°C, calculate the change in its internal energy. $\alpha = 7.0 \times 10^{-6}/{}^\circ\text{C}$, $\rho = 8.92 \times 10^3\ \text{kg/m}^3$ and $c = 387\ \text{J/kg-}^\circ\text{C}$

21. One mole of an ideal monoatomic gas occupies a volume of $1.0 \times 10^{-2}\ \text{m}^3$ at a pressure of $2.0 \times 10^5\ \text{N/m}^2$.

(a) What is the temperature of the gas?

(b) The gas undergoes an adiabatic compression until its volume is decreased to $5.0 \times 10^{-3}\ \text{m}^3$. What is the new gas temperature?

(c) How much work is done on the gas during the compression?

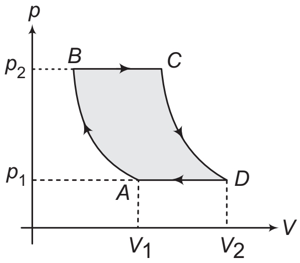
(d) What is the change in the internal energy of the gas?

22. A bullet of mass 10 g travelling horizontally at 200 m/s strikes and embeds in a pendulum bob of mass 2.0 kg.

(a) How much mechanical energy is dissipated in the collision?

(b) Assuming that C_v for the bob plus bullet is $3R$, calculate the temperature increase of the system due to the collision. Take the molecular mass of the system to be 200 g/mol.

23. An ideal gas is carried through a thermodynamic cycle consisting of two isobaric and two isothermal processes as shown in figure. Show that the net work done in the entire cycle is given by the equation. $W_{\text{net}} = p_1 (V_2 - V_1) \ln \dfrac{p_2}{p_1}$



24. An ideal gas is enclosed in a cylinder with a movable piston on top. The piston has mass of 8000 g and an area of $5.00\ \text{cm}^2$ and is free to slide up and down, keeping the pressure of the gas constant. How much work is done as the temperature of 0.200 mol of the gas is raised from 200°C to 300°C?

LEVEL 2

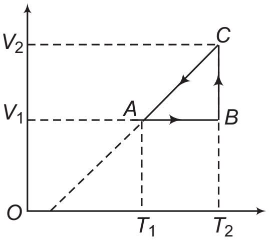
Single Correct Option

1. The equation of a state of a gas is given by $p(V-b)=nRT$. If 1 mole of a gas is isothermally expanded from volume V and $2V$, the work done during the process is

(a) $RT\ln\left|\frac{2V-b}{V-b}\right|$ (b) $RT\ln\left|\frac{V-b}{V}\right|$

(c) $RT\ln\left|\frac{V-b}{2V-b}\right|$ (d) $RT\ln\left|\frac{V}{V-b}\right|$

2. A cyclic process for 1 mole of an ideal gas is shown in the V-T diagram. The work done in AB, BC and CA respectively is



(a) $0, RT_2\ln\left|\frac{V_2}{V_1}\right|, R(T_1-T_2)$

(b) $R(T_1-T_2), 0, RT_1\ln\left|\frac{V_1}{V_2}\right|$

(c) $0, RT_1\ln\left|\frac{V_1}{V_2}\right|, R(T_1-T_2)$

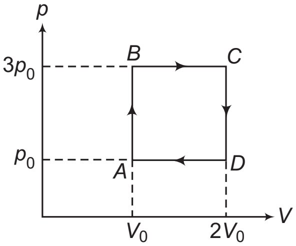
(d) $0, RT_2\ln\left|\frac{V_2}{V_1}\right|, R(T_2-T_1)$

3. Ten moles of a diatomic perfect gas are allowed to expand at constant pressure. The initial volume and temperature are V_0 and T_0, respectively. If $\frac{7}{2}RT_0$ heat is transferred to the gas, then the final volume and temperature are

(a) $1.1\,V_0, 1.1\,T_0$ (b) $0.9\,V_0, 0.9\,T_0$

(c) $1.1\,V_0, \frac{10}{11}T_0$ (d) $0.9\,V_0, \frac{10}{9}T_0$

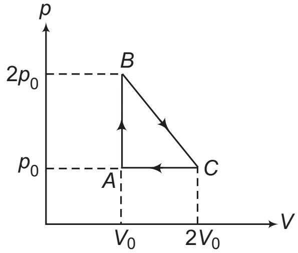
4. An ideal monoatomic gas is carried around the cycle $ABCDA$ as shown in the figure. The efficiency of the gas cycle is



(a) $\frac{4}{21}$ (b) $\frac{2}{21}$

(c) $\frac{4}{31}$ (d) $\frac{2}{31}$

5. In the process shown in figure, the internal energy of an ideal gas decreases by $\frac{3p_0V_0}{2}$ in going from point C to A. Heat transfer along the process CA is



(a) $(-3\,p_0V_0)$
(b) $(-5\,p_0V_0/2)$
(c) $(-3\,p_0V_0/2)$
(d) zero

6. One mole of an ideal monoatomic gas at temperature T_0 expands slowly according to the law $\frac{p}{V}$ = constant. If the final temperature is $2T_0$, heat supplied to the gas is

(a) $2RT_0$
(b) $\frac{3}{2}RT_0$
(c) RT_0
(d) $\frac{1}{2}RT_0$

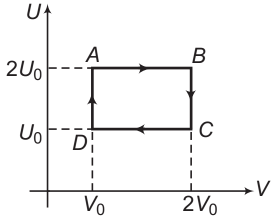
7. A mass of gas is first expanded isothermally and then compressed adiabatically to its original volume. What further simplest operation must be performed on the gas to restore it to its original state?

(a) An isobaric cooling to bring its temperature to initial value
(b) An isochoric cooling to bring its pressure to its initial value
(c) An isothermal process to take its pressure to its initial value
(d) An isochoric heating to bring its temperature to initial value

8. A monatomic ideal gas, initially at temperature T_1, is enclosed in a cylinder fitted with a frictionless piston. The gas is allowed to expand adiabatically to a temperature T_2 by releasing the piston suddenly. If L_1 and L_2 are the lengths of gas column before and after expansion respectively, then $\frac{T_1}{T_2}$ is given by

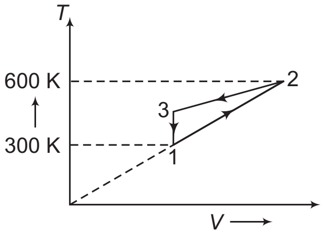
(a) $\left(\frac{L_1}{L_2}\right)^{\frac{2}{3}}$
(b) $\frac{L_1}{L_2}$
(c) $\frac{L_2}{L_1}$
(d) $\left(\frac{L_2}{L_1}\right)^{\frac{2}{3}}$

9. One mole of an ideal gas is taken through a cyclic process. The minimum temperature during the cycle is 300 K. Then, net exchange of heat for complete cycle is



(a) $600\,R\ln 2$
(b) $300\,R\ln 2$
(c) $-300\,R\ln 2$
(d) $900\,R\ln 2$

10. Two moles of an ideal gas are undergone a cyclic process 1-2-3-1. If net heat exchange in the process is 300 J, the work done by the gas in the process 2-3 is



(a) –500 J (b) – 5000 J (c) – 3000 J (d) None of these

11. Two cylinders fitted with pistons contain equal amount of an ideal diatomic gas at 300 K. The piston of *A* is free to move, while that of *B* is held fixed. The same amount of heat is given to the gas in each cylinder. If the rise in temperature of the gas in *A* is 30 K, then the rise in temperature of gas in *B* is

(a) 30 K (b) 18 K
(c) 50 K (d) 42 K

12. A gas follows a process TV^{n-1} = constant, where T = absolute temperature of the gas and V = volume of the gas. The bulk modulus of the gas in the process is given by

(a) $(n-1)p$ (b) $p/(n-1)$
(c) np (d) p/n

13. One mole of an ideal gas at temperature T_1 expands slowly according to the law $\frac{p}{V}$ = constant. Its final temperature is T_2. The work done by the gas is

(a) $R(T_2-T_1)$ (b) $2R(T_2-T_1)$
(c) $\frac{R}{2}(T_2-T_1)$ (d) $\frac{2R}{3}(T_2-T_1)$

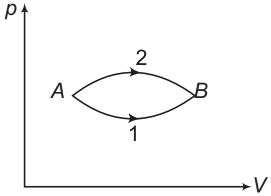
14. 600 J of heat is added to a monoatomic gas in a process in which the gas performs a work of 150 J. The molar heat capacity for the process is

(a) $3R$ (b) $4R$
(c) $2R$ (d) $6R$

15. The internal energy of a gas is given by $U = 2pV$. It expands from V_0 to $2V_0$ against a constant pressure p_0. The heat absorbed by the gas in the process is

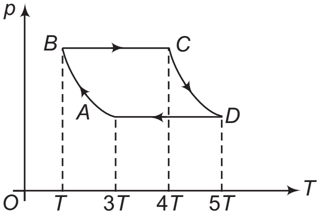
(a) $2p_0V_0$ (b) $4p_0V_0$
(c) $3p_0V_0$ (d) p_0V_0

16. The figure shows two paths for the change of state of a gas from *A* to *B*. The ratio of molar heat capacities in path 1 and path 2 is



(a) < 1 (b) > 1
(c) 1 (d) Data insufficient

17. p-T diagram of one mole of an ideal monatomic gas is shown. Processes AB and CD are adiabatic. Work done in the complete cycle is

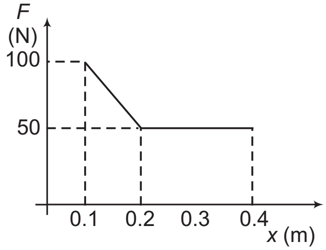


(a) $2.5\,RT$ (b) $-2\,RT$
(c) $1.5\,RT$ (d) $-3.5\,RT$

18. An ideal monoatomic gas undergoes a process in which its internal energy U and density ρ vary as $U\rho$ = constant. The ratio of change in internal energy and the work done by the gas is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$
(c) $\frac{1}{3}$ (d) $\frac{3}{5}$

19. The given figure shows the variation of force applied by ideal gas on a piston which undergoes a process during which piston position changes from 0.1 to 0.4 m. If the internal energy of the system at the end of the process is 2.5 J higher, then the heat absorbed during the process is



(a) 15 J (b) 17.5 J
(c) 20 J (d) 22.5 J

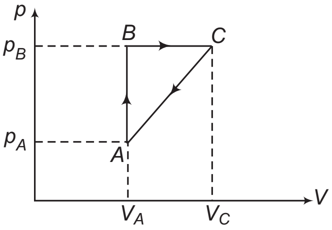
20. A gas can expand through two processes : (i) isobaric, (ii) $\frac{p}{V}$ = constant. Assuming that the initial volume is same in both processes and the final volume which is two times the initial volume is also same in both processes, which of the following is true?

(a) Work done by gas in process (i) is greater than the work done by the gas in process (ii)
(b) Work done by gas in process (i) is smaller than the work done by the gas in process (ii)
(c) Final pressure is greater in process (i)
(d) Final temperature is greater in process (i)

21. An ideal gas of adiabatic exponent γ is expanded so that the amount of heat transferred to the gas is equal to the decrease of its internal energy. Then, the equation of the process in terms of the variables T and V is

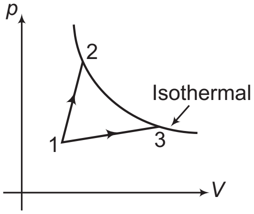
(a) $TV^{\frac{(\gamma-1)}{2}} = C$ (b) $TV^{\frac{(\gamma-2)}{2}} = C$
(c) $TV^{\frac{(\gamma-1)}{4}} = C$ (d) $TV^{\frac{(\gamma-2)}{4}} = C$

22. A thermodynamical process is shown in the figure with $p_A = 3\times p_{\text{atm}}$, $V_A = 2\times 10^{-4}\ \text{m}^3$, $p_B = 8\times p_{\text{atm}}$, $V_C = 5\times 10^{-4}\ \text{m}^3$. In the process AB and BC, 600 J and 200 J heat are added to the system. Find the change in internal energy of the system in the process CA. [$1\ p_{\text{atm}} = 10^5\ \text{N/m}^2$]



(a) 560 J
(b) –560 J
(c) –240 J
(d) +240 J

23. A gas takes part in two processes in which it is heated from the same initial state 1 to the same final temperature. The processes are shown on the p-V diagram by the straight lines 1-3 and 1-2. 2 and 3 are the points on the same isothermal curve. Q_1 and Q_2 are the heat transfer along the two processes. Then,

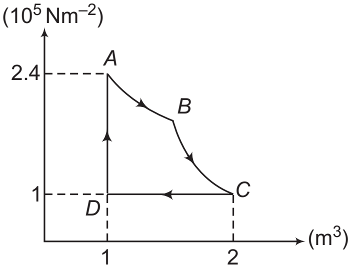


(a) $Q_1 = Q_2$
(b) $Q_1 < Q_2$
(c) $Q_1 > Q_2$
(d) Insufficient data

24. A closed system receives 200 kJ of heat at constant volume. It then rejects 100 kJ of heat while it has 50 kJ of work done on it at constant pressure. If an adiabatic process can be found which will restore the system to its initial state, the work done by the system during this process is

(a) 100 kJ
(b) 50 kJ
(c) 150 kJ
(d) 200 kJ

25. 100 moles of an ideal monatomic gas undergoes the thermodynamic process as shown in the figure

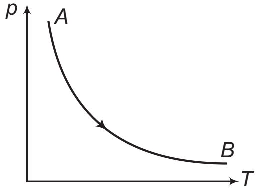


$A\rightarrow B$: isothermal expansion
$B\rightarrow C$: adiabatic expansion
$C\rightarrow D$: isobaric compression
$D\rightarrow A$: isochoric process

The heat transfer along the process AB is 9×10^4 J.The net work done by the gas during the cycle is [Take $R = 8\ \text{JK}^{-1}\ \text{mol}^{-1}$)

(a) -0.5×10^4 J
(b) $+0.5\times 10^4$ J
(c) -5×10^4 J
(d) $+5\times 10^4$ J

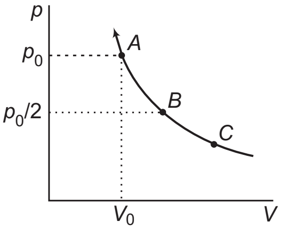
26. Two moles of an ideal monoatomic gas are expanded according to the equation pT = constant from its initial state (p_0, V_0) to the final state due to which its pressure becomes half of the initial pressure. The change in internal energy is



(a) $\frac{3p_0V_0}{4}$ (b) $\frac{3p_0V_0}{2}$

(c) $\frac{9p_0V_0}{2}$ (d) $\frac{5p_0V_0}{2}$

27. The state of an ideal gas is changed through an isothermal process at temperature T_0 as shown in figure. The work done by the gas in going from state B to C is double the work done by gas in going from state A to B. If the pressure in the state B is $\frac{p_0}{2}$, then the pressure of the gas in state C is



a) $\frac{p_0}{3}$ (b) $\frac{p_0}{4}$

(c) $\frac{p_0}{6}$ (d) $\frac{p_0}{8}$

More than One Correct Options

1. An ideal gas is taken from the state A (pressure p, volume V) to the state B (pressure $\frac{p}{2}$, volume $2V$) along a straight line path in the p-V diagram. Select the correct statement(s) from the following.

(a) The work done by the gas in the process A to B is negative

(b) In the T-V diagram, the path AB becomes a part of a parabola

(c) In the p-T diagram, the path AB becomes a part of a hyperbola

(d) In going from A to B, the temperature T of the gas first increases to a maximum value and then decreases

2. In the process pV^2 = constant, if temperature of gas is increased, then

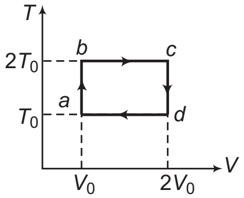
(a) change in internal energy of gas is positive

(b) work done by gas is positive

(c) heat is given to the gas

(d) heat is taken out from the gas

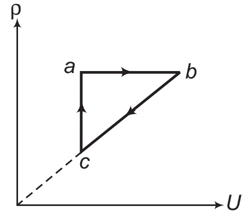
3. T-V diagram of two moles of a monoatomic gas is as shown in figure.



For the process *abcda* choose the correct options given below

(a) $\Delta U = 0$
(b) work done by gas > 0
(c) heat given to the gas is $4RT_0$
(d) heat given to the gas is $2RT_0$

4. Density (ρ) *versus* internal energy (U) graph of a gas is as shown in figure. Choose the correct options.



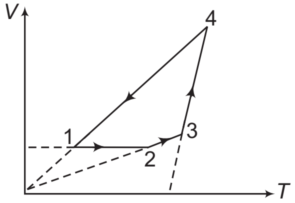
(a) $Q_{bc} = 0$ (b) $W_{bc} = 0$ (c) $W_{ca} < 0$ (d) $Q_{ab} > 0$

Here, W is work done by gas and Q is heat given to the gas.

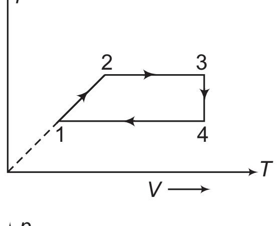
5. Temperature of a monatomic gas is increased from T_0 to $2T_0$ in three different processes : isochoric, isobaric and adiabatic. Heat given to the gas in these three processes are Q_1, Q_2 and Q_3 respectively. Then, choose the correct option.

(a) $Q_1 > Q_3$
(b) $Q_2 > Q_1$
(c) $Q_2 > Q_3$
(d) $Q_3 = 0$

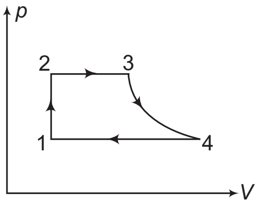
6. A cyclic process 1-2-3-4-1 is depicted on V-T diagram. The p-T and p-V diagrams for this cyclic process are given below. Select the correct choices (more than one options is/are correct)



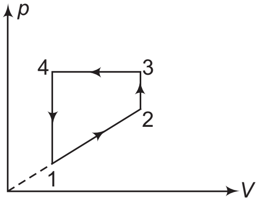
(a)



(b)



(c)

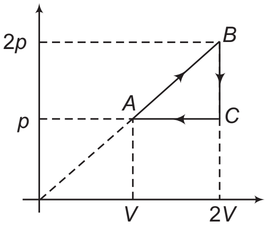


(d) None of these

Comprehension Based Questions

Passage I (Q. No. 1 and 2)

One mole of a monatomic ideal gas is taken along the cycle ABCA as shown in the diagram.



1. The net heat absorbed by the gas in the given cycle is

(a) pV (b) $\frac{pV}{2}$

(c) $2pV$ (d) $4pV$

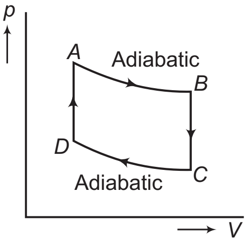
2. The ratio of specific heat in the process CA to the specific heat in the process BC is

(a) 2 (b) $\frac{5}{3}$

(c) 4 (d) None of these

Passage II (Q. No. 3 to 5)

One mole of a monatomic ideal gas is taken through the cycle ABCDA as shown in the figure. $T_A = 1000\,K$ *and* $2p_A = 3p_B = 6p_C$.



$$\left[Assume \left(\frac{2}{3}\right)^{0.4} = 0.85 \text{ and } R = \frac{25}{3} JK^{-1}\,mol^{-1} \right]$$

3. The temperature at B is

(a) 350 K (b) 1175 K

(c) 850 K (d) 577 K

4. Work done by the gas in the process $A \rightarrow B$ is

(a) 5312 J (b) 1875 J

(c) 6251 J (d) 8854 J

5. Heat lost by the gas in the process $B \rightarrow C$ is

(a) 5312 J (b) 1875 J

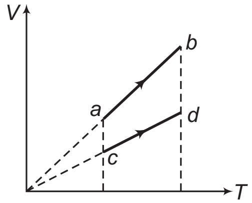
(c) 6251 J (d) 8854 J

Match the Columns

1. Temperature of 2 moles of a monatomic gas is increased from T to $2T$. Match the following two columns.

Column I	Column II
(a) Work done by gas in isobaric process	(p) $3RT$
(b) Change in internal energy in isobaric process	(q) $4RT$
(c) Work done by gas in adiabatic process	(r) $2RT$
(d) Change in internal energy in an adiabatic process	(s) None of these

2. For V-T diagrams of two processes a - b and c - d are shown in figure for same gas. Match the following two columns.



Column I	Column II
(a) Work done	(p) is more in process ab
(b) Change in internal energy	(q) is more in process ca
(c) Heat exchange	(r) is same in both processes
(d) Molar heat capacity	(s) Can't say anything

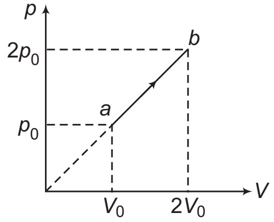
3. Temperature of a monatomic gas is increased by ΔT in process $p^2V =$ constant. Match the following two columns.

Column I	Column II
(a) Work done by gas	(p) $2nR\Delta T$
(b) Change in internal energy of gas	(q) $5nR\Delta T$
(c) Heat taken by the gas	(r) $3nR\Delta T$
(d) Work done on the gas	(s) None of these

4. Match the following two columns.

Column I	Column II
(a) Isobaric expansion	(p) $W > \Delta U$
(b) Isochoric cooling	(q) $W < \Delta U$
(c) Adiabatic expansion	(r) $Q = \Delta U$
(d) Isothermal expansion	(s) $Q < \Delta U$

5. Heat taken by a gas in process a-b is $6p_0V_0$. Match the following columns.



Column I	Column II
(a) W_{ab}	(p) $2p_0V_0$
(b) ΔU_{ab}	(q) $4p_0V_0$
(c) Molar heat capacity in given process	(r) $2R$
(d) C_V of gas	(s) None of these

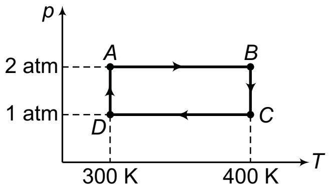
Subjective Questions

1. Two moles of helium gas undergo a cyclic process as shown in figure. Assuming the gas to be ideal, calculate the following quantities in this process.

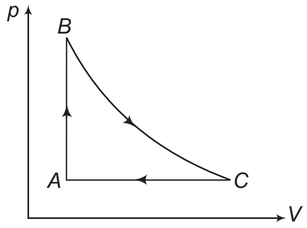
(a) The net change in the heat energy.

(b) The net work done.

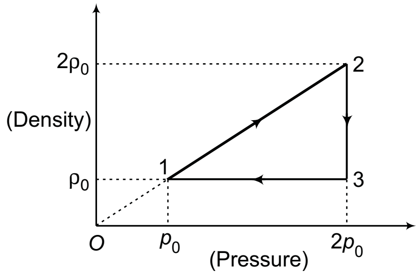
(c) The net change in internal energy.



2. 1.0 k-mol of a sample of helium gas is put through the cycle of operations shown in figure. BC is an isothermal process and $p_A = 1.00$ atm, $V_A = 22.4\ \text{m}^3$, $p_B = 2.00$ atm. What are T_A, T_B and V_C?



3. The density (ρ) *versus* pressure (p) graph of one mole of an ideal monoatomic gas undergoing a cyclic process is shown in figure. Molecular mass of gas is M.



(a) Find work done in each process.

(b) Find heat rejected by gas in one complete cycle.

(c) Find the efficiency of the cycle.

4. An ideal gas goes through the cycle *abc*. For the complete cycle 800 J of heat flows out of the gas. Process *ab* is at constant pressure and process *bc* is at constant volume . In process *c-a*, $p \propto V$. States *a* and *b* have temperatures $T_a = 200\,\text{K}$ and $T_b = 300\,\text{K}$. (a) Sketch the p-V diagram for the cycle. (b) What is the work done by the gas for the process *ca*?

5. A cylinder of ideal gas is closed by an 8 kg movable piston of area 60 cm^2 . The atmospheric pressure is 100 kPa. When the gas is heated from 30° C to 100° C, the piston rises 20 cm. The piston is then fastened in the place and the gas is cooled back to 30° C. If ΔQ_1 is the heat added to the gas during heating and ΔQ_2 is the heat lost during cooling, find the difference.

6. Three moles of an ideal gas $\left(C_p = \frac{7}{2} R\right)$ at pressure p_0 and temperature T_0 is isothermally expanded to twice its initial volume. It is then compressed at a constant pressure to its original volume.

(a) Sketch p-V and p-T diagram for complete process.

(b) Calculate net work done by the gas.

(c) Calculate net heat supplied to the gas during complete process.

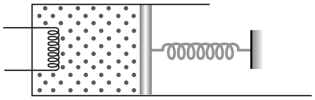
(Write your answer in terms of gas constant = R)

7. Two moles of a gas ($\gamma = 5/3$) are initially at temperature 27°C and occupy a volume of 20 litres. The gas is first expanded at constant pressure until the volume is doubled. Then, it is subjected to an adiabatic change until the temperature returns to its initial value.

(a) Sketch the process on a p-V diagram.

(b) What are final volume and pressure of the gas?

(c) What is the work done by the gas?



8. An ideal monoatomic gas is confined by a spring loaded massless piston of cross-section $8.0 \times 10^{-3}\,\text{m}^2$. Initially, the gas is at 300 K and occupies a volume of $2.4 \times 10^{-3}\,\text{m}^3$ and the spring is in its relaxed state. The gas is heated by an electric heater until the piston moves out slowly without friction by 0.1 m. Calculate

(a) the final temperature of the gas and

(b) the heat supplied by the heater.

The force constant of the spring is 8000 N/m, atmospheric pressure is $1.0 \times 10^5\,\text{N/m}^2$. The cylinder and the piston are thermally insulated.

9. An ideal diatomic gas $\left(\gamma = \frac{7}{5}\right)$ undergoes a process in which its internal energy relates to the volume as $U = \alpha\sqrt{V}$, where α is a constant.

(a) Find the work performed by the gas to increase its internal energy by 100 J.

(b) Find the molar specific heat of the gas.

10. For an ideal gas the molar heat capacity varies as $C = C_V + 3aT^2$. Find the equation of the process in the variables (T, V) where a is a constant.

11. One mole of an ideal monatomic gas undergoes the process $p = \alpha T^{1/2}$, where α is a constant.

(a) Find the work done by the gas if its temperature increases by 50 K.

(b) Also, find the molar specific heat of the gas.

12. One mole of a gas is put under a weightless piston of a vertical cylinder at temperature T. The space over the piston opens into atmosphere. Initially, piston was in equilibrium. How much work should be performed by some external force to increase isothermally the volume under the piston to twice the volume? (Neglect friction of piston).

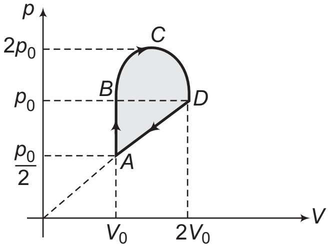
13. An ideal monatomic gas undergoes a process where its pressure is inversely proportional to its temperature.

(a) Calculate the molar specific heat for the process.

(b) Find the work done by two moles of gas if the temperature changes from T_1 to T_2.

14. The volume of one mole of an ideal gas with the adiabatic exponent γ is changed according to the relation $V = \frac{a}{T}$, where a is a constant. Find the amount of heat absorbed by the gas in the process, if the temperature is increased by ΔT.

15. Two moles of a monatomic ideal gas undergo a cyclic process *ABCDA* as shown in figure. *BCD* is a semicircle. Find the efficiency of the cycle.



16. Pressure p, volume V and temperature T for a certain gas are related by

$$p = \frac{\alpha T - \beta T^2}{V}$$

where, α and β are constants. Find the work done by the gas if the temperature changes from T_1 to T_2 while the pressure remains the constant.

17. An ideal gas has a specific heat at constant pressure $C_p = \frac{5R}{2}$. The gas is kept in a closed vessel of volume V_0 at temperature T_0 and pressure p_0. An amount of $10\ p_0V_0$ of heat is supplied to the gas.

(a) Calculate the final pressure and temperature of the gas.

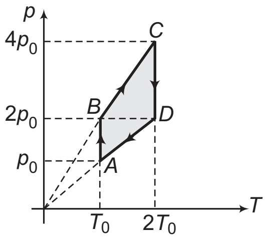
(b) Show the process on *p*-*V* diagram.

18. Pressure *versus* temperature (p-T) graph of n moles of an ideal gas is shown in figure. Plot the corresponding:

(a) density *versus* volume (ρ-V) graph,

(b) pressure *versus* volume (p-V) graph and

(c) density *versus* pressure (ρ-p) graph.



19. Three moles of an ideal gas being initially at a temperature $T_i = 273$ K were isothermally expanded 5 times its initial volume and then isochorically heated so that the pressure in the final state becomes equal to that in the initial state. The total heat supplied in the process is 80 kJ. Find $\gamma \left(= \frac{C_p}{C_V}\right)$ of the gas.

Answers

Introductory Exercise 21.1

1. $a = 80$ J, $b = 0$, $c = 50$ J, $d = 40$ J, $e = 40$ J, $f = 40$ J, $g = 40$ J **2.** 327 J

Introductory Exercise 21.2

1. (a) -6.8×10^4 J (b) 1.78×10^5 J, out of gas (c) No

2. (a) Positive (b) Into the system (c) In loop 1, into the system, In loop 2, out of the system

3. 2.67×10^{-2} mol **4.** $W = 0, Q = \Delta U = 1800\,R$ **5.** (i) $2\,p_0V_0$ (ii) zero (iii) $\frac{3}{2}\,p_0V_0$

6. $-600\,R$ **7.** $\frac{1}{2}\,p_0V_0$

Introductory Exercise 21.3

1. (a) 316 K (b) 309.6 K **2.** Into gas

3.

Process	Q	W	ΔU
BC	–	0	–
CA	–	–	–
AB	+	+	+

4. (a) No (b) No (c) No **5.** γ and $\frac{\gamma}{\gamma - 1}$ **6.** $Q_1 > Q_2$

7. 1.18 MJ **8.** 0.6 kJ, 1.0 kJ, 1.6

Introductory Exercise 21.4

1. 2.72×10^5 cal, 72.72 % **2.** 60 % **3.** (a) 25% (b) 6.25×10^5 J (c) 18.75×10^5 J

4. 2×10^6 cal **5.** 117° C and 52° C **6.** Refrigerator A **7.** 35 W **8.** 18.18%

Exercises

LEVEL 1

Assertion and Reason

1. (b) **2.** (d) **3.** (d) **4.** (a) **5.** (b) **6.** (d) **7.** (c) **8.** (b) **9.** (a) **10.** (a or b)

Objective Questions

1. (a) **2.** (c) **3.** (c) **4.** (a) **5.** (d) **6.** (c) **7.** (d) **8.** (a) **9.** (b) **10.** (c)
11 (d) **12.** (c) **13.** (c) **14.** (c) **15.** (d)

Subjective Questions

1. 2.67×10^{-2} mol **2.** $\Delta U = \frac{nR\Delta T}{\gamma - 1}$ for all processes

3. (a) absorbs (b) 7200 J (c) liberates, 7200 J **4.** (a) 0.50 J (b) 0.50 J

5. (a) positive (b) I→ positive, II→ negative (c) into the system (d) I→ into the system II→ out of the system. **6.** 1.68×10^3 W

7. (a) 316 K (b) 310 K **8.** 10^4 J **9.** (a) p_0V_0 (b) $\frac{5}{2}p_0V_0, 3p_0V_0$ (c) $\frac{p_0V_0}{2}$ (d) $\frac{25p_0V_0}{8R}$

10. $C = 3.1R$ **11.** 2.49 kJ **12.** 113°C

13. 33600.2 J **14.** 2.75×10^4 J

15. (a) 169 J (b) 2087 J **16.** (a) -1.15×10^5 J (b) 2.55×10^5 J, out of gas (c) No

17. (a) $Q_{ABC} = -2P_0V_0$ (b) $\Delta U_{ABC} = 0$ **18.** (a) AC (b) 150 J (c) 10 J

19. (a) – 150 J (b) – 400 J **20.** 11609.99287 J

21. (a) 241 K (b) 383 K (c) 1770 J (d) 1770 J **22.** (a) 200 J (b) 0.80°C

24. 166 J

LEVEL 2

Single Correct Option

1.(a) **2.**(a) **3.**(a) **4.**(a) **5.**(b) **6.**(a) **7.**(b) **8.**(d) **9.**(b) **10.**(d)
11(d) **12.**(c) **13.**(c) **14.**(c) **15.**(c) **16.**(a) **17.**(a) **18.**(a) **19.**(c) **20.**(b)
21.(a) **22.**(b) **23.**(c) **24.**(c) **25.**(d) **26.**(b) **27.**(d)

More than One Correct Options

1.(b,d) **2.**(a,c) **3.**(a,b) **4.**(c,d) **5.**(a,b,c,d) **6.**(a,b)

Comprehension Based Questions

1.(b) **2.**(b) **3.**(c) **4.**(b) **5.**(a)

Match the Columns

1. (a) → r (b) → p (c) → s (d) → p
2. (a) → s (b) → s (c) → s (d) → r
3. (a) → p (b) → s (c) → s (d) → s
4. (a) → q (b) → p,r (c) → p (d) → p
5. (a) → s (b) → s (c) → r (d) → s

Subjective Questions

1. (a) 1153 J (b) 1153 J (c) zero

2. $T_A = 273$ K, $T_B = 546$ K, $V_C = 44.8$ m^3

3. (a) $W_{12} = \frac{-p_0M}{\rho_0}\ln(2)$, $W_{23} = \frac{p_0M}{\rho_0}$, $W_{31} = 0$ (b) $\frac{p_0M}{\rho_0}\left(\frac{3}{2} + \ln 2\right)$ (c) $\frac{2}{5}(1 - \ln 2)$

4. (b) – 4000 J **5.** 136 J

6. (b) $3RT_0\ln(2) - \frac{3}{2}RT_0$ (c) $3RT_0\ln(2) - \frac{21}{4}RT_0$ **7.** (b) 113.1 L, 0.44×10^5 N/m^2 (c) 12479 J

8. (a) 800 K (b) 720 J **9.** (a) 80 J (b) $\frac{9R}{2}$

10. $Ve^{-\left(\frac{3a}{2R}\right)T^2}$ = constant **11.** (a) 207.75 J (b) $2R$

12. $RT(1 - \ln 2)$

13. (a) $\frac{7R}{2}$ (b) $4R(T_2 - T_1)$

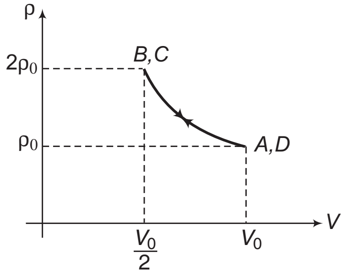
14. $\frac{(2 - \gamma)R\Delta T}{(\gamma - 1)}$

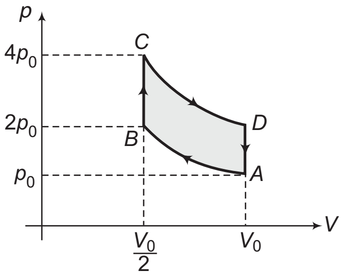
15. 25.8%

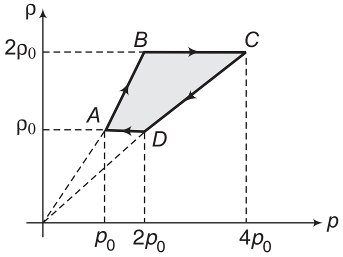
16. $\alpha(T_2 - T_1) - \beta(T_2^2 - T_1^2)$

17. (a) $\frac{23}{3} p_0, \frac{23}{3} T_0$

18.


ρ
2ρ0
ρ0
B,C
A,D
V
V0/2
V0


p
4p0
2p0
p0
C
D
B
A
V
V0/2
V0


ρ
2ρ0
ρ0
B
C
A
D
p
p0
2p0
4p0

19. 1.4

22

Calorimetary and Heat Transfer

Chapter Contents

22.1 Calorimetry

22.2 Heat Transfer

22.1 Calorimetry

Heat transfer between two substances, due to a temperature difference (may be in same or different states) comes under the topic calorimetry. Let us discuss this topic pointwise.

Specific Heat (c)

The amount of heat required to raise the temperature of unit mass of a substance by 1°C (or 1 K) is called its specific heat. Thus,

$$\boxed{c = \frac{Q}{m\Delta\theta} \quad \text{or} \quad \frac{Q}{m\Delta T}} \qquad \ldots\text{(i)}$$

The SI unit of specific heat is J/kg-K. Because heat is so frequently measured in calories, the unit cal/g-°C is also used quite often.

Heat Required to Change The Temperature

From Eq. (i), we can write

$$\boxed{Q = mc\Delta\theta \quad \text{or} \quad mc\Delta T} \qquad \ldots\text{(ii)}$$

Thus Eq. (ii) is used when temperature of a substance changes without change in state.

Note (i) *In general c is a function of temperature. But normally variation in c is small. So, it is assumed to be constant for wide range of temperature. For example*

"Specific heat of water is 1 cal/g-°C between 14.5 °C to 15.5 °C". This implies that 1 cal of heat will be required to raise the temperature of 1 g of water from 14.5 °C to 15.5 °C. From 15.5 °C to 16.5 °C, a different amount of heat (≠ 1 cal) will be required. But normally this variation is very small. So, it is assumed constant (= 1 cal/g-°C) from 0 °C to 100 °C.

(ii) *If c is given a function of temperature, then Eq. (ii) cannot be applied directly for calculation of Q. Rather integration will be used. Thus,*

$$\boxed{Q = m\int_{T_i}^{T_f} c\,dT \quad \text{or} \quad m\int_{\theta_i}^{\theta_f} c\,d\theta} \qquad \ldots\text{(iii)}$$

In the above expression, c is a function of T (or θ).

(iii) *The specific heat of water is much larger than that of most other substances. Consequently, for the same amount of added heat, the temperature change of a given mass of water is generally less than that for the same mass of another substance. For this reason a large body of water moderates the climate of nearby land. In the winter, the water cools off more slowly than the surrounding land and tends to warm the land. In the summer, the opposite effect occurs as the water heats up more slowly than the land.*

(iv) *Specific heat is also represented by s or S. So, in different problems we have taken all notations for it.*

Latent Heat (L)

The amount of heat required to change the state (or phase) of unit mass of a substance at constant temperature is called latent heat L. Thus,

$$\boxed{L = \frac{Q}{m}} \qquad \ldots\text{(iv)}$$

For a solid-liquid transition, the latent heat is known as the **latent heat of fusion (L_f)** and for the liquid-gas transition, it is known as the **latent heat of vaporization (L_v).**

For water at 1 atmosphere, latent heat of fusion is 80.0 cal/g. This simply means 80.0 cal of heat are required to melt 1.0 g of water or 80.0 cal heat is liberated when 1.0 g of water freezes at 0°C. Similarly, latent heat of vaporization for water at 1 atmosphere is 539 cal/g.

Heat Required to Change the State (or Phase) at Constant Temperature

From Eq. (iv), we can see that

$$\boxed{Q = mL} \qquad \text{...(v)}$$

So, this is the heat required to change the state (or phase) at constant temperature.

Extra Points to Remember

- In some problems, two or more than two substances at different temperatures are mixed. Heat is given by the hot bodies and taken by the cold bodies. Finally, all substances reach to a common equilibrium temperature. All such problems can be solved by a single equation,

  Heat given by hot bodies = Heat taken by cold bodies

  But note that, on both sides of this equation, we have to take the positive signs. To make them positive, always write

  $$\Delta\theta = \theta_{\text{Higher}} - \theta_{\text{Lower}}$$

  in the equation $Q = mc\Delta\theta$ when there is a change of temperature without change in state.
- **Water equivalent of a container** Normally, a liquid is heated in a container. So, some heat is wasted in heating the container also. Suppose water equivalent of a container is 10 g, then it implies that heat required to increase the temperature of this container is equal to heat required to increase the temperature of 10 g of water.

Example 22.1 *How much heat is required to convert* 8.0 *g of ice at* – 15°C *to steam at* 100°C? *(Given,* $c_{ice} = 0.53$ *cal/g-°C,* $L_f = 80$ *cal/g and* $L_v = 539$ *cal/g, and* $c_{water} = 1$ *cal/g-°C)*

Solution

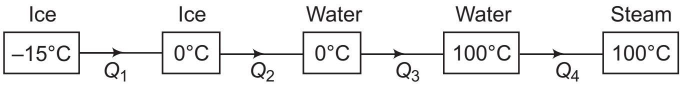


Fig. 22.1

$$Q_1 = mc_{\text{ice}}\,(T_f - T_i)$$
$$= (8.0)\,(0.53)\,[0 - (-15)] = 63.6\,\text{cal}$$
$$Q_2 = mL_f = (8)\,(80) = 640\,\text{cal}$$
$$Q_3 = mc_{\text{water}}\,(T_f - T_i)$$
$$= (8.0)\,(1.0)\,[100 - 0] = 800\,\text{cal}$$
$$Q_4 = mL_v = (8.0)\,(539) = 4312\,\text{cal}$$

∴ Net heat required, $Q = Q_1 + Q_2 + Q_3 + Q_4$

$= 5815.6$ cal **Ans.**

Example 22.2 *In the above problem if heat is supplied at a constant rate of $q = 10$ cal/min, then plot temperature versus time graph.*

Solution

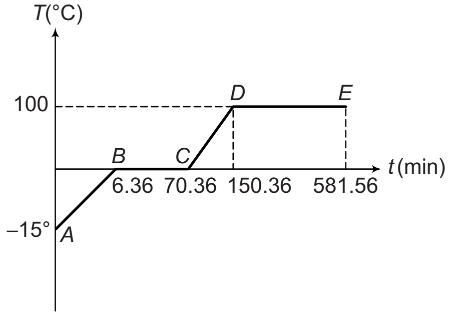


Fig. 22.2

From *A* to *B*

(i) Temperature of ice will increase from –15° C to 0°C.

(ii)
$$t_{AB} = \frac{\text{Total heat required}}{\text{Heat supplied per minute}} = \frac{Q_1}{q}$$
$$= \frac{63.6}{10} = 6.36 \text{ min}$$

(iii) Between *A* and *B* we will get only ice.

From *B* to *C*

(i) Temperature of (ice + water) mixture will remain constant at 0°C.

(ii)
$$t_{BC} = \frac{Q_2}{q} = \frac{640}{10} = 64 \text{ min}$$

$$\therefore \quad t_{\text{Total}} = t_{AB} + t_{BC} = 70.36 \text{ min}$$

(iii) Between *B* and *C* we will get both ice and water.

From *C* to *D*

(i) Temperature of water increases from 0°C to 100°C

(ii)
$$t_{CD} = \frac{Q_3}{q} = \frac{800}{10} = 80 \text{ min}$$

$$\therefore \quad t_{\text{Total}} = t_{AB} + t_{BC} + t_{CD} = 150.36 \text{ min}$$

(iii) Between *C* and *D* we will get only water.

From *D* to *E*

(i) Temperature of (water + steam) mixture will remain constant at 100°C.

(ii)
$$t_{DE} = \frac{Q_4}{q} = \frac{4312}{10} = 431.2 \text{ min}$$

$$\therefore \quad t_{\text{Total}} = t_{AB} + t_{BC} + t_{CD} + t_{DE} = 581.56 \text{ min}$$

(iii) Between *D* and *A* we will get both water and steam.

The corresponding graph is as shown in Fig. 22.2.

Exercise *AB* and *CD* correspond to temperature change without change in state. *BC* and *DE* correspond to state change without change in temperature.

In the given condition, prove that *AB* and *CD* are straight lines and slope of these lines is inversely proportional to specific heat in that state. Further, prove that lengths of lines *BC* and *DE* are proportional to latent heat in that state.

Example 22.3 *10 g of water at 70°C is mixed with 5 g of water at 30°C. Find the temperature of the mixture in equilibrium. Specific heat of water is 1 cal/g-°C*

Solution Let $t°$ C be the temperature of the mixture. From energy conservation,

Heat given by 10 g of water = Heat taken by 5 g of water

or $$m_1 c_{\text{water}} |\Delta t_1| = m_2 c_{\text{water}} |\Delta t_2|$$

∴ $$(10)(1)(70 - t) = 5(1)(t - 30)$$

∴ $$t = 56.67° \text{ C}$$ **Ans.**

Example 22.4 *The temperature of equal masses of three different liquids A, B and C are 12°C, 19°C and 28°C respectively. The temperature when A and B are mixed is 16°C and when B and C are mixed it is 23°C. What should be the temperature when A and C are mixed?*

Solution Let m be the mass of each liquid and s_A, s_B, s_C specific heats of liquids A, B and C respectively. When A and B are mixed. The final temperature is 16° C.

∴ Heat gained by A = Heat lost by B

i.e. $$ms_A (16 - 12) = ms_B (19 - 16)$$

i.e. $$s_B = \frac{4}{3} s_A \qquad \text{...(i)}$$

When B and C are mixed.

Heat gained by B = Heat lost by C

i.e. $$ms_B (23 - 19) = ms_C (28 - 23)$$

i.e. $$s_C = \frac{4}{5} s_B \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), $$S_C = \frac{4}{5} \times \frac{4}{3} S_A = \frac{16}{15} S_A$$

When A and C are mixed, let the final temperature be θ

Heat gained by A = Heat lost by C

∴ $$ms_A (\theta - 12) = ms_C (28 - \theta)$$

i.e. $$\theta - 12 = \frac{16}{15}(28 - \theta)$$

By solving, we get

$$\theta = \frac{628}{31} = 20.26°\text{C}$$ **Ans.**

Example 22.5 *In a container of negligible mass 30 g of steam at 100°C is added to 200 g of water that has a temperature of 40°C. If no heat is lost to the surroundings, what is the final temperature of the system? Also find masses of water and steam in equilibrium. Take* $L_v = 539$ *cal/g and* $c_{water} = 1$ *cal/g-°C.*

Solution Let Q be the heat required to convert 200 g of water at 40°C into 100°C, then

$$Q = mc\Delta T = (200)(1.0)(100 - 40) = 12000 \text{ cal}$$

Now, suppose m_0 mass of steam converts into water to liberate this much amount of heat, then

$$m_0 = \frac{Q}{L} = \frac{12000}{539} = 22.26 \text{ g}$$

Since, it is less than 30 g, the temperature of the mixture is 100°C. **Ans.**

Mass of steam in the mixture = 30 – 22.26 = 7.74 g **Ans.**

and mass of water in the mixture = 200 + 22.26 = 222.26 g **Ans.**

Example 22.6 *In an insulated vessel, 0.05 kg steam at 373 K and 0.45 kg of ice at 253 K are mixed. Find the final temperature of the mixture (in kelvin).*

Given,

$$L_{fusion} = 80 \text{ cal/g} = 336 \text{ J/g}$$

$$L_{vaporization} = 540 \text{ cal/g} = 2268 \text{ J/g}$$

$$s_{ice} = 2100 \text{ J/kg-K} = 0.5 \text{ cal/g-K}$$

and

$$s_{water} = 4200 \text{ J/kg-K} = 1 \text{ cal/g-K}$$

(JEE 2006)

Solution

$$0.05 \text{ kg steam at } 373 \text{ K} \xrightarrow{Q_1} 0.05 \text{ kg water at } 373 \text{ K}$$

$$0.05 \text{ kg water at } 373 \text{ K} \xrightarrow{Q_2} 0.05 \text{ kg water at } 273 \text{ K}$$

$$0.45 \text{ kg ice at } 253 \text{ K} \xrightarrow{Q_3} 0.45 \text{ kg ice at } 273 \text{ K}$$

$$0.45 \text{ kg ice at } 273 \text{ K} \xrightarrow{Q_4} 0.45 \text{ kg water at } 273 \text{ K}$$

$$Q_1 = (50)(540) = 27{,}000 \text{ cal} = 27 \text{ kcal}$$

$$Q_2 = (50)(1)(100) = 5000 \text{ cal} = 5 \text{ kcal}$$

$$Q_3 = (450)(0.5)(20) = 4500 \text{ cal} = 4.5 \text{ kcal}$$

$$Q_4 = (450)(80) = 36000 \text{ cal} = 36 \text{ kcal}$$

Now, since $Q_1 + Q_2 > Q_3$ but $Q_1 + Q_2 < Q_3 + Q_4$ ice will come to 273 K from 253 K, but whole ice will not melt. Therefore, temperature of the mixture is 273 K.

Example 22.7 *An ice cube of mass 0.1 kg at 0°C is placed in an isolated container which is at 227°C. The specific heat s of the container varies with temperature T according to the empirical relation* $s = A + BT$, *where* $A = 100$ *cal/kg-K and* $B = 2 \times 10^{-2}$ *cal/kg-K*^{2}. *If the final temperature of the container is 27°C, determine the mass of the container.*

(Latent heat of fusion for water $= 8 \times 10^4$ *cal/kg, specific heat of water* $= 10^3$ *cal/kg-K).*

(JEE 2001)

Solution Let m be the mass of the container.

Initial temperature of container,

$$T_i = (227 + 273) = 500\ \text{K}$$

and final temperature of container,

$$T_f = (27 + 273) = 300\ \text{K}$$

Now, heat gained by the ice cube = heat lost by the container

$$\therefore \qquad (0.1)(8 \times 10^4) + (0.1)(10^3)(27) = -m\int_{500}^{300} (A + BT)\,dT$$

or

$$10700 = -m\left[AT + \frac{BT^2}{2}\right]_{500}^{300}$$

After substituting the values of A and B and the proper limits, we get

$$m = 0.495\ \text{kg}$$ **Ans.**

INTRODUCTORY EXERCISE 22.1

(Take $c_{\text{ice}} = 0.53$ cal/g-°C, $c_{\text{water}} = 1.0$ cal/g-°C, $(L_f)_{\text{water}} = 80$ cal/g and $(L_v)_{\text{water}} = 529$ cal/g unless given in the question.)

1. 10 g ice at 0°C is converted into steam at 100°C. Find total heat required. ($L_f = 80$ cal/g, $S_w = 1$ cal/g-°C, $L_v = 540$ cal/g)
2. Three liquids A, B and C of specific heats 1 cal/g -°C, 0.5 cal/g -°C and 0.25 cal/g -°C are at temperatures 20 °C, 40 °C and 60 °C respectively. Find temperature in equilibrium if they are mixed together. Their masses are equal.
3. Equal masses of ice (at 0°C) and water are in contact. Find the temperature of water needed to just melt the complete ice.
4. A lead bullet just melts when stopped by an obstacle. Assuming that 25 per cent of the heat is absorbed by the obstacle, find the velocity of the bullet if its initial temperature is 27°C. (Melting point of lead = 327°C, specific heat of lead = 0.03 cal/g-°C, latent heat of fusion of lead = 6 cal/g, $J = 4.2$ J/cal). **(JEE 1981)**
5. The temperature of 100 g of water is to be raised from 24°C to 90°C by adding steam to it. Calculate the mass of the steam required for this purpose. **(JEE 1996)**
6. 15 g ice at 0°C is mixed with 10 g water at 40°C. Find the temperature of mixture. Also, find mass of water and ice in the mixture.
7. Three liquids P, Q and R are given 4 kg of P at 60 °C and 1 kg of R at 50 °C when mixed produce a resultant temperature 55 °C. A mixture of 1 kg of P at 60 °C and 1 kg of Q at 50 °C shows a temperature of 55 °C. What will be the resulting temperature when 1 kg of Q at 60 °C is mixed with 1 kg of R at 50 °C?

22.2 Heat Transfer

Heat can be transferred from one place to the other by any of three possible ways : **conduction, convection** and **radiation.** In the first two processes, a medium is necessary for the heat transfer. Radiation, however, does no have this restriction. This is also the fastest mode of heat transfer, in which heat is transferred from one place to the other in the form of electromagnetic radiation.

Conduction

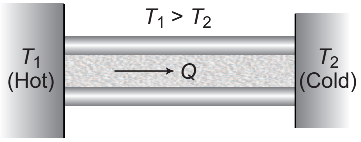


Fig. 22.3

Figure shows a rod whose ends are in thermal contact with a hot reservoir at temperature T_1 and a cold reservoir at temperature T_2. The sides of the rod are covered with insulating medium, so the transport of heat is along the rod, not through the sides. The molecules at the hot reservoir have greater vibrational energy. This energy is transferred by collisions to the atoms at the end face of the rod. These atoms in turn transfer energy to their neighbours further along the rod. Such transfer of heat through a substance in which heat is transported without direct mass transport is called conduction.

Most metals use another, more effective mechanism to conduct heat. The free electrons, which move throughout the metal can rapidly carry energy from the hotter to cooler regions, so metals are generally good conductors of heat. The presence of 'free' electrons also causes most metals to be good electrical conductors. A metal rod at 5°C feels colder than a piece of wood at 5°C because heat can flow more easily from your hand into the metal.

Heat transfer occurs only between regions that are at different temperatures, and the rate of heat flow is $\frac{dQ}{dt}$. This rate is also called the **heat current**, denoted by H. Experiments show that the heat current is proportional to the cross-section area A of the rod and to the temperature gradient $\frac{dT}{dx}$, which is the rate of change of temperature with distance along the bar. In general,

$$\boxed{H = \frac{dQ}{dt} = -KA\frac{dT}{dx}} \qquad \ldots\text{(i)}$$

The negative sign is used to make $\frac{dQ}{dt}$ a positive quantity since $\frac{dT}{dx}$ is negative. The constant K, called the **thermal conductivity** is a measure of the ability of a material to conduct heat. A substance with a large thermal conductivity k is a good heat conductor. The value of k depends on the temperature, increasing slightly with increasing temperature, but K can be taken to be practically constant throughout a substance if the temperature difference between its ends is not too great. Let us apply Eq. (i) to a rod of length l and constant cross-sectional area A in which a steady state has been reached. In a steady state, the temperature at each point is constant in time. Hence,

$$-\frac{dT}{dx} = T_1 - T_2$$

Therefore, the heat ΔQ transferred in time Δt is

$$\boxed{\Delta Q = KA\left(\frac{T_1 - T_2}{l}\right)\Delta t} \qquad \ldots\text{(ii)}$$

Thermal Resistance (R)

Eq. (ii) in differential form can be written as

$$\boxed{\frac{dQ}{dt} = H = \frac{\Delta T}{l/KA} = \frac{\Delta T}{R}} \qquad \ldots\text{(iii)}$$

Here, ΔT = temperature difference (TD) and $R = \frac{l}{KA}$ = thermal resistance of the rod.

Extra Points to Remember

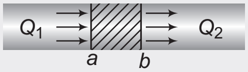
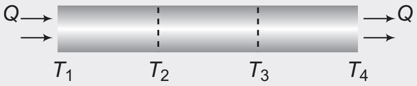


Fig. 22.4

- Consider a section *ab* of a rod as shown in figure. Suppose Q_1 heat enters into the section at '*a*' and Q_2 leaves at '*b*', then $Q_2 < Q_1$. Part of the energy $Q_1 - Q_2$ is utilized in raising the temperature of section *ab* and the remaining is lost to atmosphere through *ab*. If heat is continuously supplied from the left end of the rod, a stage comes when temperature of the section becomes constant. In that case, $Q_1 = Q_2$ if rod is insulated from the surroundings (or loss through *ab* is zero). This is called the **steady state** condition. Thus, in steady state temperature of different sections of the rod becomes constant (but not same). Hence, in the figure :



Insulated rod in steady state

Fig. 22.5

$$T_1 = \text{constant}, \; T_2 = \text{constant etc. and} \; T_1 > T_2 > T_3 > T_4$$

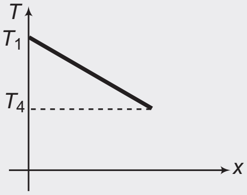


Fig. 22.6

Now, a natural question arises, why the temperature of whole rod not becomes equal when heat is being continuously supplied? The answer is : there must be a temperature difference in the rod for the heat flow, same as we require a potential difference across a resistance for the current flow through it.

In steady state, the temperature varies linearly with distance along the rod if it is insulated.

- Comparing equation number (iii), i.e. heat current

$$H = \frac{dQ}{dt} = \frac{\Delta T}{R} \qquad \left(\text{where, } R = \frac{l}{KA}\right)$$

with the equation, of current flow through a resistance,

$$i = \frac{dq}{dt} = \frac{\Delta V}{R} \qquad \left(\text{where, } R = \frac{l}{\sigma A}\right)$$

We find the following similarities in heat flow through a rod and current flow through a resistance.

Table 22.1

S.No	Heat flow through a conducting rod	Current flow through a resistance
1.	Conducting rod	Electrical resistance
2.	Heat flows	Charge flows
3.	TD is required	PD is required
4.	Heat current $H = \frac{dQ}{dt}$ = rate of heat flow	Electric current $i = \frac{dq}{dt}$ = rate of charge flow
5.	$H = \frac{\Delta T}{R} = \frac{\text{TD}}{R}$	$i = \frac{\Delta V}{R} = \frac{\text{PD}}{R}$
6.	$R = \frac{l}{KA}$	$R = \frac{l}{\sigma A}$
7.	K = thermal conductivity	σ = electrical conductivity

From the above table it is evident that flow of heat through rods in series and parallel is analogous to the flow of current through resistance in series and parallel. This analogy is of great importance in solving complicated problems of heat conduction.

Convection

Although conduction does occur in liquids and gases also, heat is transported in these media mostly by convection. In this process, the actual motion of the material is responsible for the heat transfer. Familiar examples include hot-air and hot-water home heating systems, the cooling system of an automobile engine and the flow of blood in the body.

You probably have warmed your hands by holding them over an open flame. In this situation, the air directly above the flame is heated and expands. As a result, the density of this air decreases and then air rises. When the movement results from differences in density, as with air around fire, it is referred to as **natural convection.** Air flow at a beach is an example of natural convection. When the heated substance is forced to move by a fan or pump, the process is called **forced convection.** If it were not for convection currents, it would be very difficult to boil water. As water is heated in a kettle, the heated water expands and rises to the top because its density is lowered. At the same time, the denser, cool water at the surface sinks to the bottom of the kettle and is heated. Heating a room by a radiator is an example of forced convection.

Radiation

The third means of energy transfer is radiation which does not require a medium. The best known example of this process is the radiation from sun. All objects radiate energy continuously in the form of electromagnetic waves.

Now, let us define few terms before studying the other topics.

Radiant Energy

All bodies radiate energy in the form of electromagnetic waves by virtue of their temperature. This energy is called the radiant energy.

Absorptive Power '*a*'

"It is defined as the ratio of the radiant energy absorbed by it in a given time to the total radiant energy incident on it in the same interval of time."

$$\boxed{a = \frac{\text{energy absorbed}}{\text{energy incident}}}$$

As a perfectly black body absorbs all radiations incident on it, the absorptive power of a perfectly black body is maximum and unity.

Spectral Absorptive Power 'a_λ'

The absorptive power 'a' refers to radiations of all wavelengths (or the total energy) while the spectral absorptive power is the ratio of radiant energy absorbed by a surface to the radiant energy incident on it for a particular wavelength λ. It may have different values for different wavelengths for a given surface. Let us take an example, suppose $a = 0.6$, $a_\lambda = 0.4$ for 1000 Å and $a_\lambda = 0.7$ for 2000 Å for a given surface. Then, it means that this surface will absorb only 60% of the total radiant energy incident on it. Similarly, it absorbs 40% of the energy incident on it corresponding to 1000 Å and 70% corresponding to 2000 Å. The spectral absorptive power a_λ is related to absorptive power a through the relation

$$\boxed{a = \int_0^\infty a_\lambda \, d\lambda}$$

Emissive Power '*e*'

(Do not confuse it with the emissivity e which is different from it, although both have the same symbol e).

"For a given surface it is defined as the radiant energy emitted per second per unit area of the surface." It has the units of W/m^2 or J/s-m^2. For a black body $e = \sigma T^4$.

Spectral Emissive Power 'e_λ'

"It is emissive power for a particular wavelength λ." Thus,

$$\boxed{e = \int_0^\infty e_\lambda \, d\lambda}$$

Stefan's Law

The rate at which an object radiates energy is proportional to the fourth power of its absolute temperature. This is known as the **Stefan's law** and is expressed in equation form as

$$\boxed{P = \sigma A e T^4}$$

Here, P is the power in watts (J/s) radiated by the object, A is the surface area in m^2, e is a fraction between 0 and 1 called the **emissivity** of the object and σ is a universal constant called **Stefan's constant**, which has the value

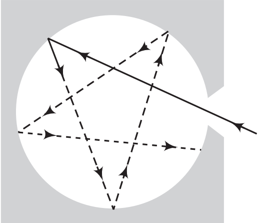
$$\boxed{\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{-K}^4}$$

Note *Emissivity e is also sometimes denoted by* e_r.

Perfectly Black Body

A body that absorbs all the radiation incident upon it and has an emissivity equal to 1 is called a perfectly black body. A black body is also an ideal radiator. It implies that if a black body and an identical another body are kept at the same temperature, then the black body will radiate maximum power as is obvious from equation $P = eA\sigma T^4$ also. Because $e = 1$ for a perfectly black body while for any other body $e < 1$.

Materials like black velvet or lamp black come close to being ideal black bodies, but the best practical realization of an ideal black body is a small hole leading into a cavity, as this absorbs 98% of the radiation incident on them.



Cavity approximating an ideal black body. Radiation entering the cavity has little chance of leaving before it is completely absorbed.

Fig. 22.7

Note *(i) Energy radiated by a body per unit area per unit time is called emissive power and given by*

$$\boxed{e = e_r \sigma T^4} \qquad (e_r = \text{emissivity})$$

It has the units of W/m^2 *or* $J/s\text{-}m^2$.

Energy radiated per unit time is called power and it is called power and it is given by

$$\boxed{P = e_r \sigma T^4 A}$$

It has the unit J/s or watt.

Total energy radiated by the body in time t is

$$\boxed{E = e_r \sigma T^4 At}$$

(ii) *For a given body, values of* e_r *and a are same. But their meanings are different. For example if,* $e_r = 0.6$ *then a is also* 0.6.

Further, $e_r = 0.6$ *means if this body and perfectly black body are kept under identical conditions then this body will radiate* 60% *of the energy radiated by a perfectly black body. But* $a = 0.6$ *means this body will absorb* 60% *of the total energy incident on its surface.*

(iii) *Absorptive power (a), spectral absorptive power* (a_λ) *and emissivity* (e_r) *are the constants which are temperature independent. But emissive power (e) and spectral emissive power* (e_λ) *are temperature dependent. As,*

$$e = e_r \sigma T^4 \quad \Rightarrow \quad e \propto T^4$$

Kirchhoff's Law

"According to this law the ratio of emissive power to absorptive power is same for all surfaces at the same temperature."

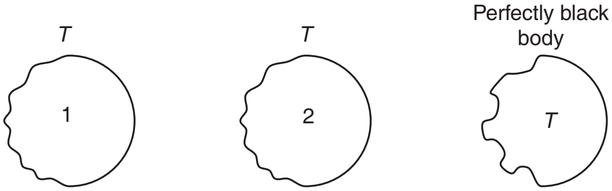


Fig. 22.8

Hence, $$\frac{e_1}{a_1} = \frac{e_2}{a_2} = \left(\frac{e}{a}\right)_{\text{perfectly black body}}$$

but $$(a)_{\text{black body}} = 1$$

and $$(e)_{\text{black body}} = E \quad \text{(say)}$$

Then, $$\boxed{\left(\frac{e}{a}\right)_{\text{for any surface}} = \text{constant} = E}$$

Similarly, for a particular wavelength λ,

$$\boxed{\left(\frac{e_\lambda}{a_\lambda}\right)_{\text{for any body}} = E_\lambda}$$

Here, E = emissive power of black body at temperature T

$= \sigma T^4$

From the above expression, we can see that

$$\boxed{e_\lambda \propto a_\lambda}$$

i.e. good absorbers for a particular wavelength are also good emitters of the same wavelength.

Note *According to Kirchhoff's law, at a given temperature, the ratio* $\frac{e}{a}$ *is constant for all bodies and this constant is equal to e of perfectly black body at that temperature. Similarly, the ratio* $\frac{e_\lambda}{a_\lambda}$ *is constant for all bodies at given temperature and value of this constant is* e_λ *of perfectly black body at that temperature.*

Cooling by Radiation

Consider a hot body at temperature T placed in an environment at a lower temperature T_0. The body emits more radiation than it absorbs and cools down while the surroundings absorb radiation from the body and warm up. The body is losing energy by emitting radiations at a rate

$$P_1 = eA\sigma T^4$$

and is receiving energy by absorbing radiations at a rate

$$P_2 = aA\sigma T_0^4$$

Here, 'a' is a pure number between 0 and 1 indicating the relative ability of the surface to absorb radiation from its surroundings. Note that this 'a' is different from the absorptive power 'a'. In thermal equilibrium, both the body and the surroundings have the same temperature (say T_c) and,

$$P_1 = P_2 \quad \text{or} \quad eA\sigma T_c^4 = aA\sigma T_c^4 \quad \text{or} \quad e = a$$

Thus, when $T > T_0$, the net rate of heat transfer from the body to the surroundings is

$$\frac{dQ}{dt} = eA\sigma\,(T^4 - T_0^4) \quad \text{or} \quad mc\left(-\frac{dT}{dt}\right) = eA\sigma\,(T^4 - T_0^4)$$

Rate of cooling

$$\boxed{\left(-\frac{dT}{dt}\right) = \frac{eA\sigma}{mc}\,(T^4 - T_0^4)} \quad \text{or} \quad -\frac{dT}{dt} \propto (T^4 - T_0^4)$$

Newton's Law of Cooling

According to this law, if the temperature T of the body is not very different from that of the surroundings T_0, then rate of cooling $-\frac{dT}{dt}$ is proportional to the temperature difference between them. To prove it, let us assume that

$$T = T_0 + \Delta T$$

So that
$$T^4 = (T_0 + \Delta T)^4 = T_0^4\left(1 + \frac{\Delta T}{T_0}\right)^4$$

$$\approx T_0^4\left(1 + \frac{4\Delta T}{T_0}\right) \qquad \text{(from binomial expansion)}$$

$$\therefore \quad (T^4 - T_0^4) = 4T_0^3\,(\Delta T)$$

$$\text{or} \quad (T^4 - T_0^4) \propto \Delta T \qquad (\text{as } T_0 = \text{constant})$$

Now, we have already shown that rate of cooling

$$\left(-\frac{dT}{dt}\right) \propto (T^4 - T_0^4)$$

and here we have shown that

$$(T^4 - T_0^4) \propto \Delta T$$

if the temperature difference is small.

Thus, rate of cooling

$$\boxed{-\frac{dT}{dt} \propto \Delta T} \quad \text{or} \quad \boxed{-\frac{d\theta}{dt} \propto \Delta\theta}$$

as $$dT = d\theta \quad \text{or} \quad \Delta T = \Delta\theta$$

Variation of Temperature of a Body According to Newton's Law

Suppose a body has a temperature θ_i at time $t = 0$. It is placed in an atmosphere whose temperature is θ_0. We are interested in finding the temperature of the body at time t. Assuming Newton's law of cooling to hold good or by assuming that the temperature difference is small. As per this law,

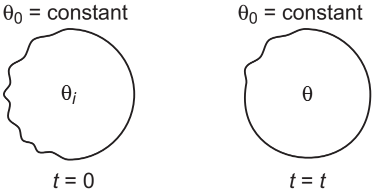


Fig. 22.9

Rate of cooling ∝ temperature difference

or $$\left(-\frac{d\theta}{dt}\right) = \left(\frac{eA\sigma}{mc}\right)(4\theta_0^3)(\theta - \theta_0)$$

or $$\left(-\frac{d\theta}{dt}\right) = \alpha(\theta - \theta_0)$$

Here, $$\alpha = \left(\frac{4eA\sigma\theta_0^3}{mc}\right)$$ (is a constant)

$\therefore$ $$\int_{\theta_i}^{\theta} \frac{d\theta}{\theta - \theta_0} = -\alpha \int_0^t dt$$

$\therefore$ $$\boxed{\theta = \theta_0 + (\theta_i - \theta_0)e^{-\alpha t}}$$

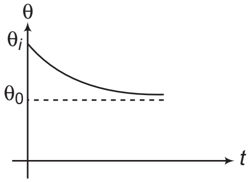


Fig. 22.10

From this expression we see that $\theta = \theta_i$ at $t = 0$ and $\theta = \theta_0$ at $t = \infty$, i.e. temperature of the body varies exponentially with time from θ_i to θ_0 ($< \theta_i$). The temperature *versus* time graph is as shown in Fig.22.10.

Note *If the body cools by radiation from* θ_1 *to* θ_2 *in time* t, *then taking the approximation*

$$\left(-\frac{d\theta}{dt}\right) = \frac{\theta_1 - \theta_2}{t} \quad \text{and} \quad \theta = \theta_{av} = \left(\frac{\theta_1 + \theta_2}{2}\right)$$

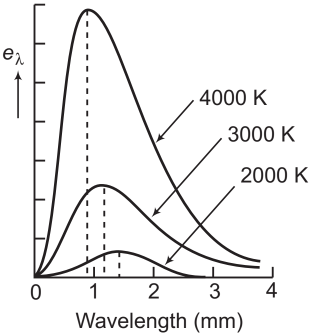
The equation $\left(-\frac{d\theta}{dt}\right) = \alpha(\theta - \theta_0)$ *becomes*

$$\boxed{\frac{\theta_1 - \theta_2}{t} = \alpha\left(\frac{\theta_1 + \theta_2}{2} - \theta_0\right)}$$

This form of the law helps in solving numerical problems related to ***Newton's law of cooling****.*

Wien's Displacement Law

At ordinary temperatures (below about 600°C) the thermal radiation emitted by a body is not visible, most of it is concentrated in wavelengths much longer than those of visible light.



Power of black body radiation *versus* wavelength at three temperatures. Note that the amount of radiation emitted (the area under a curve) increase with increasing temperature.

Fig. 22.11

Figure shows how the energy of a black body radiation varies with temperature and wavelength. As the temperature of the black body increases, two distinct behaviors are observed. The first effect is that the peak of the distribution shifts to shorter wavelengths. This shift is found to obey the following relationship called **Wien's displacement law.**

$$\boxed{\lambda_m T = b}$$

Here, b is a constant called Wien's constant. The value of this constant for perfectly black body in SI unit is 2.898×10^{-3} m-K. Thus,

$$\boxed{\lambda_m \propto \frac{1}{T}}$$

Here, λ_m is the wavelength corresponding to the maximum spectral emissive power e_λ.

The second effect is that the total amount of energy the black body emits per unit area per unit time $(=\sigma T^4)$ increases with fourth power of absolute temperature T. This is also known as the emissive power. We know

$$e = \int_0^\infty e_\lambda \, d\lambda = \text{Area under } e_\lambda \text{ - } \lambda \text{ graph} = \sigma T^4$$

or $$\boxed{\text{Area} \propto T^4} \quad \Rightarrow \quad A_2 = (2)^4 A_1 = 16 A_1$$

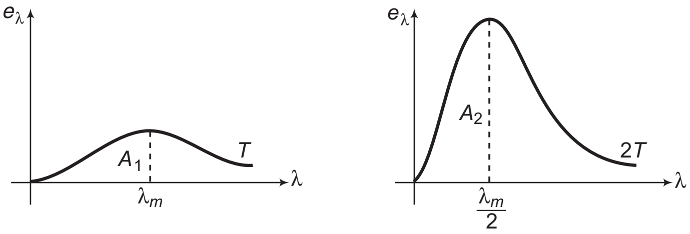


Fig. 22.12

Thus, if the temperature of the black body is made two fold, λ_m remains half while the area becomes 16 times.

Example 22.8 *In which of the following process, convection does not take place primarily?* (JEE 2005)

(a) Sea and land breeze
(b) Boiling of water
(c) Warming of glass of bulb due to filament
(d) Heating air around a furnace

Solution (c) Glass of bulb heats due to filament by radiation.

Example 22.9 *A copper rod 2 m long has a circular cross-section of radius 1 cm. One end is kept at 100°C and the other at 0°C. The surface is insulated so that negligible heat is lost through the surface. In steady state, find*

(a) the thermal resistance of the bar
(b) the thermal current H
(c) the temperature gradient $\frac{dT}{dx}$ and
(d) the temperature at a distance 25 cm from the hot end.

Thermal conductivity of copper is 401 W/m-K.

Solution (a) Thermal resistance, $R = \frac{l}{KA} = \frac{l}{K(\pi r^2)}$

or
$$R = \frac{(2)}{(401)(\pi)(10^{-2})^2}$$
$$= 15.9\ \text{K/W}$$ **Ans.**

(b) Thermal current, $H = \frac{\Delta T}{R} = \frac{\Delta\theta}{R} = \frac{100}{15.9}$

or $H = 6.3\ \text{W}$ **Ans.**

(c) Temperature gradient
$$= \frac{0-100}{2} = -50\ \text{K/m}$$
$$= -50°\ \text{C/m}$$ **Ans.**

(d) Let θ be the temperature at 25 cm from the hot end, then

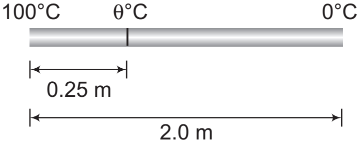


Fig. 22.13

$$(\theta - 100) = (\text{temperature gradient}) \times (\text{distance})$$

or $\theta - 100 = (-50)(0.25)$

or $\theta = 87.5°\ \text{C}$ **Ans.**

Example 22.10 *Three rods made of the same material and having the same cross-section have been joined as shown in the figure. Each rod is of the same length. The left and right ends are kept at 0°C and 90°C, respectively. The temperature of junction of the three rods will be* **(JEE 2001)**

(a) $45^\circ C$ (b) $60^\circ C$

(c) $30^\circ C$ (d) $20^\circ C$

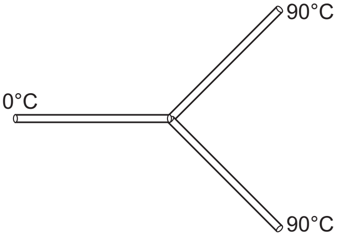


Fig. 22.14

Solution Let θ be the temperature of the junction (say B). Thermal resistance of all the three rods is equal. Rate of heat flow through AB + Rate of heat flow through CB = Rate of heat flow through BD

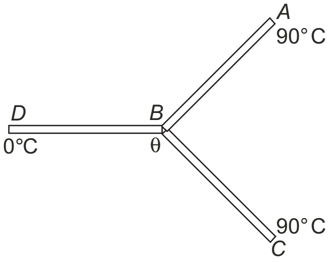


Fig. 22.15

$$\therefore \qquad \frac{90^\circ - \theta}{R} + \frac{90^\circ - \theta}{R} = \frac{\theta - 0}{R}$$

Solving this equation, we get $\theta = 60^\circ \text{C}$

∴ The correct option is (b).

Here, R = Thermal resistance

Note *Rate of heat flow,* $H = \dfrac{\text{Temperature difference (TD)}}{\text{Thermal resistance } (R)}$

where, $R = \dfrac{l}{KA}$

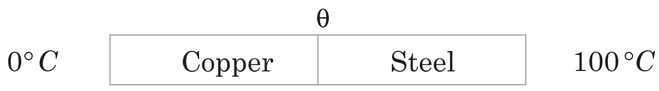
K = Thermal conductivity of the rod.

This is similar to the current flow through a resistance (R) where current (i) = Rate of flow of charge

$$= \frac{\text{Potential difference (PD)}}{\text{Electrical resistance } (R)}$$

Here, $R = \dfrac{l}{\sigma A}$ where σ = Electrical conductivity.

Example 22.11 *Figure shows a copper rod joined to a steel rod. The rods have equal length and equal cross- sectional area. The free end of the copper rod is kept at $0^\circ C$ and that of the steel rod is kept at $100^\circ C$. Find the temperature θ at the junction of the rods. Conductivity of copper $= 390\ W/m\text{-}^\circ C$ and that of steel $= 46\ W/m\text{-}^\circ C$.*



Solution $H_{\text{steel}} = H_{\text{copper}}$ (two rods are in series)

$$\therefore \qquad \frac{(\text{TD})_S}{(l/KA)_S} = \frac{(\text{TD})_C}{(l/KA)_C}$$

or $$K_S\ (\text{TD})_S = K_C\ (\text{TD})_C$$

or $$46\,(100 - \theta) = 390\,(\theta - 0)$$

Solving we get, $\theta = 10.6^\circ$ C **Ans.**

Note *In the above problem heat flows from right to left or from higher temperature to lower temperature.*

Example 22.12 *A spherical black body with a radius of 12 cm radiates 450 W power at 500 K. If the radius were halved and the temperature doubled, the power radiated in watt would be* **(JEE 1997)**

(a) 225 *(b)* 450

(c) 900 *(d)* 1800

Solution Power radiated = $e_r \sigma T^4 A$ or Power radiated $\propto$ (surface area) $(T)^4$. The radius is halved, hence, surface area will become $\frac{1}{4}$ times. Temperature is doubled, therefore, T^4 becomes 16 times.

$$\text{New power} = (450)\left(\frac{1}{4}\right)(16) = 1800\,\text{W}.$$

∴ The correct option is (d).

Example 22.13 *A cylinder of radius R made of a material of thermal conductivity K_1 is surrounded by a cylindrical shell of inner radius R and outer radius 2R made of a material of thermal conductivity K_2. The two ends of the combined system are maintained at two different temperatures. There is no loss of heat across the cylindrical surface and the system is in steady state. The effective thermal conductivity of the system is* **(JEE 1998)**

(a) $K_1 + K_2$ *(b)* $K_1K_2/(K_1 + K_2)$

(c) $(K_1 + 3K_2)/4$ *(d)* $(3K_1 + K_2)/4$

Solution Let R_1 and R_2 be the thermal resistances of inner and outer portions. Since, temperature difference at both ends is same, the resistances are in parallel. Hence,

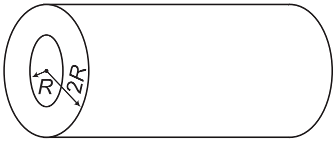


Fig. 22.16

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$R = \frac{l}{KA}$$

or $$\frac{1}{R} = \frac{KA}{l}$$

∴ $$\frac{K(4\pi R^2)}{l} = \frac{K_1\,(\pi R^2)}{l} + \frac{K_2\,(3\pi R^2)}{l}$$

∴ $$K = \frac{K_1 + 3K_2}{4}$$

∴ The correct option is (c).

Example 22.14 *A body cools in 10 minutes from 60°C to 40°C. What will be its temperature after next 10 minutes? The temperature of the surroundings is 10°C.*

Solution According to Newton's law of cooling,

$$\left(\frac{\theta_1 - \theta_2}{t}\right) = \alpha\left[\left(\frac{\theta_1 + \theta_2}{2}\right) - \theta_0\right]$$

For the given conditions,

$$\frac{60 - 40}{10} = \alpha\left[\frac{60 + 40}{2} - 10\right] \qquad \ldots\text{(i)}$$

Let θ be the temperature after next 10 minutes. Then,

$$\frac{40 - \theta}{10} = \alpha\left[\frac{40 + \theta}{2} - 10\right] \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\theta = 28^\circ \text{ C}$$ **Ans.**

INTRODUCTORY EXERCISE 22.2

1. A rod is heated at one end as shown in figure. In steady state temperature of different sections becomes constant but not same. Why so?

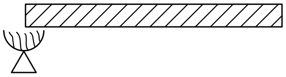


Fig. 22.17

2. Show that the SI units of thermal conductivity are W/m-K.
3. Find SI units of thermal resistance.
4. Suppose a liquid in a container is heated at the top rather than at the bottom. What is the main process by which the rest of the liquid becomes hot?
5. Three rods each of same length and cross-section are joined in series. The thermal conductivity of the materials are $K, 2K$ and $3K$ respectively. If one end is kept at 200 °C and the other at 100 °C. What would be the temperature of the junctions in the steady state? Assume that no heat is lost due to radiation from the sides of the rods.
6. A rod *CD* of thermal resistance 5.0 K/W is joined at the middle of an identical rod *AB* as shown in figure. The ends *A*, *B* and *D* are maintained at 100°C and 25°C respectively. Find the heat current in *CD*.

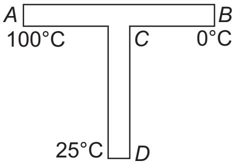


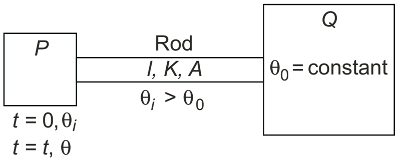
Fig. 22.18

7. A liquid takes 5 minutes to cool from 80 °C to 50 °C. How much time will it take to cool from 60 °C to 30°C? The temperature of surroundings is 20 °C.

Final Touch Points

1. **Cooling by conduction or radiation**

(i) **By conduction** A body P of mass m and specific heat c is connected to a large body Q (of specific heat infinite) through a rod of length l, thermal conductivity K and area of cross-section A. Temperature of Q is $\theta_0 (< \theta_i)$. This temperature will remain constant as its specific heat is very high. Heat will flow from P to Q through the rod. If we neglect the loss of heat due to radiation then due to this heat transfer, temperature of P will decrease but temperature of Q will remain almost constant. At time t, suppose temperature of P becomes θ then due to temperature difference heat transfer through the rod.



$$\frac{dQ}{dt} = H = \frac{\text{TD}}{R} = \frac{\theta - \theta_0}{R} \qquad \text{...(i)}$$

Here, $$R = \frac{l}{KA}$$

Now, if we apply equation of calorimetry in P, then

$$Q = mc(-\Delta\theta) \quad \text{or} \quad \frac{dQ}{dt} = mc\left(\frac{-d\theta}{dt}\right) \qquad \text{...(ii)}$$

Equating Eqs. (i) and (ii), we have

$$\boxed{-\frac{d\theta}{dt} = \frac{\text{TD}}{mcR} = \frac{\theta - \theta_0}{mcR} = \text{Rate of cooling}} \qquad \text{...(iii)}$$

So, this is the rate of cooling by conduction.

or $$\boxed{-\frac{d\theta}{dt} \propto \text{TD}} \qquad \text{...(iv)}$$

(ii) **By radiation** In article 22.2, we have already derived the expression of rate of cooling,

$$\boxed{-\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4) \quad \text{or} \quad -\frac{dT}{dt} \propto (T^4 - T_0^4)} \qquad \text{...(v)}$$

Further in Newton's law of cooling, we have also seen that, if temperature difference between body and atmosphere is less than this rate of cooling,

$$\boxed{-\frac{dT}{dt} \propto \Delta T \quad \text{or} \quad \text{TD}} \qquad \text{...(vi)}$$

In Newton's law of cooling, we have also seen that temperature of the body falls exponentially, if

$$\text{rate of cooling, } -\frac{dT}{dt} \quad \text{or} \quad -\frac{d\theta}{dt} \propto \text{TD}$$

In Eq. (iv) of conduction and Eq. (vi) of radiation (special case when TD is small)

$$-\frac{dT}{dt} \quad \text{or} \quad -\frac{d\theta}{dt} \propto \text{TD}$$

So, in both cases temperature of the body will fall exponentially like,

$$\theta = \theta_0 + (\theta_i - \theta_0)\, e^{-\alpha t} \quad \text{or} \quad T = T_0 + (T_i - T_0)\, e^{-\alpha t}$$

In radiation, $\alpha = \dfrac{4eA\sigma T_0^3}{mc}$ and in conduction $\alpha = \dfrac{1}{mcR}$

2. For emissivity, we have used the term e or e_r. This is sometimes confused with emissive power e. Emissivity is unitless and always less than or equal to one. But emissive power has the units J/s-m^2 or watt/m^2.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on calculation of thermal resistance*

Concept

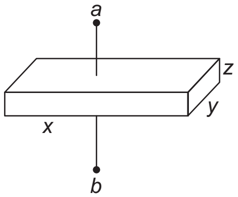
(i) A thermal resistance of a conducting rod is calculated/required between two points or two surfaces (say a and b). The formula of thermal resistance is

$$R = \frac{l}{KA}$$

Here, l is that dimension of conductor which is parallel to a and b and A is that cross-sectional area which is perpendicular to a and b.

(ii) From a to b if A is uniform at every point then direct formula $R = \frac{l}{KA}$ can be applied otherwise we will have to use integration.

Example 1 *Thermal conductivity of the conductor shown in figure is K. Find thermal resistance between points a and b.*

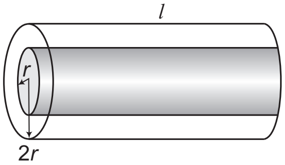


Solution From a to b area of cross-section perpendicular to ab is uniform ($A = xy$) and length along ab is z. Therefore, using the formula

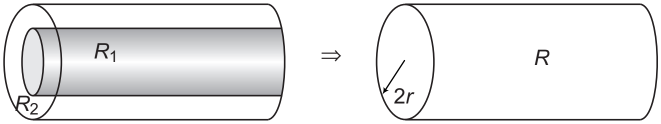
$$R = \frac{l}{KA}, \text{ we have}$$

$$R = \frac{z}{Kxy}$$ **Ans.**

Example 2 *Thermal conductivity of inner core of radius r is K and of the outer one of radius 2r is 2K. Find equivalent value of thermal conductivity between its two ends.*



Solution The meaning of equivalent value of thermal conductivity means, if a single material of K_e is used with same dimensions then value of thermal resistance should remain unchanged.



Between two ends R_1 and R_2 are in parallel. Hence,

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{R} \qquad \text{...(i)}$$

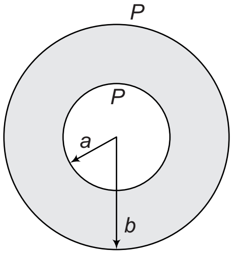
R is given by $\frac{l}{KA}$. Therefore, $\frac{1}{R}$ will be given by $\frac{KA}{l}$. Using this in Eq. (i), we have

$$\frac{K(\pi r^2)}{l}+\frac{2K[(\pi)(2r)^2-\pi r^2]}{l}=\frac{K_e(\pi)(2r)^2}{l}$$

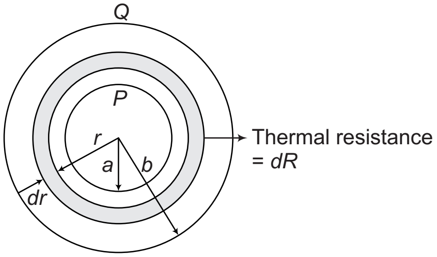
Solving this equation, we get

$$K_e=\frac{7}{4}K \qquad \textbf{Ans.}$$

Example 3 *A spherical body of radius 'b' has a concentric cavity of radius 'a' as shown. Thermal conductivity of the material is K. Find thermal resistance between inner surface P and outer surface Q.*



Solution As we move from P to Q surface perpendicular to PQ is spherical and its size keeps on increasing (just like different layers of a spherical onion). So, first we will calculate thermal resistance of one layer at a distance r from centre and thickness dr by using the formula $R=\frac{l}{KA}$.



In this formula, dimension of the layer along PQ is dr and the surface area perpendicular to PQ is $4\pi r^2$

$$\therefore \qquad dR = \frac{dr}{K(4\pi r^2)}$$

Now, if we integrate dR from $r = a$ to $r = b$, we will get the total thermal resistance between P and Q. Thus,

$$R = \int_a^b dR = \int_a^b \frac{dr}{K(4\pi r^2)}$$

Solving this expression, we get

$$R = \frac{1}{4\pi K}\left(\frac{1}{a} - \frac{1}{b}\right) \qquad \textbf{Ans.}$$

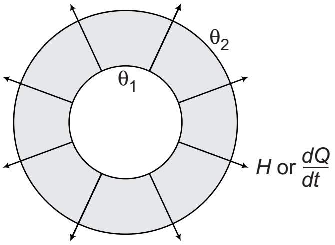
Example 4 *In the above example, if temperature of inner surface P is kept constant at θ_1 and of the outer surface Q at $\theta_2 (< \theta_1)$. Then,*

Find

(a) rate of heat flow or heat current from inner surface to outer surface.

(b) temperature θ at a distance $r(a < r < b)$ from centre.

Solution (a)

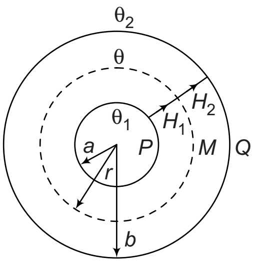


$$H \quad \text{or} \quad \frac{dQ}{dt} = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$$

$$= \frac{(\theta_1 - \theta_2)(4\pi K)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$= \frac{(\theta_1 - \theta_2)(ab)(4\pi K)}{(b - a)} \qquad \textbf{Ans.}$$

(b)



In the figure, we can see that

$$\text{Heat current } H_1 = \text{Heat current } H_2$$

$$\therefore \qquad \frac{(\text{TD})_{\text{PM}}}{R_{\text{PM}}} = \frac{(\text{TD})_{\text{MQ}}}{R_{\text{MQ}}} \qquad \ldots(\text{i})$$

Using the result obtained in Example-3 of thermal resistance, we can find

$$R_{\text{PM}} = \frac{1}{4\pi K}\left(\frac{1}{a} - \frac{1}{r}\right)$$

and $$R_{\text{MQ}} = \frac{1}{4\pi K}\left(\frac{1}{r} - \frac{1}{b}\right)$$

Substituting the values in Eq. (i), we have

$$\frac{\theta_1 - \theta}{\frac{1}{4\pi K}\left(\frac{1}{a} - \frac{1}{r}\right)} = \frac{\theta - \theta_2}{\frac{1}{4\pi K}\left(\frac{1}{r} - \frac{1}{b}\right)}$$

Solving this equation we can find θ.

Type 2. *Mixed problems of calorimetry and conduction*

Concept

(i) In calorimetry, we have two equations,

$$Q = mc\Delta\theta \qquad \text{(when temperature changes without change in state)}$$

and $$Q = mL \qquad \text{(when state changes without change in temperature)}$$

The above two equations in differential form can be written as

$$\frac{dQ}{dt} = mc\left(\pm\frac{d\theta}{dt}\right) \quad \text{or} \quad L\left(\frac{dm}{dt}\right) \qquad \text{...(i)}$$

In the above equation $\frac{dm}{dt}$ is rate of conversion of mass from one state to another state.

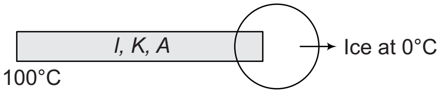
(ii) In conduction, we have the equation,

$$\frac{dQ}{dt} \quad \text{or} \quad H = \frac{\text{TD}}{R} \qquad \text{...(ii)}$$

where, $$R = \frac{l}{KA}$$

In these types of problems $\frac{dQ}{dt}$ of calorimetry is equated with $\frac{dQ}{dt}$ of conduction.

Example 5 *One end of the rod of length l, thermal conductivity K and area of cross-section A is maintained at a constant temperature of 100 °C. At the other end large quantity of ice is kept at 0 °C. Due to temperature difference, heat flows from left end to right end of the rod. Due to this heat ice will start melting. Neglecting the radiation losses find the expression of rate of melting of ice.*



Solution Putting,

$$\left(\frac{dQ}{dt}\right)_{\text{conduction}} = \left(\frac{dQ}{dt}\right)_{\text{calorimetry}}$$

$$\therefore \qquad \frac{\text{TD}}{R} = L \cdot \frac{dm}{dt}$$

$$\Rightarrow \qquad \frac{dm}{dt} = \frac{\text{TD}}{RL}$$ **Ans.**

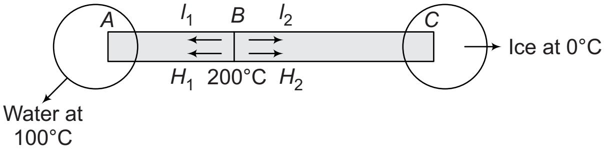
So, this is the desired expression of $\frac{dm}{dt}$.

In the above expression,

$$\text{TD} = 100^\circ\text{C}, R = \frac{l}{KA}$$

and L = latent heat of fusion

Example 6 *B point of the rod shown in figure is maintained at* $200^\circ C$. *At left end A, there is water at* $100^\circ C$ *and at right end C there is ice at* $0^\circ C$. *Heat currents* H_1 *and* H_2 *will flow on both sides. Due to* H_1, *water will convert into steam and due to* H_2 *ice will be melted. If latent heat of vaporization is* 540 *cal/g and latent heat of fusion is* 80 *cal/g then neglecting the radiation losses find* $\frac{l_1}{l_2}$ *so that rate of melting of ice is two times the rate of conversion of water into steam.*



Solution Using the relation of $\frac{dm}{dt}$ derived in above example,

$$\frac{dm}{dt} = \frac{\text{TD}}{RL} = \frac{(\text{TD})KA}{lL} \qquad \left(\text{as } R = \frac{l}{KA}\right)$$

Given that,

$$\left(\frac{dm}{dt}\right)_{\text{RHS}} = 2\left(\frac{dm}{dt}\right)_{\text{LHS}}$$

or

$$\left[\frac{(\text{TD})KA}{lL}\right]_{\text{RHS}} = 2\left[\frac{(\text{TD})KA}{lL}\right]_{\text{LHS}}$$

K and A are same on both sides. Hence,

$$\left(\frac{\text{TD}}{lL}\right)_{\text{RHS}} = 2\left(\frac{\text{TD}}{lL}\right)_{\text{LHS}}$$

Substituting the proper values, we have

$$\frac{200}{l_2 \times 80} = 2\left[\frac{100}{l_1 \times 540}\right]$$

$$\therefore \qquad \frac{l_1}{l_2} = \frac{80}{540} = \frac{4}{27}$$ **Ans.**

Type 3. *Mixed problems of Stefan's law and Wien's displacement law*

Concept

Stefan's law

Energy radiated per unit time = emissive power = $e_r \sigma T^4$

Energy radiated per unit time called power = $e_r \sigma T^4 A$

Total energy radiated in time 't' = $e_r \sigma T^4 At$

Above all three terms $\propto T^4$

Wien's displacement law

$$\lambda_m \propto \frac{1}{T}$$

From Wien's law normally we find the ratio of temperatures of two bodies and then this ratio is used in Stefan's law.

Example 7 *Three discs, A, B and C having radii 2m, 4m and 6m respectively are coated with carbon black on their outer surfaces. The wavelengths corresponding to maximum intensity are 300 nm, 400 nm and 500 nm, respectively. The power radiated by them are Q_A, Q_B and Q_C respectively* **(JEE 2004)**

(a) Q_A is maximum *(b) Q_B is maximum* *(c) Q_C is maximum* *(d) $Q_A = Q_B = Q_C$*

Solution (b) $Q \propto AT^4$ and $\lambda_m T$ = constant.

Hence,

$$Q \propto \frac{A}{(\lambda_m)^4} \quad \text{or} \quad Q \propto \frac{r^2}{(\lambda_m)^4}$$

$$Q_A : Q_B : Q_C = \frac{(2)^2}{(3)^4} : \frac{(4)^2}{(4)^4} : \frac{(6)^2}{(5)^4} = \frac{4}{81} : \frac{1}{16} : \frac{36}{625} = 0.05 : 0.0625 : 0.0576$$

i.e. Q_B is maximum.

Type 4. *Problems of cooling either by conduction or radiation*

Concept

In final touch points we have seen that if a body cools by conduction, then

$$\text{rate of cooling, } -\frac{d\theta}{dt} = \frac{\text{TD}}{mcR} \qquad \text{...(i)}$$

Here, $mc = C$ = heat capacity of the body

This C is the heat required to raise the temperature of whole body by 1°C or 1K and R is the thermal resistance of conducting rod. So, Eq. (i) can also be written as

$$-\frac{d\theta}{dt} = \frac{\text{TD}}{CR}$$

If the body cools by radiation, then

$$\text{rate of cooling or } -\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4) \qquad \text{...(ii)}$$

For small temperature difference, Newton's law of cooling can be applied and

$$-\frac{dT}{dt} \propto \text{TD} \qquad \text{...(iii)}$$

Further, if rate of cooling ∝ TD as Eqs. (i) or (iii), temperature of the body falls exponentially.

Example 8 *Two spheres, one solid and other hollow are kept in atmosphere at same temperature. They are made of same material and their radii are also same. Which sphere will cool at a faster rate initially?*

Solution In the equation,

$$-\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4)$$

Only mass is different. Therefore,

$$-\frac{dT}{dt} \propto \frac{1}{m}$$

Mass of hollow sphere is less. So, hollow sphere will cool at a faster rate initially.

Example 9 *The ratio of specific heats of two spheres is* $2:3$, *radii* $1:2$, *emissivity* $3:1$ *and density* $1:1$. *Initially, they are kept at same temperatures in atmosphere. Which sphere will cool at a faster rate initially.*

Solution In the equation,

$$-\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4)$$

$$A = \text{surface area } = 4\pi R^2$$

$$m = \text{Volume} \times \text{density} = \frac{4}{3}\pi R^3 \rho$$

Substituting in the above equation, we have

$$-\frac{dT}{dt} = \frac{3e\sigma}{R\rho c}(T^4 - T_0^4) \quad \text{or} \quad -\frac{dT}{dt} \propto \frac{e}{R\rho c}$$

Now,

$$\left(\frac{e}{R\rho c}\right)_{\text{sphere-1}} = \frac{3}{1 \times 1 \times 2} = \frac{3}{2}$$

and

$$\left(\frac{e}{R\rho c}\right)_{\text{sphere-2}} = \frac{1}{2 \times 1 \times 3} = \frac{1}{6}$$

$$\left(\frac{e}{R\rho c}\right) \text{ for sphere-1 is more.}$$

So, first sphere will cool at faster rate initially.

Example 10 *Two metallic spheres* S_1 *and* S_2 *are made of the same material and have got identical surface finish. The mass of* S_1 *is thrice that of* S_2. *Both the spheres are heated to the same high temperature and placed in the same room having lower temperature but are thermally insulated from each other. The ratio of the initial rate of cooling of* S_1 *to that of* S_2 *is* **(JEE 1995)**

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{3}}{1}$ (d) $\left(\frac{1}{3}\right)^{1/3}$

Solution $-\dfrac{dT}{dt} = \dfrac{eA\sigma}{mc}(T^4 - T_o^4)$

$$m_1 = 3m_2$$

$$\therefore \quad R_1 = (3)^{\frac{1}{3}} R_2$$

$$\left(-\frac{dT}{dt}\right) \propto \frac{A}{m} \propto \frac{R^2}{R^3} \propto \frac{1}{R}$$

$$\therefore \quad \frac{(-dT/dt)_1}{(-dT/dt)_2} = \frac{R_2}{R_1} = \left(\frac{1}{3}\right)^{\frac{1}{3}}$$

∴ The correct option is (d).

Example 11 *Two identical conducting rods are first connected independently to two vessels, one containing water at 100°C and the other containing ice at 0°C. In the second case, the rods are joined end to end and connected to the same vessels. Let q_1 and q_2 gram per second be the rate of melting of ice in the two cases respectively. The ratio $\dfrac{q_1}{q_2}$ is* **(JEE 2004)**

(a) $\dfrac{1}{2}$ *(b)* $\dfrac{2}{1}$ *(c)* $\dfrac{4}{1}$ *(d)* $\dfrac{1}{4}$

Solution (c) $\dfrac{dQ}{dt} = L\left(\dfrac{dm}{dt}\right)$ or $\dfrac{\text{Temperature difference}}{\text{Thermal resistance}} = L\left(\dfrac{dm}{dt}\right)$

or $$\frac{dm}{dt} \propto \frac{1}{\text{Thermal resistance}} \quad \Rightarrow \quad q \propto \frac{1}{R}$$

In the first case rods are in parallel and thermal resistance is $\dfrac{R}{2}$ while in second case rods are in series and thermal resistance is $2R$.

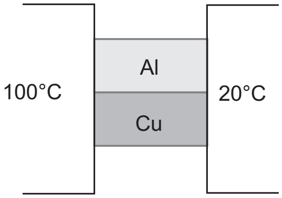
$$\frac{q_1}{q_2} = \frac{2R}{R/2} = \frac{4}{1}$$

Miscellaneous Examples

Example 12 *Two metal cubes with 3 cm-edges of copper and aluminium are arranged as shown in figure. Find*

(a) the total thermal current from one reservoir to the other

(b) the ratio of the thermal current carried by the copper cube to that carried by the aluminium cube. Thermal conductivity of copper is 401 W/m-K and that of aluminium is 237 W/m-K.



Solution (a) Thermal resistance of aluminium cube, $R_1 = \dfrac{l}{KA}$

or $$R_1 = \frac{(3.0 \times 10^{-2})}{(237)(3.0 \times 10^{-2})^2}$$

$$= 0.14 \text{ K/W}$$

and thermal resistance of copper cube, $R_2 = \dfrac{l}{KA}$

or $$R_2 = \frac{(3.0 \times 10^{-2})}{(401)\,(3.0 \times 10^{-2})^2} = 0.08 \text{ K/W}$$

As these two resistance are in parallel, their equivalent resistance will be

$$R = \frac{R_1 R_2}{R_1 + R_2} = \frac{(0.14)\,(0.08)}{(0.14) + (0.08)} = 0.05 \text{ K/W}$$

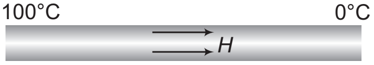
∴ $$\text{Thermal current, } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}} = \frac{(100 - 20)}{0.05} = 1.6 \times 10^3 \text{ W}$$ **Ans.**

(b) In parallel thermal current distributes in the inverse ratio of resistance. Hence,

$$\frac{H_{\text{Cu}}}{H_{\text{Al}}} = \frac{R_{\text{Al}}}{R_{\text{Cu}}} = \frac{R_1}{R_2} = \frac{0.14}{0.08} = 1.75$$ **Ans.**

Example 13 *One end of a copper rod of length 1 m and area of cross-section 4.0×10^{-4} m² is maintained at 100°C. At the other end of the rod ice is kept at 0°C. Neglecting the loss of heat from the surroundings, find the mass of ice melted in 1 h. Given, $K_{Cu} = 401$ W/m-K and $L_f = 3.35 \times 10^5$ J/kg.*

Solution Thermal resistance of the rod,



$$R = \frac{l}{KA} = \frac{1.0}{(401)\,(4 \times 10^{-4})} = 6.23 \text{ K/W}$$

∴ $$\text{Heat current, } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}} = \frac{(100 - 0)}{6.23} = 16 \text{ W}$$

Heat transferred in 1 h,

$$Q = Ht \qquad \left(\because\ H = \frac{Q}{t}\right)$$

$$= (16)\,(3600) = 57600 \text{ J}$$

Now, let m mass of ice melts in 1 h, then

$$m = \frac{Q}{L} \qquad (Q = mL)$$

$$= \frac{57600}{3.35 \times 10^5} = 0.172 \text{ kg}$$

or $= 172$ g **Ans.**

Example 14 *Two bodies A and B have thermal emissivities of 0.01 and 0.81 respectively. The outer surface areas of the two bodies are same. The two bodies emit total radiant power at the same rate. The wavelength λ_B corresponding to maximum spectral radiancy from B is shifted from the wavelength corresponding to maximum spectral radiancy in the radiation from A by 1.0 μm. If the temperature of A is 5802 K, calculate (a) the temperature of B, (b) wavelength λ_B.*

Solution (a)

$$P_A = P_B$$

$$\therefore \quad e_A \sigma A_A T_A^4 = e_B \sigma A_B T_B^4$$

$$\therefore \quad T_B = \left(\frac{e_A}{e_B}\right)^{1/4} T_A \qquad (\text{as } A_A = A_B)$$

Substituting the values,

$$T_B = \left(\frac{0.01}{0.81}\right)^{1/4} (5802) = 1934 \text{ K}$$ **Ans.**

(b) According to Wien's displacement law,

$$\lambda_A T_A = \lambda_B T_B$$

$$\therefore \quad \lambda_B = \left(\frac{T_A}{T_B}\right)\lambda_A = \left(\frac{5802}{1934}\right)\lambda_A$$

or $$\lambda_B = 3\lambda_A$$

Also, $$\lambda_B - \lambda_A = 1 \,\mu\text{m}$$

or $$\lambda_B - \left(\frac{1}{3}\right)\lambda_B = 1 \,\mu\text{m}$$

or $$\lambda_B = 1.5 \,\mu\text{m}$$ **Ans.**

Example 15 *5 g of water at 30°C and 5 g of ice at – 20°C are mixed together in a calorimeter. Find the final temperature of mixture. Water equivalent of calorimeter is negligible, specific heat of ice = 0.5 cal/g-°C and latent heat of ice = 80 cal/g.*

Solution In this case heat is given by water and taken by ice

Heat available with water to cool from 30° C to 0° C

$$= ms\Delta\theta = 5 \times 1 \times 30$$

$$= 150 \text{ cal}$$

Heat required by 5 g ice to increase its temperature upto 0°C

$$ms\Delta\theta = 5 \times 0.5 \times 20$$

$$= 50 \text{ cal}$$

Out of 150 cal heat available, 50 cal is used for increasing temperature of ice from – 20° C to 0° C. The remaining heat 100 cal is used for melting the ice.

If mass of ice melted is m g, then

$$m \times 80 = 100$$

$$\Rightarrow \quad m = 1.25 \text{ g}$$

Thus, 1.25 g ice out of 5 g melts and mixture of ice and water is at 0° C.

Example 16 *A bullet of mass* 10 *g moving with a speed of* 20 *m/s hits an ice block of mass* 990 *g kept on a frictionless floor and gets stuck in it. How much ice will melt if* 50% *of the lost kinetic energy goes to ice?* (*Temperature of ice block* = 0°*C*).

Solution Velocity of bullet + ice block,

$$V = \frac{(10\text{ g}) \times (20\text{ m/s})}{1000\text{ g}} = 0.2\text{ m/s} \qquad (P_i = P_f)$$

$$\text{Loss of KE} = \frac{1}{2}mv^2 - \frac{1}{2}(m+M)V^2$$

$$= \frac{1}{2}[0.01 \times (20)^2 - 1 \times (0.2)^2]$$

$$= \frac{1}{2}[4 - 0.04] = 1.98\text{ J}$$

$\therefore$ Heat received by ice block $= \dfrac{1.98}{4.2 \times 2}$ cal

$= 0.24$ cal

$\therefore$ Mass of ice melted $= \dfrac{(0.24\text{ cal})}{(80\text{ cal/ g})}$

$= 0.003$ g **Ans.**

Example 17 *At* 1 *atmospheric pressure,* 1.000 *g of water having a volume of* 1.000 *cm*^{3} *becomes* 1671 *cm*^{3} *of steam when boiled. The heat of vaporization of water at* 1 *atmosphere is* 539 *cal/g. What is the change in internal energy during the process?*

Solution Heat spent during vaporisation,

$$Q = mL = 1.000 \times 539 = 539\text{ cal}$$

$$\text{Work done, } W = P(V_v - V_l)$$

$$= 1.013 \times 10^5 \times (1671 - 1.000) \times 10^{-6}$$

$$= 169.2\text{ J} = \frac{169.2}{4.18}\text{ cal}$$

$$= 40.5\text{ cal}$$

$\therefore$ Change in internal energy,

$\therefore$ $\Delta U = 539\text{ cal} - 40.5\text{ cal}$

$= 498.5$ cal **Ans.**

Example 18 *At* 1 *atmospheric pressure,* 1.000 *g of water having a volume of* 1.000 *cm*^{3} *becomes* 1.091 *cm*^{3} *of ice on freezing. The heat of fusion of water at* 1 *atmosphere is* 80.0 *cal/g. What is the change in internal energy during the process?*

Solution Heat given out during freezing,

$$Q = -mL = -1 \times 80 = -80\text{ cal}$$

External work done $W = P(V_{\text{ice}} - V_{\text{water}})$

$$= 1.013 \times 10^5 \times (1.091 - 1.000) \times 10^{-6}$$

$$= 9.22 \times 10^{-3} \text{ J}$$

$$= \frac{9.22 \times 10^{-3}}{4.18} \text{ cal}$$

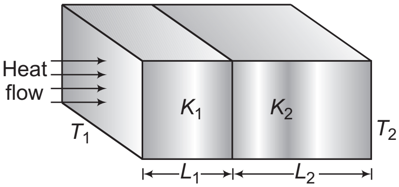
$$= 0.0022 \text{ cal}$$

∴ Change in internal energy,

$$\Delta U = Q - W = -80 - 0.0022$$

$$= -80.0022 \text{ cal}$$ **Ans.**

Example 19 *Two plates each of area A, thickness L_1 and L_2 thermal conductivities K_1 and K_2 respectively are joined to form a single plate of thickness $(L_1 + L_2)$. If the temperatures of the free surfaces are T_1 and T_2. Calculate*



(a) rate of flow of heat
(b) temperature of interface and
(c) equivalent thermal conductivity.

Solution (a) If the thermal resistance of the two plates are R_1 and R_2 respectively. Plates are in series.

$$R_S = R_1 + R_2 = \frac{L_1}{AK_1} + \frac{L_2}{AK_2}$$

as $$R = \frac{L}{KA}$$

and so $$H = \frac{dQ}{dt} = \frac{\Delta T}{R}$$

$$= \frac{(T_1 - T_2)}{(R_1 + R_2)} = \frac{A(T_1 - T_2)}{\left[\frac{L_1}{K_1} + \frac{L_2}{K_2}\right]}$$ **Ans.**

(b) If T is the common temperature of interface, then as in series rate of flow of heat remains same, i.e. $H = H_1$ $(= H_2)$

$$\frac{T_1 - T_2}{R_1 + R_2} = \frac{T_1 - T}{R_1}$$

i.e. $$T = \frac{T_1R_2 + T_2R_1}{(R_1 + R_2)}$$

or $$T = \frac{\left[T_1\frac{L_2}{K_2} + T_2\frac{L_1}{K_1}\right]}{\left[\frac{L_1}{K_1} + \frac{L_2}{K_2}\right]}$$ $\left(\text{as } R = \frac{L}{KA}\right)$

(c) If K is the equivalent conductivity of composite slab, i.e. slab of thickness $L_1 + L_2$ and cross-sectional area A, then as in series

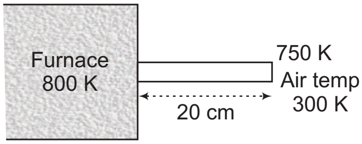
$$R_S = R_1 + R_2 \quad \text{or} \quad \frac{(L_1 + L_2)}{AK_{\text{eq}}} = R_1 + R_2$$

i.e.

$$K_{\text{eq}} = \frac{L_1 + L_2}{A(R_1 + R_2)} = \frac{L_1 + L_2}{\left[\dfrac{L_1}{K_1} + \dfrac{L_2}{K_2}\right]} \qquad \left(\text{as } R = \frac{L}{KA}\right)$$

Example 20 *One end of a rod of length 20 cm is inserted in a furnace at 800 K. The sides of the rod are covered with an insulating material and the other end emits radiation like a black body. The temperature of this end is 750 K in the steady state. The temperature of the surrounding air is 300 K. Assuming radiation to be the only important mode of energy transfer between the surroundings and the open end of the rod, find the thermal conductivity of the rod. Stefan constant* $\sigma = 6.0 \times 10^{-8}\ W/m^2\text{-}K^4$. *Take emissivity of the open end* $e = 1$

Solution Heat flowing through the rod per second in steady state,



$$\frac{dQ}{dt} = \frac{KAd\theta}{x} \qquad \text{...(i)}$$

Heat radiated from the open end of the rod per second in steady state,

$$\frac{dQ}{dt} = A\sigma\,(T^4 - T_0^4) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

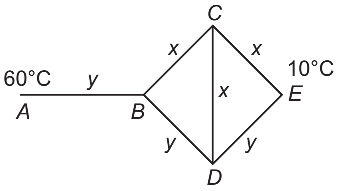
$$\frac{Kd\theta}{x} = \sigma\,(T^4 - T_0^4)$$

$$\frac{K \times 50}{0.2} = 6.0 \times 10^{-8}\ [(7.5)^4 - (3)^4] \times 10^8$$

or

$$K = 74\ \text{W/m-K}$$ **Ans.**

Example 21 *Three rods of material x and three rods of material y are connected as shown in figure. All the rods are of identical length and cross-sectional area. If the end A is maintained at 60°C and the junction E at 10°C, calculate temperature of junctions B, C and D. The thermal conductivity of x is 0.92 cal/cm-s°C and that of y is 0.46 cal/cm-s°C.*



Solution Thermal resistance, $R = \frac{l}{KA}$

$$\therefore \quad \frac{R_x}{R_y} = \frac{K_y}{K_x} \qquad \text{(as } l_x = l_y \text{ and } A_x = A_y\text{)}$$

$$= \frac{0.46}{0.92}$$

$$= \frac{1}{2}$$

So, if $R_x = R$ then $R_y = 2R$

CEDB forms a balanced Wheatstone bridge, i.e. $T_C = T_D$ and no heat flows through *CD*.

$$\therefore \quad \frac{1}{R_{BE}} = \frac{1}{R+R} + \frac{1}{2R+2R}$$

or
$$R_{BE} = \frac{4}{3}R$$

The total resistance between *A* and *E* will be

$$R_{AE} = R_{AB} + R_{BE} = 2R + \frac{4}{3}R = \frac{10}{3}R$$

∴ Heat current between *A* and *E* is

$$H = \frac{(\Delta T)_{AE}}{R_{AE}} = \frac{(60-10)}{(10/3)\,R} = \frac{15}{R}$$

Now, if T_B is the temperature at *B*,

$$H_{AB} = \frac{(\Delta T)_{AB}}{R_{AB}}$$

or
$$\frac{15}{R} = \frac{60 - T_B}{2R}$$

or
$$T_B = 30^\circ \text{C}$$ **Ans.**

Further,

$$H_{AB} = H_{BC} + H_{BD}$$

or
$$\frac{15}{R} = \frac{30 - T_C}{R} + \frac{30 - T_D}{2R} \qquad [T_C = T_D = T \text{ (say)}]$$

or
$$15 = (30 - T) + \frac{(30 - T)}{2}$$

Solving this, we get
$$T = 20^\circ \text{C}$$

or
$$T_C = T_D = 20^\circ \text{C}$$ **Ans.**

Example 22 *A hollow sphere of glass whose external and internal radii are 11 cm and 9 cm respectively is completely filled with ice at 0°C and placed in a bath of boiling water. How long will it take for the ice to melt completely? Given that density of ice = 0.9 g/cm^3, latent heat of fusion of ice = 80 cal/g and thermal conductivity of glass = 0.002 cal/cm-s°C.*

Solution In steady state, rate of heat flow (result is taken from Example 4)

$$H = \frac{4\pi K r_1 r_2 \Delta T}{r_2 - r_1}$$

Substituting the values, $$H = \frac{(4)(\pi)(0.002)(11)(9)(100-0)}{(11-9)}$$

or $$\frac{dQ}{dt} = 124.4 \text{ cal/s}$$

This rate should be equal to, $L\dfrac{dm}{dt}$

$$\therefore \quad \left(\frac{dm}{dt}\right) = \frac{dQ/dt}{L} = \frac{124.4}{80} = 1.555 \text{ g/s}$$

Total mass of ice, $$m = \rho_{\text{ice}}(4\pi r_1^2) = (0.9)(4)(\pi)(9)^2 = 916 \text{ g}$$

∴ Time taken for the ice to melt completely

$$t = \frac{m}{(dm/dt)} = \frac{916}{1.555} = 589 \text{ s}$$ **Ans.**

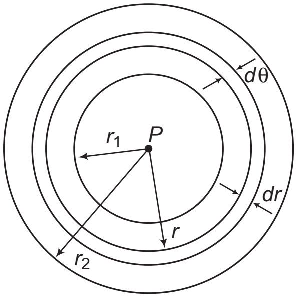
Example 23 *A point source of heat of power P is placed at the centre of a spherical shell of mean radius R. The material of the shell has thermal conductivity K. Calculate the thickness of the shell if temperature difference between the outer and inner surfaces of the shell in steady state is T.*

Solution Consider a concentric spherical shell of radius r and thickness dr as shown in figure. In steady state, the rate of heat flow (heat current) through this shell will be

$$H = \frac{\Delta T}{R} = \frac{(-d\theta)}{\dfrac{dr}{(k)(4\pi r^2)}} \qquad \left(R = \frac{l}{KA}\right)$$

or $$H = -(4\pi K r^2)\frac{d\theta}{dr}$$

Here, negative sign is used because with increase in r, θ decreases



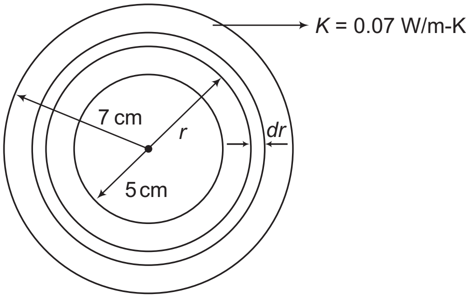
$$\therefore \quad \int_{r_1}^{r_2}\frac{dr}{r^2} = -\frac{4\pi K}{H}\int_{\theta_1}^{\theta_2} d\theta$$

This equation gives, $$H = \frac{4\pi K r_1 r_2(\theta_1-\theta_2)}{(r_2-r_1)}$$

In steady state, $$H = P,\ r_1 r_2 \approx R^2 \text{ and } \theta_1-\theta_2 = T$$

∴ Thickness of shell, $$r_2 - r_1 = \frac{4\pi K R^2 T}{P}$$ **Ans.**

Example 24 *A steam cylindrical pipe of radius* 5 *cm carries steam at* 100°*C. The pipe is covered by a jacket of insulating material* 2 *cm thick having a thermal conductivity* 0.07 *W/m-K. If the temperature at the outer wall of the pipe jacket is* 20°*C, how much heat is lost through the jacket per metre length in an hour?*



Solution Thermal resistance per metre length of an element at distance r of thickness dr is

$$dR = \frac{dr}{K\,(2\pi r)} \qquad \left(R = \frac{l}{KA}\right)$$

∴ Total resistance

$$R = \int_{r_1 = 5\text{ cm}}^{r_2 = 7\text{ cm}} dR$$

$$= \frac{1}{2\pi K}\int_{5.0\times10^{-2}\text{ m}}^{7.0\times10^{-2}\text{ m}} \frac{dr}{r}$$

$$= \frac{1}{2\pi K}\ln\left(\frac{7}{5}\right)$$

$$= \frac{1}{(2\pi)(0.07)}\ln(1.4)$$

$$= 0.765\text{ K/W}$$

$$\text{Heat current, } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$$

$$= \frac{(100-20)}{0.765} = 104.6\text{ W}$$

∴ Heat lost in one hour = Heat current × time

$$= (104.6)(3600)\text{ J}$$

$$= 3.76\times10^{5}\text{ J}$$ **Ans.**

Exercises

LEVEL 1

Take $c_{\text{ice}} = 0.53$ cal/g-°C, $c_{\text{water}} = 1.0$ cal/g-°C, $(L_f)_{\text{water}} = 80$ cal/g and $(L_v)_{\text{water}} = 529$ cal/g unless given in the question.

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. **Assertion :** Specific heat of any substance remains constant at all temperatures.

Reason : It is given by $s = \frac{1}{m} \cdot \frac{dQ}{dT}$

2. **Assertion :** When temperature of a body is increased, in radiant energy, number of low wavelength photons get increased.

Reason : According to Wien's displacement law $\lambda_m \propto \frac{1}{T}$.

3. **Assertion :** Warming a room by a heat blower is an example of forced convection.

Reason : Natural convection takes place due to gravity.

4. **Assertion :** A conducting rod is placed between boiling water and ice. If rod is broken into two equal parts and two parts are connected side by side, then rate of melting of ice will increase to four times.

Reason : Thermal resistance will become four times.

5. **Assertion :** A normal body can radiate energy more than a perfectly black body.

Reason : A perfectly black body is always black in colour.

6. **Assertion :** According to Newton's law, good conductors of electricity are also good conductors of heat.

Reason : At a given temperature, $e_\lambda \propto a_\lambda$ for any body.

7. **Assertion :** Good conductors of electricity are also good conductors of heat due to large number of free electrons.

Reason : It is easy to conduct heat from free electrons.

8. **Assertion :** Emissivity of any body (e) is always less than its absorptive power (a).

Reason : Both the quantities are dimensionless.

9. **Assertion :** Heat is supplied at constant rate from one end of a conducting rod. In steady state, temperature of all points of the rod becomes uniform.

Reason : In steady state, temperature of rod does not increase.

10. **Assertion :** A solid sphere and a hollow sphere of same material and same radius are kept at same temperatures in atmosphere. Rate of cooling of hollow sphere will be more.

    Reason : If all other conditions are same, then rate of cooling is inversely proportional to the mass of body.

Objective Questions

1. For an enclosure maintained at 2000 K, the maximum radiation occurs at wavelength λ_m. If the temperature is raised to 3000 K, the peak will shift to

   (a) $0.5\,\lambda_m$ (b) λ_m (c) $\frac{2}{3}\lambda_m$ (d) $\frac{3}{2}\lambda_m$

2. A substance cools from 75°C to 70°C in T_1 minute, from 70°C to 65°C in T_2 minute and from 65°C to 60°C in T_3 minute, then

   (a) $T_1 = T_2 = T_3$ (b) $T_1 < T_2 < T_3$
   (c) $T_1 > T_2 > T_3$ (d) $T_1 < T_2 > T_3$

3. Two liquids are at temperatures 20°C and 40°C. When same mass of both of them is mixed, the temperature of the mixture is 32°C. What is the ratio of their specific heats?

   (a) 1/3 (b) 2/5
   (c) 3/2 (d) 2/3

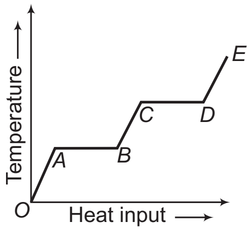
4. The specific heat of a metal at low temperatures varies according to $S = aT^3$, where a is a constant and T is absolute temperature. The heat energy needed to raise unit mass of the metal from temperature $T = 1$ K to $T = 2$ K is

   (a) $3a$ (b) $\frac{15a}{4}$ (c) $\frac{2a}{3}$ (d) $\frac{13a}{4}$

5. The intensity of radiation emitted by the sun has its maximum value at a wavelength of 510 nm and that emitted by the North star has the maximum value at 350 nm. If these stars behave like black bodies, then the ratio of the surface temperatures of the sun and the north star is

   (a) 1.46 (b) 0.69
   (c) 1.21 (d) 0.83

6. A solid material is supplied heat at a constant rate. The temperature of material is changing with heat input as shown in the figure. What does the slope of DE represent?



   (a) Latent heat of liquid (b) Latent heat of vaporization

   (c) Heat capacity of vapour (d) Inverse of heat capacity of vapour

7. Two ends of rods of length L and radius R of the same material are kept at the same temperature. Which of the following rods conducts the maximum heat?

   (a) $L = 50$ cm, $R = 1$ cm (b) $L = 100$ cm, $R = 2$ cm
   (c) $L = 25$ cm, $R = 0.5$ cm (d) $L = 75$ cm, $R = 1.5$ cm

8. 1 g of ice at 0°C is mixed with 1 g of steam at 100°C. After thermal equilibrium is achieved, the temperature of the mixture is

(a) 100°C (b) 55°C (c) 75°C (d) 0°C

9. A wall has two layers *A* and *B* each made of different materials. The layer *A* is 10 cm thick and *B* is 20 cm thick. The thermal conductivity of *A* is thrice that of *B*. Under thermal equilibrium temperature difference across the wall is 35°C. The difference of temperature across the layer *A* is

(a) 30°C (b) 14°C (c) 8.75°C (d) 5°C

10. A wall has two layers *A* and *B* each made of different materials. Both the layers have the same thickness. The thermal conductivity of material *A* is twice of *B*. Under thermal equilibrium the temperature difference across the layer *B* is 36°C. The temperature difference across layer *A* is

(a) 6°C (b) 12°C (c) 18°C (d) 24°C

11. The end of two rods of different materials with their thermal conductivities, area of cross-section and lengths all in the ratio 1 : 2 are maintained at the same temperature difference. If the rate of flow of heat in the first rod is 4 cal/s. Then, in the second rod rate of heat flow in cal/s will be

(a) 1 (b) 2 (c) 8 (d) 16

Subjective Questions

1. A thin square steel plate 10 cm on a side is heated in a black smith's forge to temperature of 800° C. If the emissivity is 0.60, what is the total rate of radiation of energy?

2. A lead bullet penetrates into a solid object and melts. Assuming that 50% of its kinetic energy was used to heat it, calculate the initial speed of the bullet. The initial temperature of the bullet is 27° C and its melting point is 327° C. Latent heat of fusion of lead $= 2.5 \times 10^4$ J/kg and specific heat capacity of lead = 125 J/kg-K.

3. A ball is dropped on a floor from a height of 2.0 m . After the collision it rises upto a height of 1.5 m. Assume that 40% of the mechanical energy lost goes as thermal energy into the ball. Calculate the rise in the temperature of the ball in the collision. Specific heat of the ball is 800 J/K.

4. A nuclear power plant generates 500 MW of waste heat that must be carried away by water pumped from a lake. If the water temperature is to rise by 10°C, what is the required flow rate in kg/s?

5. The emissivity of tungsten is 0.4. A tungsten sphere with a radius of 4.0 cm is suspended within a large evacuated enclosure whose walls are at 300 K. What power input is required to maintain the sphere at a temperature of 3000 K if heat conduction along supports is neglected? Take, $\sigma = 5.67 \times 10^{-8}$ Wm^2-K^4.

6. A pot with a steel bottom 1.2 cm thick rests on a hot stove. The area of the bottom of the pot is 0.150 m^2. The water inside the pot is at 100°C and 0.440 kg are evaporated every 5.0 minute. Find the temperature of the lower surface of the pot, which is in contact with the stove. Take $L_v = 2.256 \times 10^6$ J/kg and $k_{\text{steel}} = 50.2$ W/m-K

7. A carpenter builds an outer house wall with a layer of wood 2.0 cm thick on the outside and a layer of an insulation 3.5 cm thick as the inside wall surface. The wood has $K = 0.08$ W/m-K and the insulation has $K = 0.01$ W/m-K. The interior surface temperature is 19° C and the exterior surface temperature is – 10° C.

(a) What is the temperature at the plane where the wood meets the insulation?

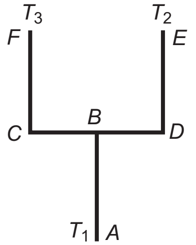
(b) What is the rate of heat flow per square metre through this wall?

8. A closely thermally insulated vessel contains 100 g of water at 0°C. If the air from this vessel is rapidly pumped out, intensive evaporation will produce cooling and as a result of this, water freeze. How much ice will be formed by this method? If latent heat of fusion is 80 cal/g and of evaporation 560 cal/g.
[**Hint** If m gram ice is formed, $mL_f = (100 - m)\,L_v$]

9. In a container of negligible mass 140 g of ice initially at $-15°$ C is added to 200 g of water that has a temperature of 40°C. If no heat is lost to the surroundings, what is the final temperature of the system and masses of water and ice in mixture?

10. A certain amount of ice is supplied heat at a constant rate for 7 minutes. For the first one minute the temperature rises uniformly with time. Then, it remains constant for the next 4 minutes and again the temperature rises at uniform rate for the last two minutes. Calculate the final temperature at the end of seven minutes.
(Given, L of ice $= 336 \times 10^3$ J/kg and specific heat of water = 4200 J/kg-K)

11. Four identical rods AB, CD, CF and DE are joined as shown in figure. The length, cross-sectional area and thermal conductivity of each rod are l, A and K respectively. The ends A, E and F are maintained at temperatures T_1, T_2 and T_3 respectively. Assuming no loss of heat to the atmosphere. Find the temperature at B, the mid-point of CD.



12. The ends of a copper rod of length 1 m and area of cross-section 1 cm^2 are maintained at 0° C and 100° C. At the centre of the rod there is a source of heat of power 25 W. Calculate the temperature gradient in the two halves of the rod in steady state. Thermal conductivity of copper is 400 Wm^{-1}K^{-1}.

13. A copper sphere is suspended in an evacuated chamber maintained at 300 K. The sphere is maintained at a constant temperature of 500 K by heating it electrically. A total of 210 W of electric power is needed to do it. When the surface of the copper sphere is completely blackened, 700 W is needed to maintain the same temperature of the sphere. Calculate the emissivity of copper.

LEVEL 2

Single Correct Options

1. A cylindrical rod with one end in a steam chamber and the other end in ice results in melting of 0.1 g of ice per second. If the rod is replaced by another with half the length and double the radius of the first and if the thermal conductivity of material of second rod is 1/4 that of first, the rate at which ice melts in g/s will be

(a) 0.4
(b) 0.05
(c) 0.2
(d) 0.1

2. Two sheets of thickness d and $3d$, are touching each other. The temperature just outside the thinner sheet is T_1 and on the side of the thicker sheet is T_3. The interface temperature is T_2. T_1, T_2 and T_3 are in arithmetic progression. The ratio of thermal conductivity of thinner sheet to thicker sheet is

(a) 1 : 3 (b) 3 : 1
(c) 2 : 3 (d) 3 : 9

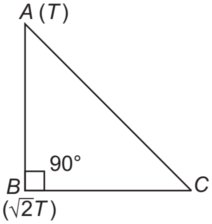
3. A long rod has one end at 0°C and other end at a high temperature. The coefficient of thermal conductivity varies with distance from the low temperature end as $K = K_0(1 + ax)$, where $K_0 = 10^2$ SI unit and $a = 1\ \text{m}^{-1}$. At what distance from the first end the temperature will be 100°C ? The area of cross-section is $1\ \text{cm}^2$ and rate of heat conduction is 1 W.

(a) 2.7 m (b) 1.7 m
(c) 3 m (d) 1.5 m

4. Two rods are of same material and having same length and area. If heat ΔQ flows through them for 12 min when they are joined side by side. If now both the rods are joined in parallel, then the same amount of heat ΔQ will flow in

(a) 24 min (b) 3 min
(c) 12 min (d) 6 min

5. Three rods of identical cross-sectional area and made from the same metal form the sides of an isosceles triangle ABC right angled at B as shown in the figure. The points A and B are maintained at temperatures T and $\sqrt{2}\,T$ respectively in the steady state. Assuming that only heat conduction takes place, temperature of point C will be

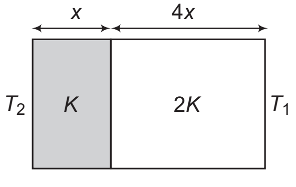


(a) $\dfrac{T}{\sqrt{2}-1}$ (b) $\dfrac{T}{\sqrt{2}+1}$
(c) $\dfrac{3T}{\sqrt{2}+1}$ (d) $\dfrac{T}{\sqrt{3}(\sqrt{2}-1)}$

6. A kettle with 2 litre water at 27°C is heated by operating coil heater of power 1 kW. The heat is lost to the atmosphere at constant rate 160 J/s, when its lid is open. In how much time will water heated to 77°C with the lid open? (specific heat of water = 4.2 kJ/° C-kg)

(a) 8 min 20 s (b) 6 min 2 s
(c) 14 min (d) 7 min

7. The temperature of the two outer surfaces of a composite slab consisting of two materials having coefficients of thermal conductivity K and $2K$ and thickness x and $4x$ respectively are T_2 and $T_1 (T_2 > T_1)$. The rate of heat transfer through the slab in steady state is $\dfrac{AK(T_2 - T_1)}{x} f$. where, f is equal to



(a) 1 (b) 1/2
(c) 2/3 (d) 1/3

More than One Correct Options

1. A solid sphere and a hollow sphere of the same material and of equal radii are heated to the same temperature

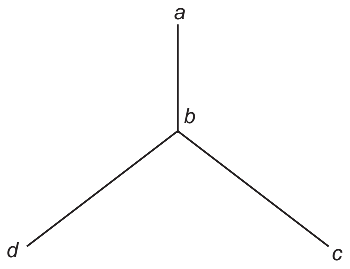
(a) both will emit equal amount of radiation per unit time in the beginning

(b) both will absorb equal amount of radiation per second from the surroundings in the beginning

(c) the initial rate of cooling will be the same for both the spheres

(d) the two spheres will have equal temperatures at any instant

2. Three identical conducting rods are connected as shown in figure. Given that $\theta_a = 40^\circ$ C, $\theta_c = 30^\circ$ C and $\theta_d = 20^\circ$ C. Choose the correct options.



(a) temperature of junction b is 15°C

(b) temperature of junction b is 30°C

(c) heat will flow from c to b

(d) heat will flow from b to d

3. Two liquids of specific heat ratio 1 : 2 are at temperatures 2θ and θ

(a) if equal amounts of them are mixed, then temperature of mixture is $1.5\,\theta$

(b) if equal amounts of them are mixed, then temperature of mixture is $\frac{4}{3}\theta$

(c) for their equal amounts, the ratio of heat capacities is 1 : 1

(d) for their equal amounts, the ratio of their heat capacities is 1 : 2

4. Two conducting rods when connected between two points at constant but different temperatures separately, the rate of heat flow through them is q_1 and q_2

(a) When they are connected in series, the net rate of heat flow will be $q_1 + q_2$

(b) When they are connected in series, the net rate of heat flow is $\frac{q_1 q_2}{q_1 + q_2}$

(c) When they are connected in parallel, the net rate of heat flow is $q_1 + q_2$

(d) When they are connected in parallel, the net rate of heat flow is $\frac{q_1 q_2}{q_1 + q_2}$

5. Choose the correct options.

(a) Good absorbers of a particular wavelength are good emitters of same wavelength. This statement was given by Kirchhoff

(b) At low temperature of a body the rate of cooling is directly proportional to temperature of the body. This statement was given by the Newton

(c) Emissive power of a perfectly black body is 1

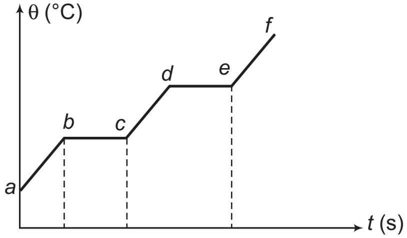
(d) Absorptive power of a perfectly black body is 1

Match the Columns

1. Match the following two columns.

Column I	Column II
(a) Stefan's constant	(p) $[L\theta]$
(b) Wien's constant	(q) $[ML^2T^{-3}\theta^{-2}]$
(c) Emissive power	(r) $[MT^{-3}]$
(d) Thermal resistance	(s) None of these

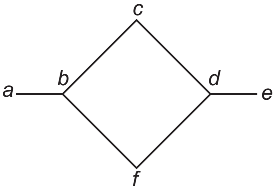
2. Heat is supplied to a substance in solid state at a constant rate. Its temperature varies with time as shown in figure. Match the following two columns.



Column I	Column II
(a) Slope of line *ab*	(p) *de*
(b) Length of line *bc*	(q) *cd*
(c) Solid + liquid state	(r) directly proportional to mass
(d) Only liquid state	(s) None of these

3. Six identical conducting rods are connected as shown in figure. In steady state temperature of point *a* is fixed at 100°C and temperature of *e* at – 80° C. Match the following two columns.

Column I	Column II
(a) Temperature of *b*	(p) 10°C
(b) Temperature of *c*	(q) 40°C
(c) Temperature of *f*	(r) – 20°C
(d) Temperature of *d*	(s) None of these



4. Three liquids A, B and C having same specific heats have masses m, $2m$ and $3m$. Their temperatures are, θ, 2θ and 3θ respectively. For temperature of mixture, match the following two columns.

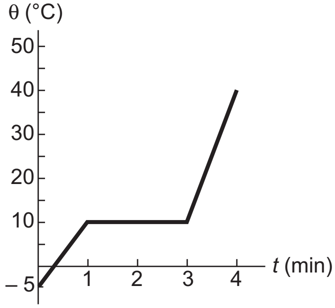
Column I	Column II
(a) When A and B are mixed	(p) $\frac{5}{2}\theta$
(b) When A and C are mixed	(q) $\frac{5}{3}\theta$
(c) When B and C are mixed	(r) $\frac{7}{3}\theta$
(d) When A, B and C all are mixed	(s) $\frac{13}{5}\theta$

5. Match the following two columns.

Column I	Column II
(a) Specific heat	(p) watt
(b) Heat capacity	(q) J/kg-°C
(c) Heat current	(r) J/s
(d) Latent heat	(s) None of these

Subjective Questions

1. As a physicist, you put heat into a 500 g solid sample at the rate of 10.0 kJ/ min, while recording its temperature as a function of time. You plot your data and obtain the graph shown in figure.



(a) What is the latent heat of fusion for this solid?
(b) What is the specific heat of solid state of the material?

2. A hot body placed in air is cooled according to Newton's law of cooling, the rate of decrease of temperature being k times the temperature difference from the surroundings. Starting from $t = 0$, find the time in which the body will lose half the maximum temperature it can lose.

3. Three rods of copper, brass and steel are welded together to form a Y-shaped structure. The cross-sectional area of each rod is 4 cm^2. The end of copper rod is maintained at 100 °C and the ends of the brass and steel rods at 80 °C and 60 °C respectively. Assume that there is no loss of heat from the surfaces of the rods. The lengths of rods are : copper 46 cm, brass 13 cm and steel 12 cm.
(a) What is the temperature of the junction point?
(b) What is the heat current in the copper rod?
K (copper) = 0.92, K (steel) = 0.12 and K (brass) = 0.26 cal/ cm-s °C

4. Ice at 0°C is added to 200 g of water initially at 70°C in a vacuum flask. When 50 g of ice has been added and has all melted the temperature of the flask and contents is 40°C. When a further 80 g of ice has been added and has all melted the temperature of the whole becomes 10°C. Find the latent heat of fusion of ice.

5. A copper cube of mass 200 g slides down a rough inclined plane of inclination 37° at a constant speed. Assuming that the loss in mechanical energy goes into the copper block as thermal energy. Find the increase in temperature of the block as it slides down through 60 cm. Specific heat capacity of copper is equal to 420 J/ kg-K. (Take, $g = 10$ m/ s^2)

6. A cylindrical block of length 0.4 m and area of cross-section 0.04 m^2 is placed coaxially on a thin metal disc of mass 0.4 kg and of the same cross-section. The upper face of the cylinder is maintained at a constant temperature of 400 K and the initial temperature of the disc is 300 K. If the thermal conductivity of the material of the cylinder is 10 watt/m-K and the specific heat of the material of the disc is 600 J/kg-K, how long will it take for the temperature of the disc to increase to 350 K? Assume for purpose of calculation the thermal conductivity of the disc to be very high and the system to be thermally insulated except for the upper face of the cylinder.

7. A metallic cylindrical vessel whose inner and outer radii are r_1 and r_2 is filled with ice at 0° C. The mass of the ice in the cylinder is m. Circular portions of the cylinder is sealed with completely adiabatic walls. The vessel is kept in air. Temperature of the air is 50° C. How long will it take for the ice to melt completely. Thermal conductivity of the cylinder is K and its length is l. Latent heat of fusion is L.

8. An electric heater is placed inside a room of total wall area 137 m^2 to maintain the temperature inside at 20° C . The outside temperature is $-10°$ C. The walls are made of three composite materials. The inner most layer is made of wood of thickness 2.5 cm the middle layer is of cement of thickness 1 cm and the exterior layer is of brick of thickness 2.5 cm. Find the power of electric heater assuming that there is no heat losses through the floor and ceiling . The thermal conductivities of wood, cement and brick are 0.125 W/m°-C, 1.5 W/m°-C and 1.0 W/m°-C respectively.

9. A 2 m long wire of resistance 4 Ω and diameter 0.64 mm is coated with plastic insulation of thickness 0.66 mm. A current of 5 A flows through the wire. Find the temperature difference across the insulation in the steady state. Thermal conductivity of plastic is 0.16×10^{-2} cal/s cm°-C.

10. Two chunks of metal with heat capacities C_1 and C_2 are interconnected by a rod of length l and cross-sectional area A and fairly low conductivity K. The whole system is thermally insulated from the environment. At a moment $t = 0$, the temperature difference between two chunks of metal equals $(\Delta T)_0$. Assuming the heat capacity of the rod to be negligible, find the temperature difference between the chunks as a function of time .

11. A rod of length l with thermally insulated lateral surface consists of material whose heat conductivity coefficient varies with temperature as $k = a/T$, where a is a constant. The ends of the rod are kept at temperatures T_1 and T_2. Find the function $T(x)$, where x is the distance from the end whose temperature is T_1.

12. One end of a uniform brass rod 20 cm long and 10 cm^2 cross-sectional area is kept at 100° C. The other end is in perfect thermal contact with another rod of identical cross-section and length 10 cm. The free end of this rod is kept in melting ice and when the steady state has been reached, it is found that 360 g of ice melts per hour. Calculate the thermal conductivity of the rod, given that the thermal conductivity of brass is 0.25 cal/s cm° C and $L = 80$ cal/g.

13. Heat flows radially outward through a spherical shell of outside radius R_2 and inner radius R_1. The temperature of inner surface of shell is θ_1 and that of outer is θ_2. At what radial distance from centre of shell the temperature is just half way between θ_1 and θ_2?

14. A layer of ice of thickness y is on the surface of a lake. The air is at a constant temperature $-\theta°$ C and the ice water interface is at 0°C. Show that the rate at which the thickness increases is given by

$$\frac{dy}{dt} = \frac{K\theta}{L\rho y}$$

where, K is the thermal conductivity of the ice, L the latent heat of fusion and ρ is the density of the ice.

Answers

Introductory Exercise 22.1

1. 7200 cal **2.** 31.43 °C **3.** 80°C **4.** 12.96 m/s **5.** 12 g
6. 0°C, $m_w = 15$ g, $m_i = 10$ g **7.** 52°C

Introductory Exercise 22.2

1. For heat flow to take place, there must be a temperature difference along the rod.
3. KW^{-1} **4.** Conduction **5.** 145.5°C, 118.2°C **6.** 4 W **7.** 9 min

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (b) **4.** (c) **5.** (c) **6.** (d) **7.** (a) **8.** (d) **9.** (d) **10.** (a)

Objective Questions

1.(c) **2.**(b) **3.**(d) **4.**(b) **5.**(b) **6.**(d) **7.**(b) **8.**(a) **9.**(d) **10.**(c)
11.(c)

Subjective Questions

1. 900 W **2.** 500 m/s **3.** 2.5×10^{-3}°C **4.** 1.2×10^4 kg/s **5.** 3.7×10^4 W **6.** 105°C
7. (a) –8.1°C (b) 7.7 W/m^2 **8.** 87.5 g **9.** 0°C, mass of ice is 54 g and that of water is 286 g
10. 40°C **11.** $\dfrac{3T_1 + 2(T_2 + T_3)}{7}$ **12.** 424°C/m, 212°C/m **13.** 0.3

LEVEL 2

Single Correct Option

1.(c) **2.**(a) **3.**(b) **4.**(b) **5.**(c) **6.**(a) **7.**(d)

More than One Correct Options

1.(a,b) **2.**(b,d) **3.**(b,d) **4.**(b,c) **5.**(a,d)

Match the Columns

1. (a) → s (b) → p (c) → r (d) → s
2. (a) → s (b) → r (c) → s (d) → q
3. (a) → q (b) → p (c) → p (d) → r
4. (a) → q (b) → p (c) → s (d) → r
5. (a) → q (b) → s (c) → p,r (d) → s

Subjective Questions

1. (a) 40 kJ/kg (b) 1.33 kJ/kg-°C **2.** $\ln(2)/k$ **3.** (a) 84°C (b) 1.28 cal/s **4.** 90 cal/g
5. 8.6×10^{-3}°C **6.** 166 s **7.** $t = mL \ln(r_2 / r_1)/100\pi Kl$ **8.** 17647 W **9.** 2.23°C
10. $\Delta T = (\Delta T)_0 e^{-\alpha t}$, where $\alpha = \dfrac{KA(C_1 + C_2)}{lC_1C_2}$ **11.** $T = T_1\left(\dfrac{T_2}{T_1}\right)^{x/l}$ **12.** 0.222 cal/cm-s-°C **13.** $\dfrac{2R_1R_2}{R_1 + R_2}$

17. Wave Motion

INTRODUCTORY EXERCISE 17.1

1. $\dfrac{\partial^2 y}{\partial t^2} = -\omega^2 a \sin \omega t$

and $\dfrac{\partial^2 y}{\partial x^2} = 0$

Since, $\dfrac{\partial^2 y}{\partial t^2} \neq \dfrac{\partial^2 y}{\partial x^2}$ (constant)

Hence, the given equation does not represent a wave equation.

2. Speed of wave $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x} = \dfrac{c}{b}$

3. The converse is not true means if the function can be represented in the form $y = f(x \pm vt)$, it does not necessarily express a travelling wave. As the essential condition for a travelling wave is that the vibrating particle must have finite displacement value for all x and t.

4. (a) A has the dimensions of y.

$\dfrac{x}{a}$ and $\dfrac{t}{T}$ are dimensionless.

(b) $v = \dfrac{\text{coefficient of } t}{\text{coefficient of } x}$

(c) $\dfrac{x}{a}$ and $\dfrac{t}{T}$ are of same sign

(d) $y_{\max} = A$

INTRODUCTORY EXERCISE 17.2

1. Comparing the given equation with

$$y = \sin (kx - \omega t)$$

we find,

(a) Amplitude $A = 5$ mm

(b) Angular wave number $k = 1\ \text{cm}^{-1}$

(c) Wavelength, $\lambda = \dfrac{2\pi}{k} = 2\pi$ cm

(d) Frequency, $\nu = \dfrac{\omega}{2\pi} = \dfrac{60}{2\pi}$ Hz

$= \dfrac{30}{\pi}$ Hz

(e) Time period, $T = \dfrac{1}{\nu} = \dfrac{\pi}{30}$ s

(f) Wave velocity, $v = \nu\lambda = 60\ \text{cm s}^{-1}$

2. (a) $T = \dfrac{2\pi}{\omega}, \lambda = \dfrac{2\pi}{k}$

(b) Find $\dfrac{\partial y}{\partial t}$ and then substitute the given values of x and t.

(c) and (d) Procedure is same.

INTRODUCTORY EXERCISE 17.3

1. (a) coefficient of x and t are of same sign.

(b) $v = \dfrac{314}{31.4} = 10\ \text{m/s} \Rightarrow \lambda = \dfrac{2\pi}{k}, f = \dfrac{\omega}{2\pi}$

(c) $v_{\max} = \omega A$

2. (a) The velocity of the particle at x at time t is

$$v = \frac{\partial y}{\partial t} = (3.0\ \text{cm})(-314\ \text{s}^{-1}) \cos [(3.14\ \text{cm}^{-1}) x - (314\ \text{s}^{-1})t]$$

$$= (-9.4\ \text{ms}^{-1}) \cos[(3.14 \text{cm}^{-1})x - (314\ \text{s}^{-1})t]$$

The maximum velocity of a particle will be

$$v = 9.4\ \text{ms}^{-1}$$

(b) The acceleration of the particle at x at time t is

$$a = \frac{\partial v}{\partial t} = -(9.4\ \text{ms}^{-1})(314\ \text{s}^{-1}) \sin [(3.14\ \text{cm}^{-1}) x - (314\ \text{s}^{-1})t]$$

$$= -(2952\ \text{ms}^{-1}) \sin [(3.14\ \text{cm}^{-1})x - (314\ \text{s}^{-1})t].$$

The acceleration of the particle at $x = 6.0$ cm at time $t = 0.11$ s is

$$a = -(2952\ \text{ms}^{-2}) \sin [6\pi - 11\pi] = 0$$

3. (a) Wave velocity $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$= \dfrac{1/0.01}{1/0.05} = 5\ \text{m/s}$

Since, coefficient of t and coefficient of x are of same sign. Hence, wave is travelling in negative x-direction.

or $v = -5\ \text{m/s}$

(b) Velocity of particle,

$$v_P = \frac{\partial y}{\partial t} = 2 \cos \left(\frac{x}{0.05} + \frac{t}{0.01} \right)$$

Substituting $x = 0.2$ and $t = 0.3$, we have

$$v_P = 2 \cos (34) = 2(-0.85)$$

$$= -1.7\ \text{m/s}$$

4. (a) $$\Delta\phi = \frac{2\pi}{\lambda}\Delta x$$

$$\therefore \quad \Delta x = \frac{\lambda\Delta\phi}{2\pi} = \frac{v\,(\Delta\phi)}{2\pi f}$$

$$= \frac{(350)\,(\pi/3)}{(2\pi)\,(500)}$$

$$= 0.116 \text{ m}$$

(b) $$\Delta\phi = (\omega)\Delta t$$

$$= (2\pi f)\,(\Delta t)$$

$$= (2\pi)\,(500)(10^{-3})$$

$$= \pi \text{ or } 180°$$

INTRODUCTORY EXERCISE 17.4

1. $$f_{\max} = \frac{c}{\lambda_{\min}}$$

$$f_{\min} = \frac{c}{\lambda_{\max}}$$

2. $\lambda = \frac{v}{f}$ for both

INTRODUCTORY EXERCISE 17.5

1. The tension in the string is $F = mg = 10$ N. The mass per unit length is $\mu = 1.0 \text{ g cm}^{-1} = 0.1 \text{ kg m}^{-1}$. The wave velocity is, therefore, $v = \sqrt{F\mu} = \sqrt{\frac{10 \text{ N}}{0.1 \text{ kg m}^{-1}}} = 10 \text{ ms}^{-1}$. The time taken by the pulse in travelling through 50 cm is, therefore 0.05 s.

2. $$v = \sqrt{\frac{T}{\mu}}$$

3. $T_{CD} = 3.2 \text{ g}, T_{AB} = 6.4 \text{ g}$

and $$v = \sqrt{\frac{T}{\mu}}$$

4. $$v = \sqrt{\frac{T}{\mu}}, \quad \mu = \frac{m}{l} \quad \text{and} \quad t = \frac{d}{v}$$

5. $$v = \sqrt{\frac{T}{\rho S}} = \sqrt{\frac{T}{\rho\,(\pi d^2/A)}}$$

$$= \sqrt{\frac{4T}{\rho\pi d^2}} \qquad (T = mg)$$

$$= \sqrt{\frac{4 \times 20}{(8920)\,(\pi)\,(2.4 \times 10^{-3})^2}}$$

$$= 22 \text{ m/s}$$

6. (a) $$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{mg}{\mu}} \qquad \text{...(i)}$$

$$= \sqrt{\frac{1.5 \times 9.8}{0.055}} = 16.3 \text{ m/s}$$

(b) $\lambda = \frac{v}{f} = \frac{16.3}{120} = 0.136 \text{ m}$

(c) From Eq. (i), we can see that $v \propto \sqrt{m}$. If m is doubled, then v will become $\sqrt{2}$ times. Hence, from the relation $\lambda = \frac{v}{f} \cdot \lambda$ will also become $\sqrt{2}$ times. As f remains unchanged.

INTRODUCTORY EXERCISE 17.6

1. $$I = \frac{P}{4\pi r^2} = \frac{1.0}{4\pi\,(1)^2} = \frac{1}{4\pi} \text{ W/m}^2$$

2. Suppose power of line source is P. Then, at distance r, surface area is $2\pi rl$.

$$\therefore \quad I = \frac{P}{S} = \frac{P}{2\pi rl}$$

or $$I \propto \frac{1}{r}$$

Further, $$I \propto A^2$$

$$\therefore \quad A \propto \frac{1}{\sqrt{r}}$$

3. Energy density, $u = \frac{1}{2}\rho\omega^2 A^2$

Total energy $= (u)\,(V)$

$$= \frac{1}{2}(\rho V)\,(2\pi f)^2 A^2$$

$$= 2\pi^2\, mf^2 A^2$$

$$= (2)(\pi)^2(0.08)(120)^2(0.16 \times 10^{-3})^2$$

$$= 581 \times 10^{-3} \text{J}$$

$$= 0.58 \text{ mJ}$$

4. Power $= \frac{1}{2}\rho\omega^2 A^2 Sv$

But, $$\rho S = \mu$$

and $$v = \sqrt{\frac{T}{\mu}}$$

$$\therefore \quad \rho Sv = \sqrt{\mu T}$$

Hence, Power $= \frac{1}{2}\,(2\pi f)^2 A^2 \sqrt{T\mu}$

$$= 2\pi^2 f^2\, A^2 \sqrt{T\mu}$$

$$= (2)\,(\pi)^2 (60)^2\,(0.06)^2 \sqrt{80 \times 5 \times 10^{-2}}$$

$$= 512 \text{ W}$$

5. (a) As derived in above problem, power

$$= 2\pi^2 f^2 A^2 \sqrt{T\mu}$$
$$= (2)\,(\pi)^2\,(200)^2\,(10^{-3})^2\,\sqrt{60 \times (0.006)}$$
$$= 0.47 \text{ W}$$

(b) Energy density, $u = \dfrac{1}{2}\rho\omega^2 A^2$

$$\text{Total energy} = (u)\,(V)$$
$$= \left(\frac{1}{2}\rho\omega^2 A^2\right)(lS)$$

But, $\rho S = \mu$

$$\therefore \text{ Total energy} = \frac{1}{2}\mu l\,(2\pi f)^2\,A^2$$
$$= 2\pi^2 f^2 A^2 \mu l$$
$$= (2)\,(\pi)^2\,(200)^2\,(10^{-3})^2\,(0.006)\,(2.0)$$
$$= 9.4 \times 10^{-3}\text{ J} = 9.4 \text{ mJ}$$

6. $P = \dfrac{1}{2}\rho\omega^2 A^2 S v$ ...(i)

But, $\rho S = \mu$

and $v = \sqrt{\dfrac{T}{\mu}} = \sqrt{\dfrac{T}{\rho S}}$

$\therefore$ $\rho S = \dfrac{T}{v^2}$ or $\rho S v = \dfrac{T}{v}$

Substituting in Eq. (i), we have

$$P = \frac{1}{2}\,(2\pi f)^2 A^2 \frac{T}{v}$$
$$= \frac{2\pi^2 f^2 A^2 T}{v}$$
$$= \frac{(2)\,(\pi)^2\,(100)^2\,(0.0005)^2\,(100)}{100}$$
$$= 0.049 \text{ J} = 49 \text{ mJ}$$

Exercise

LEVEL 1

Assertion and Reason

3. If both waves ωt and kx are of opposite signs, then both are travelling in positive directions.

4. If f is doubled (which is source dependant), then λ will automatically become half, so that speed remains same. Because speed is only medium dependant.

5. Longitudinal or sound wave cannot travel in vacuum.

7. Electromagnetic wave which can travel with or without medium.

8. Just by observation we cannot say that $\mu_2 > \mu_1$ and hence $v_2 > v_1$.

If their densities are different, then μ_2 may be less than μ_1 also.

9. At mean position kinetic energy is maximum and there is a maximum stretch in string. Because one side particles are moving up and the other side particles are moving down. Hence, potential energy is also maximum.

10. In $v_P = -v\,\dfrac{\partial y}{\partial x}$,

v_P is negative and $\dfrac{\partial y}{\partial x}$ (the slope) is also negative.

Hence, the velocity v should also be negative.

Objective Questions

1. $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{\pi/9} = 18 \text{ m}$

2. $\Delta\phi = \dfrac{2\pi}{\lambda}\,\Delta x = k\Delta x$

$$= (10\pi \times 0.01)\,(10) = \pi$$

3. At $t = 0$, $y_{\max} = \dfrac{1}{2}$ at $x = 0$

At $t = 2$ s, $y_{\max} = \dfrac{1}{2}$ at $x = 2$ m

$\therefore$ In 2 s, $y_{\max}$ has travelled 2 m in positive direction

$$\therefore \quad v = +\frac{2}{3} = +1 \text{ m/s}$$

4. Wave velocity and particle velocity are two different things.

5. $\Delta\phi = \dfrac{2\pi}{\lambda}\,(\Delta x) = \dfrac{2\pi\,\phi(\Delta x)}{v}$ $\quad\left(\text{as } \lambda = \dfrac{v}{f}\right)$

$$= \frac{2\pi\,(25)\,(16 - 10)}{300} = \pi$$

6. $v = \dfrac{\omega}{k} = \sqrt{\dfrac{T}{\mu}}$

$$\therefore \quad T = \mu\left(\frac{\omega}{k}\right)^2 = (1.3 \times 10^{-4})\left(\frac{30}{1}\right)^2$$
$$= 0.12 \text{ N}$$

7. $\because \quad \omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{(t/N)} = \dfrac{2\pi N}{t}$

$$= \frac{(2\pi)(150)}{60}$$

$$= (5\pi)\ \text{rad/s}$$

Subjective Questions

1. (b) $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{2\pi/28}$

$$= 28\ \text{cm}$$

(c) $f = \dfrac{\omega}{2\pi} = \dfrac{2\pi/0.036}{2\pi}$

$$= 27.8\ \text{Hz}$$

(d) $v = f\lambda = 778.4\ \text{cm/s}$

$$= 7.8\ \text{m/s}$$

Since, ωt and kx have opposite signs. Hence, wave is travelling in positive direction.

2. (a) Put $t = 0$, $x = 2\,\text{cm}$

(b) $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{(30\pi/240)} = 16\ \text{cm}$

(c) Wave velocity $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$$= \frac{1}{1/240}$$

$$= 240\ \text{cm/s}$$

(d) $f = \dfrac{\omega}{2\pi} = \dfrac{30\pi}{2\pi} = 15\ \text{Hz}$

3. At $t = 0$, y is maximum at $x = 0$.

At $t = 2\,\text{s}$, y is maximum at $x = 1\,\text{m}$. Hence, in 2 s wave has travelled 2 m in positive x-direction

$\therefore \quad v = +\dfrac{1}{2}\ \text{m/s}$

$$= +\ 0.5\ \text{m/s}$$

4. Since, coefficient of t and x are opposite signs, then wave is travelling along negative x-direction.

Further, speed of wave $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$$= \frac{2}{1} = 2\ \text{m/s}$$

Amplitude = maximum value of y

$$= \frac{10}{5} = 2\ \text{m}$$

5. Wave speed, $\quad v = \dfrac{\omega}{k}$

and maximum particle speed,

$$(v_P)_{\max} = \omega A$$

From these two expressions, we can see that

$$(v_P)_{\max} = (kA)\,V$$

6. Mass per unit length, $\mu = \dfrac{m}{l}$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{Tl}{m}} = \sqrt{\frac{500 \times 2}{0.06}}$$

$$= 129.1\ \text{m/s}$$

7. $\quad v = \sqrt{\dfrac{T}{\rho S}}$ as mass per unit length $= \rho S$

$\therefore \quad v = \sqrt{\dfrac{0.98}{9.8 \times 10^{3} \times 10^{-6}}}$

$$= 10\ \text{m/s}$$

8. Since, wave is travelling along positive x-direction. Hence, coefficient of t and coefficient of x should have opposite signs. Further,

$$v = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

$\therefore \quad 2 = \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$\therefore$ Coefficient of $t = 2$ (coefficient of x)

$$= 2 \times 1 = 2\ \text{SI units}$$

$\therefore \quad y = \dfrac{10}{(x - 2t)^2 + 2}$

9. (a) $\dfrac{\lambda}{2} = 2\ \text{cm}$

$\Rightarrow \quad \lambda = 4\ \text{cm}$

$$f = \frac{v}{\lambda} = \frac{40}{4} = 10\ \text{Hz}$$

(b) $\Delta\phi = \dfrac{2\pi}{\lambda}\,\Delta x = \left(\dfrac{2\pi}{4}\right)(2.5) = \dfrac{5\pi}{4}$

(c) $\omega t = \theta$

or $\quad (2\pi/t) = \theta$

$\therefore \quad t = \dfrac{\theta}{2\pi f} = \dfrac{\pi/3}{(2\pi)(10)}$

$$= \frac{1}{60}\ \text{s}$$

(d) At P, particle is at mean position. So,

$$v = \text{maximum velocity}$$

$$= \omega A$$

$$= 2\pi f A$$

$$= (2\pi)(10)(2)$$

$$= (40\pi)\ \text{cm/s}$$

$$= 125.7\ \text{cm/s} = 1.26\ \text{m/s}$$

Further, $v_P = -(v)\left(\dfrac{\partial y}{\partial x}\right)$ ...(i)

Sign of v_1, the wave velocity is given positive.

Sign of $\dfrac{\partial y}{\partial x}$, slope of y - x graph is also positive.

Hence, from Eq. (i) particle velocity is negative.

$\therefore \quad v_P = -1.26 \text{ m/s}$

10. (a) Wave velocity $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$$= \frac{(1/0.01)}{(1/0.05)} = 5 \text{ m/s}$$

Since, coefficient of t and coefficient of x are of same sign. Hence, wave is travelling in negative x-direction.

or $\quad v = -5 \text{ m/s}$

(b) Particle velocity, $v_P = \dfrac{\partial y}{\partial t}$

$$= 2\cos\left(\frac{x}{0.05} + \frac{t}{0.01}\right)$$

Substituting $x = 0.2$ and $t = 0.3$, we have

$$v_P = 2\cos 34$$
$$= (2)(-0.85)$$
$$= -1.7 \text{ m/s}$$

11. (a) $\omega = 2\pi f = 2\pi\left(\dfrac{v}{\lambda}\right) = 2\pi\left(\dfrac{12}{0.4}\right)$

$$= (60\pi) \text{ rad/s}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4} = (5\pi) \text{ m}^{-1}$$

Since, wave is travelling along + ve x-direction, ωt and kx should have opposite sign. Further at $t = 0$, $x = 0$ the string has zero displacement and moving upward (in positive direction). Hence at $x = 0$, we should have $A \sin \omega t$ not $-A \sin \omega t$. Therefore, the correct expression is

$$y = A \sin(\omega t - kx)$$

or $\quad y = 0.05 \sin(60\pi t - 5\pi x)$

(b) Putting $x = 0.2$ m and $t = 0.15$ s in above equation we have,

$$y = -0.0354 \text{ m}$$
$$= -3.54 \text{ cm}$$

(c) In part (b), $y = A/\sqrt{2}$

From $\quad y = \dfrac{A}{\sqrt{2}}$ to $y = 0$, time taken is

$$t = \frac{T}{8} = \frac{2\pi}{\omega \times 8}$$
$$= \frac{2\pi}{(60\pi)(8)}$$
$$= 4.2 \times 10^{-3} \text{ s}$$
$$= 4.2 \text{ ms}$$

12. (a) $\because \quad T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi/0.01} = 0.02 \text{ s}$

$$= 20 \text{ ms}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/2}$$
$$= 4.0 \text{ cm}$$

(b) $v_P = \dfrac{\partial y}{\partial x}$

$$= \left(-\frac{\pi}{0.01}\frac{\text{mm}}{\text{s}}\right)\cos\pi\left(\frac{x}{2.0 \text{ cm}} - \frac{t}{0.01 \text{ s}}\right)$$

$$= \left(-\frac{\pi}{10}\frac{\text{m}}{\text{s}}\right)\cos\pi\left(\frac{x}{2.0 \text{ cm}} - \frac{t}{0.01 \text{ s}}\right)$$

Put $x = 1$ cm and $t = 0.01$ s in above equation We get,

$$v_P = 0$$

(c) and (d) Putting the given values in the above equation, we get the answers.

13. (a) $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{40} = 0.157 \text{ rad/cm}$

$$T = \frac{1}{f} = \frac{1}{8} = 0.125 \text{ s}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.215} = 50.3 \text{ rad/s}$$

$$v = \frac{\omega}{k} = 320 \text{ cm/s}$$

(b) Amplitude is 15 cm.

At $t = 0$, $x = 0$, $y = +A$. Hence, equation should be a cos equation. Further, wave is travelling in positive x-direction. Hence, ωt and kx should have opposite signs.

14. (a) $T_x = (m_{L-x})\,g = \mu\,(L - x)\,g$

where, μ = mass per unit length

$$\therefore \quad v_x = \sqrt{\frac{T_x}{\mu}} = \sqrt{g\,(L - x)}$$

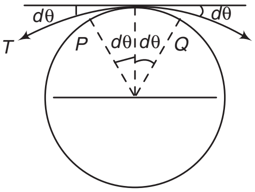
(b) $$v_x = -\frac{dx}{dt} = \sqrt{g\,(L - x)}$$

$$\therefore \quad \int_0^t dt = -\int_{x=L}^{x=0} \sqrt{g\,(L - x)}\,dx$$

Solving we get,

$$t = 2\sqrt{\frac{L}{g}}$$

15. (a) $T \cos d\theta$ components are cancelled. $T \sin d\theta$ components provide the necessary centripetal force to $P\theta$.



$$\therefore \quad 2T \sin d\theta = m_{PQ} R\omega^2$$

For small angles $\sin d\theta \approx d\theta$

$$\therefore \quad 2T\, d\theta = [\mu\,(2R)\, d\theta]\, R\omega^2$$

Solving the equation, we get

$$\sqrt{\frac{T}{\mu}} = v = R\omega$$

16.
$$M = \int_0^L kx\, dx = \frac{kL^2}{2}$$

$$\therefore \quad k = \frac{2M}{L^2}$$

$$v_x = \sqrt{\frac{T}{\mu_x}} = \sqrt{\frac{T}{kx}} = \sqrt{\frac{T}{(2M/L^2)\, x}}$$

$$= \left(L \sqrt{\frac{T}{2M}} \right) x^{-1/2} = \frac{dx}{dt}$$

$$\therefore \quad \int_0^t dt = \frac{1}{L} \sqrt{\frac{2M}{T}} \int_0^L x^{1/2}\, dx$$

$$\text{or} \quad t = \left(\frac{2}{3L} \sqrt{\frac{2M}{T}} \right) L^{3/2}$$

$$= \frac{2}{3} \sqrt{\frac{2ML}{T}}$$

LEVEL 2

Single Correct Option

1. $\because \quad v = \dfrac{\omega}{k}$

$$\therefore \quad k = \frac{\omega}{v} = \frac{600\pi}{300}$$

$$= 2\pi \text{ m}^{-1}$$

$$\therefore \quad y = 0.04 \sin (600\,\pi t - 2\pi x)$$

Now, put $t = 0.01$ s and $x = 0.75$ m

2. $\because \quad \omega = 2\pi f = (2\pi)(100)$

$$= (200\,\pi) \text{ rad/s}$$

$$v = \frac{\omega}{k}$$

$$\therefore \quad k = \frac{\omega}{v} = \omega \sqrt{\frac{\mu}{T}}$$

$$= (200\pi) \sqrt{\frac{3.5 \times 10^{-3}}{35}}$$

$$= 2\pi \text{ m}^{-1}$$

At zero displacement,

$$v_P = \omega A = -v \frac{\partial y}{\partial x}$$

$$\therefore \quad |A| = \frac{v \left(\dfrac{\partial y}{\partial x} \right)}{\omega} = \frac{\left(\sqrt{\dfrac{T}{\mu}} \right) (\text{slope})}{\omega}$$

$$= \frac{\left(\sqrt{\dfrac{35}{3.5 \times 10^{-3}}} \right) (\pi/20)}{200\pi}$$

$$= 0.025 \text{ m}$$

3. $\because \quad \omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.25}$

$$= 8\pi \text{ rad/s}$$

$$v = \frac{\omega}{k}$$

$$\therefore \quad k = \frac{\omega}{v} = \frac{8\pi}{0.48} = \left(\frac{50}{3} \right) \pi \text{ m}^{-1}$$

$$y = A \sin (\omega t - kx)$$

$$= A \sin \left(8\pi t - \frac{50}{3} \pi x \right)$$

Put $y = 3$ cm, $t = 1$ s, $x = 0.47$ m

showing we get $A = 6$ cm

4. $v = \sqrt{\dfrac{T}{\rho S}} = \sqrt{\dfrac{T}{\rho\, (\pi d^2/4)}}$

$$\therefore \quad v \propto \sqrt{\frac{T}{d^2}}$$

5. $E \propto \omega^2 A^2$

or $\quad E \propto f^2 A^2$

E is same

$\therefore \quad fA =$ constant

or $\quad A \propto \dfrac{1}{f}$

6. The wave pulse is travelling along positive x-axis. Hence, at and bx should have opposite signs. Further, wave speed

$$v = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

$$\therefore \quad 4 = \frac{\text{Coefficient of } t}{1}$$

$\therefore$ Coefficient of $t = 4\ \text{s}^{-1}$

More than One Correct Options

1. $\because$

$$v = \frac{\text{Coefficient of } t}{\text{Coefficient of } x} = \frac{6.2}{2} = 3.1\ \text{m/s}$$

$$A = \frac{2}{20} = 0.1\ \text{m}$$

3.

$$y = A\sin(\pi x + \pi t)$$

$$v_P = \frac{\partial y}{\partial t} = \pi A\cos(\pi x + \pi t)$$

$$a_P = \frac{\partial^2 y}{\partial t^2} = -\pi^2 A\sin(\pi x + \pi t)$$

Now, substitute $t = 0$ and given value of x.

Since, ωt and kx are of same sign, hence the wave is travelling in negative x-direction.

4. Speed of wave

$$= \frac{\text{Coefficient of } t}{\text{Coefficient of } x} = \frac{b}{1} = b$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{(2\pi/a)} = a$$

5. Speed of wave $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x}$

$$= \frac{1/b}{1/a} = \frac{a}{b}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{2\pi/a} = a$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi/b} = b$$

6. y - t graph is sine graph. Therefore, v - t graph is cos graph and a - t is $-$ sine graph as

$$v_P = \frac{\partial y}{\partial t} \quad \text{and} \quad a_P = \frac{\partial^2 y}{\partial t^2}$$

Match the Columns

1. (a) Wave speed $= \dfrac{\text{Coefficient of } t}{\text{Coefficient of } x} = \dfrac{b}{c}$

(b) Maximum particle speed $= \omega A = (b)(a)$

(c) $f = \dfrac{\omega}{2\pi} = \dfrac{b}{2\pi}$

(d) $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{c}$

2.

$$v_P = \frac{\partial y}{\partial t} = (4\pi)\ \text{cm/s}\cos[\pi t + 2\pi x]$$

$$a_P = \frac{\partial^2 y}{\partial t^2} = (-4\pi^2)\ \text{cm/s}^2\sin[\pi t + 2\pi x]$$

Now substitute the values of t and x.

3. For velocity, $v_P = -v\dfrac{\partial y}{\partial x}$

$\dfrac{\partial y}{\partial x}$ = Slope of y-x graph.

Sign of v is not given in the question.

Hence, direction of v_P cannot be determined.

For particle acceleration,

$$a_P \propto -y$$

i.e. a_P and y are away in opposite directions

If $y = 0$, then $a_P = 0$

4. Energy density $= \dfrac{1}{2}\rho\omega^2 A^2$

= Energy per unit volume

Power = energy transfer per unit

$$\text{Time} = \frac{1}{2}\rho\omega^2 A^2 Sv$$

Intensity = energy transfer per unit per unit area

$$= \frac{1}{2}\rho\omega^2 A^2 v = \frac{P}{S}$$

Wave number $= \dfrac{1}{\lambda}$

5. $v_P = \dfrac{\partial y}{\partial t}$, $a_P = \dfrac{\partial^2 y}{\partial t^2}$

If ωt and kx are of same sign, then wave travel in negative x-direction.

If they are of opposite signs, then wave travels in positive direction.

Subjective Questions

1. (a) $v_P = -v\left(\dfrac{dy}{dx}\right)$

As v_P and $(\text{slope})_P$ are both positive, v must be negative. Hence, the wave is moving in negative x-axis.

(b) $y = A\sin(\omega t - kx + \phi)$ …(i)

$$k = \frac{2\pi}{\lambda} = \frac{\pi}{2}\ \text{cm}^{-1}$$

$$A = 4 \times 10^{-3}\ \text{m} = 0.4\ \text{cm}$$

At $t = 0,\ x = 0,$ slope $\dfrac{dy}{dx} = +$ ve

$\therefore\quad v_P = -v(\text{slope}) = +$ ve

Further at $\quad t = 0, x = 0,\ y = +$ ve

$$\therefore\qquad \phi = \frac{\pi}{4}$$

Further, $20\sqrt{3} = -v\tan 60°$

$$\therefore\qquad v = -20 \text{ cm/s}$$

$$f = \frac{v}{\lambda} = 5 \text{ Hz}$$

$$\therefore\qquad \omega = 2\pi f = 10\pi$$

$$\therefore\quad y = (0.4 \text{ cm}) \sin\left(10\pi t + \frac{\pi}{2}x + \frac{\pi}{4}\right) \qquad \textbf{Ans.}$$

(c) $P = 2\pi^2 A^2 f^2 \mu v$

$\therefore\quad$ Energy carried per cycle

$$E = PT = \frac{P}{f} = 2\pi^2 A^2 f \mu v$$

Substituting the values, we have

$$E = 1.6 \times 10^{-5} \text{ J} \qquad \textbf{Ans.}$$

2. (a) $v = \sqrt{\dfrac{T}{\rho S}}$

$$= \sqrt{\frac{64}{12.5 \times 10^3 \times 0.8 \times 10^{-6}}}$$

$$= 80 \text{ m/s} \qquad \textbf{Ans.}$$

(b) $\omega = 2\pi f = 2\pi(20) = 40\pi$ rad/s

$$k = \frac{\omega}{v} = \frac{40\pi}{80} = \frac{\pi}{2} \text{ m}^{-1}$$

$$\therefore\quad y = (1.0 \text{ cm}) \cos\left[(40\pi s^{-1})t - \left(\frac{\pi}{2} m^{-1}\right)x\right]$$

Ans.

(c) Substituting $x = 0.5$ m and $t = 0.05$ s, we get

$$y = \frac{1}{\sqrt{2}} \text{ cm} \qquad \textbf{Ans.}$$

(d) Particle velocity at time t.

$$v_P = \frac{\partial y}{\partial t}$$

$$= -(40\pi \text{ cm/s}) \sin\left[(40\pi \text{ s}^{-1})t - \left(\frac{\pi}{2} \text{ m}^{-1}\right)x\right]$$

Substituting $x = 0.5$ m and $t = 0.05$ s, we get

$$v_P = 89 \text{ cm/s} \qquad \textbf{Ans.}$$

3. $k_{\text{eff}} = 2k = 1.0$ N/m

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m}} = \frac{10}{2\pi} \text{ s}^{-1}$$

$$v = 0.1 \text{ m/s}, \quad A = 0.02 \text{ m}$$

$$\lambda = \frac{v}{f} = \frac{0.1}{\left(\dfrac{10}{2\pi}\right)} = \frac{2\pi}{100} \text{ m}$$

$$y = A\cos\left(\frac{2\pi}{\lambda}\right)(vt - x)$$

$$= 0.02 \cos 100(0.1t - x)$$

$$= 0.02 \cos(10t - 100x) \text{ m} \qquad \textbf{Ans.}$$

The distance between two successive maxima

$$= \lambda = \frac{2\pi}{100} = 0.0628 \text{ m} \qquad \textbf{Ans.}$$

4. (a) Dimensions of A and Y are same.

Similarly, dimensions of a and x are same.

(b) As the wave is travelling towards positive x-axis, there should be negative sign between term of x and term of t.

Further, speed of wave

$$v = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

$\therefore\quad$ Coefficient of $t = (v) \times$ coefficient of x

5. From the given figure, we can see that

(a) Amplitude, $A = 1.0$ mm

(b) Wavelength, $\lambda = 4$ cm

(c) Wave number, $k = \dfrac{2\pi}{\lambda} = 1.57 \text{ cm}^{-1} \approx 1.6 \text{ cm}^{-1}$

(d) Frequency, $f = \dfrac{v}{\lambda} = \dfrac{20}{4} = 5$ Hz

6. $v = \sqrt{\dfrac{T}{\rho S}} \quad$ or $\quad v \propto \dfrac{1}{\sqrt{\rho}}$

$$\therefore\qquad \frac{v_1}{v_2} = \sqrt{\frac{\rho_2}{\rho_1}}$$

$$\therefore\qquad \frac{\rho_1}{\rho_2} = \left(\frac{v_2}{v_1}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

7. $\qquad v_1 = \sqrt{\dfrac{T_1}{\mu_1}}$

$$= \sqrt{\frac{4.8}{1.2 \times 10^{-2}}} = 20 \text{ m/s}$$

$$v_2 = \sqrt{\frac{T_2}{\mu_2}} = \sqrt{\frac{7.5}{1.2 \times 10^{-2}}}$$

$$= 25 \text{ m/s}$$

Pulses will meet when $x_A = x_B$

or $\qquad 20t = 25(t - 0.02)$

$\therefore\qquad t = 0.1$ s

and $\quad x_A$ or $x_B = 20 \times 0.1 = 2$ m

8. (a) $P = \frac{1}{2}\rho\omega^2 A^2 S v$

or $$A = \frac{1}{\omega}\sqrt{\frac{2P}{\rho S v}} \qquad \text{...(i)}$$

Here, $\rho S = \mu$ = mass per unit length

$$= \frac{6 \times 10^{-3}}{8} \text{ kg/m}$$

$$\omega = 2\pi f = \frac{2\pi v}{\lambda}$$

$$= \frac{2\pi \times 30}{0.2}$$

Substituting these values in Eq. (i), we have

$$A = \frac{0.2}{2\pi \times 30}\sqrt{\frac{2 \times 50 \times 8}{6 \times 10^{-3} \times 30}}$$

$$= 0.0707 \text{ m}$$

$$= 7.07 \text{ cm}$$

(b) $P \propto v\omega^2$

or $$P \propto v(v^2)$$

or $$P \propto v^3$$

When wave speed is doubled, then power will become eight times.

9. $-dT = (dm)x\omega^2 = \left(\frac{m}{L}dx\right)x\omega^2$

or $$-\int_0^T dT = \frac{m}{L}\omega^2 \int_{x=L}^{x} x\, dx$$

$\therefore$ $$-T = \frac{m}{L}\omega^2\left(\frac{x^2}{2} - \frac{L^2}{2}\right)$$

or $$T = \frac{m\omega^2}{2L}(L^2 - x^2)$$

$$v = \sqrt{\frac{T}{\mu}}$$

$$= \sqrt{\frac{\frac{m\omega^2}{2L}(L^2 - x^2)}{m/L}}$$

$$= \omega\sqrt{\frac{L^2 - x^2}{2}}$$

or $$\frac{dx}{dt} = \frac{\omega}{\sqrt{2}}\sqrt{L^2 - x^2}$$

$\therefore$ $$\int_0^t dt = \frac{\sqrt{2}}{\omega}\int_0^L \frac{dx}{\sqrt{L^2 - x^2}}$$

$\therefore$ $$t = \frac{\sqrt{2}}{\omega}\left[\sin^{-1}\left(\frac{x}{L}\right)\right]_0^L$$

$$= \frac{\sqrt{2}}{\omega}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{2}\omega}$$ **Ans.**

18. Superposition of Waves

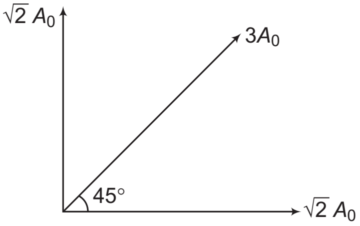
INTRODUCTORY EXERCISE 18.1

1. $\dfrac{I_{\max}}{I_{\min}} = \left(\dfrac{\sqrt{I_1/I_2}+1}{\sqrt{I_1/I_2}-1}\right)^2$

2. (a) $\dfrac{A_{\max}}{A_{\min}} = \dfrac{A_1 + A_2}{A_1 - A_2}$

(b) $\dfrac{I_{\max}}{I_{\min}} = \left(\dfrac{A_{\max}}{A_{\min}}\right)^2$

3. Path difference of $\dfrac{\lambda}{4}$ is equivalent to a phase difference of $\dfrac{\pi}{2}$.



$$A_{\text{net}} = 5A_0$$
$$I \propto A^2$$
$$\Rightarrow \qquad I_{\text{net}} = 25\,I_0$$

INTRODUCTORY EXERCISE 18.2

1. It is either 0 or π.

2. (a) $v = \dfrac{\omega}{k} = \dfrac{40\pi}{\pi/3}$

$= 120$ cm/s

(b) Distance between adjacent nodes

$$= \frac{\lambda}{2} = \frac{\pi}{k} = \frac{\pi}{(\pi/3)}$$

$= 3$ cm

(c) $v_P = \dfrac{\partial y}{\partial t} = (-\,200\,\pi)\sin\dfrac{\pi x}{3}\sin 40\pi t$

Now, substitute the given values of x and t.

3. In this case, node points will also oscillate, but standing waves are formed.

4. $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{0.4} = 15.7$ m

$$f = \frac{\omega}{2\pi} = \frac{200}{2\pi} = 31.8 \text{ Hz}$$
$$v = f\lambda = 500 \text{ m/s}$$

INTRODUCTORY EXERCISE 18.3

1. Wall will be a node (displacement). Therefore, shortest distance from the wall at which air particles have maximum amplitude of vibration (displacement antinode) should be $\lambda/4$.

Here, $\lambda = \dfrac{v}{f} = \dfrac{330}{660} = 0.5$ m

$\therefore$ Desired distance is $\dfrac{0.5}{4} = 0.125$ m.

2. $5f - 2f = 54$

$$3f = 54$$
$$\therefore \qquad f = 18 \text{ Hz}$$

3. $f \propto \sqrt{T}$

$$\therefore \qquad \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$$

or $\dfrac{220}{260} = \sqrt{\dfrac{2.2\text{g}}{(2.2 + M)\,\text{g}}}$

Solving we get,

$$M = 0.873 \text{ kg}$$

4. (a) $\dfrac{f_1}{f_2} = \dfrac{250}{300} = \dfrac{5}{6}$

So, f_1 is 5th harmonic and f_2 is 6th harmonic.

(b) $f_1 = (5) = \dfrac{v}{2l} = \dfrac{2.5\sqrt{T/\mu}}{l}$

$$\therefore \qquad T = \frac{\mu f_1^2 l^2}{(2.5)^2} = \frac{(0.036)(250)^2(1)^2}{(2.5)^2}$$

$= 360$ N

5. $6\left[\dfrac{T/\mu}{2l_1}\right] = 2\left[\dfrac{T/\mu}{2l_2}\right]$

$$\therefore \qquad \frac{l_1}{l_2} = 3$$

INTRODUCTORY EXERCISE 18.4

1. At mean position, energy is in the form of kinetic energy.

2. (a) Assuming tension to be same

$$v \propto \frac{1}{\sqrt{\mu}}$$
$$\mu_{\text{RHS}} = \frac{\mu_{\text{RHS}}}{4}$$
$$\therefore \qquad v_{\text{RHS}} = 2v_{\text{LHS}} = 20 \text{ cm/s}$$

(b) $A_r = \left(\dfrac{v_2 - v_1}{v_2 + v_1}\right) A_i$

$$\therefore \qquad \frac{A_r}{A_i} = \frac{20-10}{20+10} = \frac{1}{3}$$

$$\frac{A_t}{A_i} = \left(\frac{2v_2}{v_1 + v_2}\right) = \frac{2 \times 20}{20+10} = \frac{4}{5}$$

3. Speed of wave in first medium is,

$$v_1 = \frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

$$= \frac{50}{2} = 25 \text{ m/s}$$

$$v_2 = 50 \text{ m/s}$$

$$A_i = 2 \times 10^{-3} \text{m}$$

$$A_r = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_i = \frac{2}{3} \times 10^{-3} \text{m}$$

$$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i = \frac{8}{3} \times 10^{-3} \text{m}$$

In second medium, speed becomes two times. Therefore, λ also becomes two times. So, k remains one-half, value of ω will remain unchanged. Further, second medium is rarer medium ($v_2 > v_1$). Hence, there is no change in phase angle anywhere.

$$\therefore \qquad y_r = \frac{2}{3} \times 10^{-3} \cos \pi \; (2.0x + 50t)$$

and $$y_t = \frac{8}{3} \times 10^{-3} \cos \pi \; (x - 50t)$$

Exercises

LEVEL 1

Assertion and Reason

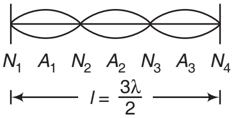
1. At $x = 0$, $y = y_1 + y_2 = A \sin \omega t + A \cos \omega t$. So, it is neither a node nor an antinode.

2. They are called stationary because net energy transfers from any section is zero, if amplitudes of constituent waves are zero.

3. $A_t = \left(\dfrac{2v_2}{v_1 + v_2}\right) A_i$

If $v_1 > v_2$ or 2 is rarer, then $A_t > A_i$

4.

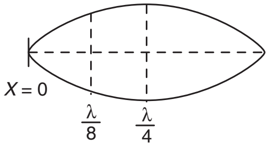


5. $f = \eta \left(\dfrac{v}{2l}\right)$

As η increases, frequency increases. Hence, λ decreases.

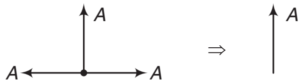
6. Amplitudes may be different also.

7.



Energy lying between $\dfrac{\lambda}{8}$ and $\dfrac{\lambda}{4}$ will be more.

9.



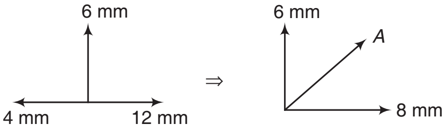
Resultant amplitude is A.

So, intensity will remain same.

10. Phase difference may be different but it should remain constant with time.

Objective Questions

2.



$$A = \sqrt{(8)^2 + (6)^2} = 10 \text{ mm}$$

3. (i) Two waves must travel in opposite directions.

(ii) At $x = 0$, $y = y_1 + y_2$ should be zero at all times.

6. $f_1 : f_2 : f_3 = 1 : 2 : 3$

$$\lambda = \frac{v}{f} \quad \text{or} \quad \lambda \propto \frac{1}{f}$$

$$\therefore \qquad \lambda_1 : \lambda_2 : \lambda_3 = 1 : \frac{1}{2} : \frac{1}{3}$$

8. All frequencies are integral multiples of 35 Hz.

9. $3\left(\dfrac{v}{2l}\right) = 300$

$$\therefore \qquad v = 200\, l = (200)\,(1)$$

$$= 200 \text{ m/s}$$

10. These are multiples of 30 Hz. Hence, fundamental frequency

$$f_0 = 30 \text{ Hz}$$

Now, $$f_0 = \frac{v}{2l}$$

$\therefore$ $$v = 2 f_0 l = 2 \times 30 \times 0.8 = 48 \text{ m/s}$$

11. $\because$ $$\Delta\phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x)$$

$$= \left(\frac{2\pi}{v/f}\right)(\Delta x) = \left(\frac{2\pi}{vT}\right)(\Delta x)$$

$$= \left(\frac{2\pi}{300 \times 0.04}\right)(16 - 10)$$

$$= \pi$$

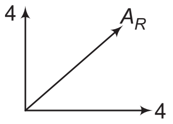
12. $$v = \frac{\omega}{k} = \sqrt{\frac{T}{\rho S}}$$

$$\therefore \quad T = \rho S \left(\frac{\omega}{k}\right)^2 = (8000 \times 10^{-6})\left(\frac{30}{1}\right)^2$$

$$= 7.2 \text{ N}$$

Subjective Questions

1. $$A_R = 4\sqrt{2} \text{ cm} = 5.66 \text{ cm}$$



2. $$v_2 = \frac{v_1}{2}$$

(a) $$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A$$

$$= \left[\frac{(v_1/2) - v_1}{v_1 + v_1/2}\right] A$$

$$= -\frac{A}{3}$$

$$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A$$

$$= \left[\frac{2 \times v_1/2}{v_1 + v_1/2}\right] A$$

$$= \frac{2}{3} A$$

(b) Since, reflected waves come in the same medium, we can say that

$$P \propto A^2$$

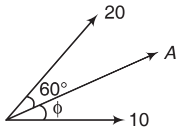
$$\frac{P_r}{P_i} = \left(\frac{A_r}{A_i}\right)^2 = \left(\frac{-A/3}{A}\right)^2 = \frac{1}{9}$$

$\therefore$ Fraction of power transmitted

$$= 1 - \frac{1}{9} = \frac{8}{9}$$

3. $$A = \sqrt{(10)^2 + (20)^2 + 2(10)(20)\cos 60^\circ}$$

$$= 26.46 \text{ cm}$$



$$\tan\phi = \frac{20 \sin 60^\circ}{10 + 20 \cos 60^\circ} = 0.866$$

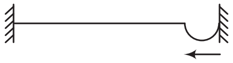
$\therefore$ $$\phi = 40.89^\circ = 0.714 \text{ rad}$$

4. Find y_1 and y_2 at given values of x and t and then simply add them to get net value of y.

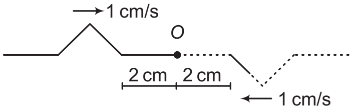
5. $$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{16}{(0.4 \times 10^{-3})/(10^{-2})}} = 20 \text{ m/s}$$

(a) $$t = \frac{d}{v} = \frac{2l}{v} = \frac{0.4 \text{ m}}{20} = 0.02 \text{ s}$$

(b) At half the time pulse is at other end and it gets a phase difference of π. Hence, the shape is as shown below.

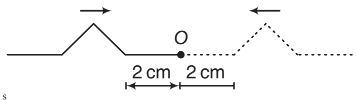


6. (a) **For fixed end.**



By the superposition of these two pulses we will get the resultant. But only to the left of point O, where string is actually present.

(b) **For free end**



7. $y = y_1 + y_2 = (6.0\text{ cm})\sin(\pi x)\cos(0.6\pi t)$

$$\therefore \quad A_x = 6.0\text{ cm}\sin(\pi x) \quad \ldots\text{(i)}$$

In parts (*a*), (*b*) and (*c*) substitute the given values of *x* and find the displacement amplitude at these locations.

(d) At antinodes

$$A_x = \text{maximum} = \pm\, 6.0\text{ cm}$$

$$\therefore \quad \pi x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

or $\quad x = 0.5\text{ cm}, 1.5\text{ cm}, 2.5\text{ cm}$

8. (a) Distance between successive antinodes

$$= \frac{\lambda}{2} = \frac{\pi}{k} = \frac{\pi}{(\pi/2)} = 2\,\text{cm}$$

(b) $A_{\max} = 2A = (2\pi)\text{ cm}$

If $x = 0$ is taken as node, we shall taken sin equation for A_x

$$\therefore \quad A_x = A_{\max} = \sin kx$$
$$= (2\pi\text{ cm})\sin kx$$

Put $k = \dfrac{\pi}{2}$ and $x = 0.5$

9.
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{20}{9\times 10^{-3}}} = 47.14\text{ m/s}$$

$$f_1 = \frac{v}{2l} = \frac{47.14}{2\times 30}$$
$$= 0.786\text{ Hz}$$

Next three frequencies are $2f_1$, $3f_1$ and $4f_1$

10. (a)
$$f = \frac{v}{2l} = \frac{\sqrt{T/\mu}}{2l}$$

$$\therefore \quad T = 4l^2 f^2 \mu$$
$$= (4)(0.7)^2(220)^2\left(\frac{1.2\times 10^{-3}}{0.7}\right)$$
$$= 163\text{ N}$$

(b) $f_3 = 3f_1 = 3\times 220 = 660\text{ Hz}$

11. Fundamental frequency,

$$f = \frac{v}{2l} = \frac{\sqrt{T/\mu}}{2l}$$
$$= \frac{\sqrt{(50)/(0.1\times 10^{-3}/10^{-2})}}{2\times 0.6}$$
$$= 58.93\text{ Hz}$$

Let, *n* th harmonic is the highest frequency, then

$$(58.93)\,n = 20000$$
$$\therefore \quad n = 339.38$$

Hence, 339 is the highest frequency.

$$\therefore \quad f_{\max} = (339)(58.93)\text{ Hz} = 19977\text{ Hz}$$

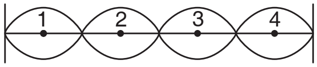
12. Fundamental frequency $= (490 - 420)$ Hz

$$f_0 = 70\text{ Hz}$$

But, $\quad f_0 = \dfrac{v}{2l} = \dfrac{\sqrt{T/\mu}}{2l}$

$$\therefore \quad l = \frac{\sqrt{T/\mu}}{2f_0} = \frac{\sqrt{(450)/0.005}}{2\times 70}$$
$$= 2.142\text{ m}$$

13. $\lambda = \dfrac{v}{f} = 0.5\text{ m}$



$$l = 4\left(\frac{\lambda}{2}\right) = 1\text{ m}$$

14. (a) $f_4 = 4f_0$

$$\therefore \quad f_0 = \frac{f_4}{4} = \frac{400}{4} = 100\text{ Hz}$$

(b) $\quad f_7 = 7f_0 = 700\text{ Hz}$

15. Fundamental frequency $\propto \dfrac{1}{l}$

$$f_1 : f_2 : f_3 = \frac{1}{l_1} : \frac{1}{l_2} : \frac{1}{l_3}$$
$$= 1 : 2 : 3$$

$$\therefore \quad l_1 : l_2 : l_3 = \frac{1}{1} : \frac{1}{2} : \frac{1}{3} = 6 : 3 : 2$$

$$l_1 = \left(\frac{6}{11}\right)(1)\text{ m} = \frac{6}{11}\text{ m}$$

$$l_2 = \left(\frac{3}{11}\right)(1)\text{ m} = \frac{3}{11}\text{ m}$$

$$l_3 = \left(\frac{2}{11}\right)(1)\text{ m} = \frac{2}{11}\text{ m}$$

16. $f \propto \dfrac{1}{l}$

$$\therefore \quad \frac{f_1}{f_2} = \frac{l_2}{l_1} \quad \text{or} \quad l_2 = \left(\frac{f_1}{f_2}\right) l_1$$
$$= \left(\frac{124}{186}\right)(90\text{ cm}) = 60\text{ cm}$$

17. (a) $\dfrac{\lambda}{2} = 15\text{ cm}$

$$\therefore \quad \lambda = 30\text{ cm}$$

$$k = \frac{2\pi}{\lambda} = \left(\frac{\pi}{15}\right)\text{cm}^{-1}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.075}\text{ s}^{-1}$$

Since, $x = 0$ is a node we will write sin equation.

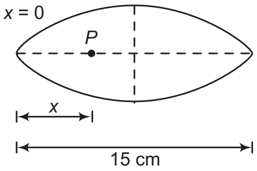
$$\therefore \qquad A_x = A_{\max} \sin kx \qquad \ldots\text{(i)}$$

and $\qquad y = A_x \sin \omega t$

(b) $v = \dfrac{\omega}{k} = \dfrac{(2\pi/0.075)}{(\pi/15)}$

$= 400 \text{ cm/s} = 4 \text{ m/s}$

(c) $x = 0$



$$x = 7.5 - 3 = 4.5 \text{ cm}$$

From Eq. (i),

$$A_x = (0.85 \text{ cm}) \sin\left(\frac{\pi}{15} \times 4.5\right)$$

$$= 0.688 \text{ cm}$$

18. (a) $f_1 = \dfrac{v}{2l} = \dfrac{48}{3}$

$= 16 \text{ Hz}$

$$\lambda_1 = \frac{v}{f_1} = 3 \text{m}$$

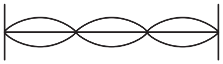
(b) Second overtone means f_3.

$$f_3 = 3 f_1 \text{ and hence } \lambda_3 = \frac{\lambda_1}{3}$$

(c) Forth harmonic means f_4.

$$f_4 = 4 f_1 \text{ and } \lambda_4 = \frac{\lambda_1}{4}$$

19. (a)



Third harmonic

(b) $A = \dfrac{5.6}{2} = 2.8 \text{ cm}$

(c) $k = 0.034 \text{ cm}^{-1}$

$$\lambda = \frac{2\pi}{k} = 184.7 \text{ cm}$$

$$l = \frac{3\lambda}{2} = 277 \text{ cm}$$

(d) $\qquad \lambda = 184.7 \text{ cm}$

$$f = \frac{\omega}{2\pi} = \frac{50}{2\pi} = 7.96 \text{ Hz}$$

$$T = \frac{1}{f} = 0.126 \text{ s}$$

$$v = f\lambda = 1470 \text{ cm/s}$$

(e) $\qquad v_{\max} = \omega A_{\max}$

$= (50)(5.60)$

$= 280 \text{ cm/s}$

(f) Frequency and hence ω will become $\dfrac{8}{3}$ times.

$$\therefore \qquad \omega' = 50 \times \frac{8}{3} = 133 \text{ rad/s}$$

$$k = \frac{\omega}{v} \quad \text{or} \quad k \propto \omega$$

Hence, k will become $\dfrac{8}{3}$ times.

$$k' = \left(\frac{8}{3}\right)(0.034) = 0.0907 \text{ rad/cm}$$

20. (a) $\dfrac{\lambda}{2} = l$ or $\lambda = 2l = 1.6 \text{ m}$

$v = f\lambda = 60 \times 1.6 = 96 \text{ m/s}$

(b) $v = \sqrt{\dfrac{T}{\mu}}$

$$\therefore \qquad T = \mu v^2 = \left(\frac{0.04}{0.8}\right)(96)^2$$

$= 461 \text{ N}$

(c) $\qquad v_{\max} = \omega A_{\max}$

$= (2\pi f) A_{\max}$

$= (2\pi)(60)(0.003)$

$= 1.13 \text{ m/s}$

$a_{\max} = \omega^2 A_{\max}$

$= (2\pi f)^2 A_{\max}$

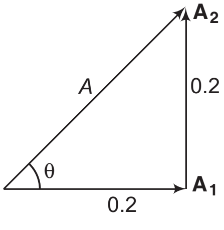
$= (2\pi \times 60)^2 (0.003)$

$= 426.4 \text{ m/s}^2$

21. (a) $y = y_1 + y_2$

$$= 0.2 \sin(x - 3.0t) + 0.2 \sin\left(x - 3.0t + \frac{\pi}{2}\right)$$

$$= A \sin(x - 3.0t + \theta)$$



Here, $A = \sqrt{(0.2)^2 + (0.2)^2}$

and $\theta = \dfrac{\pi}{4}$

$\therefore$ $y = 0.28 \sin\left(x - 3.0t + \dfrac{\pi}{4}\right)$ **Ans.**

(b) Since, the amplitude of the resulting wave is 0.32 m and $A = 0.2$ m, we have

$$0.32 = \sqrt{(0.2)^2 + (0.2)^2 + (2)(0.2)(0.2)\cos\phi}$$

Solving this, we get

$\phi = \pm 1.29$ rad **Ans.**

22. (i) $f_a = \dfrac{v}{2l} = \dfrac{\sqrt{T/\mu}}{2l}$

$$= \frac{\sqrt{(100/4 \times 10^{-2})}}{2 \times 2}$$

$= 12.5$ Hz

$$f_b = 3\left(\frac{v}{2l}\right) = 3fa = 37.5 \text{ Hz}$$

(ii) $\omega = 2\pi f$

$\therefore$ $\omega_a = 2\pi f_a = 25\pi$

$\omega_b = 2\pi f_b = 75\pi$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{100}{4 \times 10^{-2}}}$$

$= 50$ m/s

$$k = \frac{\omega}{v}$$

$\therefore$ $k_a = \dfrac{\omega a}{v} = \dfrac{25\pi}{50} = \dfrac{\pi}{2}$

$k_b = \dfrac{\omega_b}{v} = \dfrac{75\pi}{50} = \dfrac{3\pi}{2}$

23. (a) $l = \lambda_1/4$

$\therefore$ $\lambda_1 = 4l = 16$ m , $l = 3\lambda_2/4$

$\therefore$ $\lambda_2 = \dfrac{4l}{3} = 5.33$ m

$l = 5\lambda_3/4$

$\therefore$ $\lambda_3 = \dfrac{4l}{5} = 3.2$ m

(b) $v = \sqrt{\dfrac{T}{\mu}}$

$$= \sqrt{\frac{400}{(0.16/4)}} = 100 \text{ m/s}$$

Now, $f_1 = \dfrac{v}{\lambda_1} = \dfrac{100}{16} = 6.25$ Hz

Similarly, f_2 and f_3.

24. (a) $\dfrac{n\lambda_1}{2} = l$

$\therefore$ $\dfrac{n\,(0.54)}{2} = l$

or $n = \dfrac{l}{0.27}$ ...(i)

Similarly, $\dfrac{(n+1)\lambda_2}{2} = l$

or $\dfrac{(n+1)\,(0.48)}{2} = l$

or $(n+1) = \dfrac{l}{0.24}$ ...(ii)

From Eqs. (i) and (ii) we have,

$$\frac{n}{n+1} = \frac{8}{9}$$

(b) From Eq. (i),

$l = (0.27)\,n$

$= 0.27 \times 8$

$= 2.16$ m

(c) $\dfrac{\lambda}{2} = l$ (Fundamental)

$\therefore$ $\lambda = 2l = 4.32$ m

25. From fixed end there will be a phase change of π. Further, wave will start travelling in opposite direction. Hence, ωt and kx both will now become position. From free end there is no change in phase.

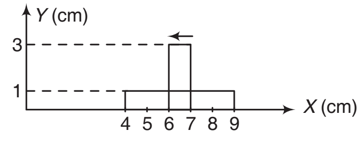
26. $t_1 = \dfrac{d_1}{v} = \dfrac{4+4}{1} = 8$ s

After one reflection from fixed boundary wave is inverted.

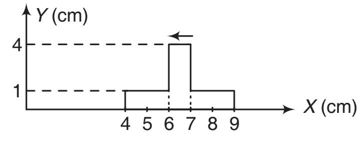
$$t_2 = \frac{d_2}{v} = \frac{4 + 10 + 16}{1} = 20 \text{ s}$$

After two times reflection from fixed boundaries wave pulse will again become upright.

27. Let us plot at $t = 3$ s



Net $y = y_1 + y_2$



Similarly, we can draw at other times.

LEVEL 2

Single Correct option

1. $\because \quad f \propto \sqrt{T}$

$$\therefore \qquad \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$$

$$\therefore \qquad \frac{3}{2} = \sqrt{\frac{T + 2.5}{T}}$$

Solving, we get $T = 2\,\text{N}$

2. $$f = \frac{v}{2l} = \frac{\sqrt{T/\mu}}{2l} = \frac{\sqrt{T/\rho S}}{l}$$

$$= \frac{\sqrt{T/\rho\,(\pi d^2/4)}}{l}$$

or $$f \propto \frac{\sqrt{T}}{dl}$$

$$\therefore \qquad \frac{f_1}{f_2} = \left(\sqrt{\frac{T_1}{T_2}}\right)\left(\frac{d_2}{d_1}\right)\left(\frac{l_2}{l_1}\right)$$

$$= \sqrt{\frac{1}{2}}\left(\frac{3}{1}\right)\left(\frac{2}{1}\right) = 3\sqrt{2}$$

3. $f \propto \dfrac{1}{l} \quad \Rightarrow \quad l = \dfrac{k}{f}$

Now, $$l = l_1 + l_2 + l_3$$

$$\therefore \qquad \frac{k}{f_0} = \frac{k}{f_1} + \frac{k}{f_2} + \frac{k}{f_3}$$

$$\therefore \qquad \frac{1}{f_0} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

4. $y_1 + y_2 = 2A \sin(\omega t - kx) = y_4$ (say)

Now, y_4 and y_3 produce standing waves where,

$A_{\max} = 2$ (Amplitude of constituent wave)

$= 2\,(2A) = 4A$

5. Tension in the string will be given by

$$T = \frac{YA\Delta l}{l} = \frac{YA}{\eta} \qquad \left(\text{as } \frac{\Delta l}{l} = \frac{1}{\eta}\right)$$

Now, $$f \propto v$$

$$\frac{f_1}{f_2} = \frac{v_1}{v_2} = \frac{\sqrt{T/\rho A}}{\sqrt{Y/\rho}}$$

$$= \sqrt{\frac{T}{YA}} = \frac{1}{\sqrt{\eta}}$$

6. $$f_1 = \frac{v}{2l} = \frac{\sqrt{T/\mu}}{2l}$$

$$= \frac{\sqrt{100/0.01}}{2} = 50\text{ Hz}$$

$$f_2 = 2f_1 = 100\text{ Hz}$$

$$f_3 = 3f_1 = 150\text{ Hz}$$

$$n_1 = \frac{v}{4l} = 25\text{ Hz}$$

$$n_2 = 3n_1 = 75\text{ Hz}$$

$$n_3 = 5n_1 = 125\text{ Hz}$$

7. $\because \qquad f \propto \dfrac{1}{l}$

$$\therefore \qquad l_1 : l_2 : l_3 = \frac{1}{f_1} : \frac{1}{f_2} : \frac{1}{f_3}$$

$$= \frac{1}{1} : \frac{1}{3} : \frac{1}{4} = 12 : 4 : 3$$

$$\therefore \qquad l_1 = \left(\frac{12}{12 + 4 + 3}\right)(114)$$

$$= 72\text{ cm}$$

$$l_2 = \left(\frac{4}{12 + 4 + 3}\right)(114)$$

$$= 24\text{ cm}$$

$$l_3 = \left(\frac{3}{12 + 4 + 3}\right)(114)$$

$$= 18\text{ cm}$$

8. $f \propto \sqrt{T}$

$$\frac{f}{f/2} = \sqrt{\frac{T_1}{T_2}} = \sqrt{\frac{W}{W - V_1}}$$

$$2 = \sqrt{\frac{v\rho g}{V\rho g - V\rho_w g}} = \sqrt{\frac{\rho}{\rho - \rho_w}}$$

Solving $\rho = \dfrac{4}{3}\rho_w$

$$\frac{f}{f/3} = \sqrt{\frac{W}{W - V_2}} = \sqrt{\frac{V\rho g}{V\rho g - V\rho_l g}}$$

$$9 = \frac{\rho}{\rho - \rho_l}$$

$$\therefore \qquad \rho_l = \frac{8}{9}\rho = \frac{32}{27}\rho_\omega$$

$\therefore$ Relative density of liquid $= \dfrac{\rho_l}{\rho_w}$

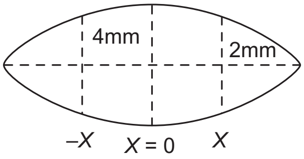
$$= \frac{32}{27} = 1.18$$

9. $\dfrac{\lambda}{2} = 1.5\text{ m}$

$$\therefore \qquad \lambda = 3.0\text{ m}$$

$$= k = \frac{2\pi}{\lambda} = \left(\frac{2\pi}{3}\right)\text{m}^{-1}$$

Let us take antinode at $x = 0$, then



$$A_x = A_{\max} \cos kx$$

$$\therefore \quad (2 \text{ mm}) = (4 \text{ mm}) \cos \left(\frac{2\pi}{3}\right) x$$

$$\therefore \quad \left(\frac{2\pi}{3}\right) x = \frac{\pi}{3}$$

or $\quad x = 0.5$ m

The asked distance is $2x$ or 1.0 m.

10. $f = 3\text{Hz}$

$$\therefore \quad \omega = 2\pi f = (6\pi) \text{ rad/s}$$

$$k = \frac{\omega}{v} = \frac{6\pi}{3} = (2\pi) \text{ rad/m}$$

$$y = A \sin (kx - \omega t)$$

$$= A \sin (2\pi x - 6\pi t)$$

Putting $\quad y = \pm A$

and $x = 3$, we get

$$(2\pi)(3) - 6\pi\, t = \frac{\pi}{2}$$

$$\therefore \quad 6\pi t = \frac{11}{2} \pi$$

$$\therefore \quad t = \frac{11}{12} \text{ s}$$

11. $\Delta\phi_1 = \pi/2$

$$\Delta\phi_2 = \left(\frac{2\pi}{\lambda}\right)(\Delta x)$$

$$= \left(\frac{2\pi}{\lambda}\right)(1.5\lambda)$$

$$= 3\pi$$

$$\therefore \quad \Delta\phi_{\text{net}} = \Delta\phi_2 - \Delta\phi_1 = \frac{5\pi}{2}$$

12. Fundamental frequency,

$$f_0 = 450 - 400$$

$$= 50 \text{ Hz}$$

$$f_0 = \frac{v}{2l} = \frac{\sqrt{T/\mu}}{2l}$$

$$\therefore \quad l = \frac{\sqrt{T/\mu}}{2 f_0} = \frac{\sqrt{490/0.1}}{2 \times 50}$$

$$= 0.7 \text{ m}$$

13. $\dfrac{f_5}{f_2} = \dfrac{5 f_1}{2 f_1} = \dfrac{5}{2}$

$$\therefore \quad f_2 = \frac{2}{5} f_5$$

$$= \frac{2}{5} \times 480$$

$$= 192 \text{ Hz}$$

14. $I \propto A^2$

$$I_r = 0.64\, I_i$$

$$\therefore \quad A_r = 0.8\, A_i$$

$$= 0.8\text{A}$$

Reflected from a denser medium. Hence, a phase change of π will occur. Reflected wave will travel in opposite direction. Hence, ωt and kx will have positive signs.

15. $\eta \left(\dfrac{v}{2l}\right)_B = \left(\dfrac{v}{2l}\right)_A$

$$\eta = \left(\frac{\sqrt{T/\rho S}}{2l}\right)_B = \left(\frac{v}{2l}\right)_A$$

or $$\eta \left(\frac{\sqrt{4T/\rho\pi}}{2ld}\right)_B = \left(\frac{\sqrt{4T/\rho\pi}}{2ld}\right)_A \quad \text{...(i)}$$

Given, $\quad T_B = 2T_A,\ l_B = 2l_A$

$$d_B = 2 d_A$$

and $\quad \rho_B = 2\rho_A$

Putting in Eq. (i), we get

$$\eta = 4$$

$\eta = 4$ means 4th harmonic or 3rd overtone.

More than One Correct Options

1. $f \propto \sqrt{T}$

$$\frac{f_2}{f_1} = \sqrt{\frac{T_2}{T_1}}$$

$$\therefore \quad \frac{f_1 + 15}{f_1} = \sqrt{\frac{1.21 T_1}{T_1}}$$

$$= 1.1$$

Solving we get $f_1 = 150$ Hz

$$v \propto \sqrt{T}$$

$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$$

or $$v_2 = \left(\sqrt{\frac{1.21\, T_1}{T_1}}\right) v_1$$

$$= 1.1\, v_1$$

Hence, increase in v is 10%

$$\frac{\lambda}{2} = l$$

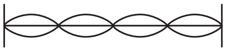
$\therefore \quad \lambda = 2l$

$\therefore$ Fundamental wavelength $= 2\lambda$ is unchanged.

4. $A_t = \left(\dfrac{2v_2}{v_1 + v_2}\right) A_i$

If $v_2 > v_1$, $A_t > A_i$

5.



$$l = 4\frac{\lambda}{2} = 2\lambda = 2\left(\frac{2\pi}{k}\right)$$

$$= \frac{4\pi}{k}$$

6. Two identical waves should travel in opposite directions.

7. $y = y_1 + y_2 = (2A\cos kx)\sin\omega t$

$= A_x \sin\omega t$

Here, $A_x = 2A\cos kx$

At $x = 0$, A_x is maximum or $2A$.

So, it is an antinode. Next antinode will occur at

$$x = \frac{\lambda}{2}, \lambda \ \dots \text{ etc.}$$

or $\quad x = \dfrac{\pi}{k}, \dfrac{2\pi}{k} \dots$ etc.

Comprehension Based Questions

1. Reflected and incident rays are in the same medium. Hence,

$$I \propto A^2$$

I_r has become 64% or 0.64 times of I_i

$\therefore \quad A_r = 0.8\,A_i = 0.8\,A$

2. $y = y_i + y_r$

$= A\sin(ax + bt + \pi/2) + 0.8\,A\sin(ax - bt + \pi/2)$

$$= \left[0.8\,A\sin\left(ax + bt + \frac{\pi}{2}\right) + 0.8\,A\sin\left(ax - bt + \frac{\pi}{2}\right)\right] + 0.2A\sin\left(ax + bt + \frac{\pi}{2}\right)$$

$= -1.6\sin ax\sin bt + 0.2A\cos(bt + ax)$

$\therefore \quad c = 0.2$

3. $A_x = -1.6\sin ax$

$\therefore \quad x = 0$ is a node

Second antinode is at a distance.

$$x = \frac{\lambda}{4} + \frac{\lambda}{2} = \frac{3\lambda}{4} = \frac{3}{4}\left(\frac{2\pi}{k}\right)$$

But, $k = a$

$$\therefore \quad x = \frac{3\pi}{2a}$$

Match the Columns

1. $v \propto \dfrac{1}{\sqrt{\mu}}$

$$\mu_2 = 9\mu_1$$

Hence, $\quad v_2 = \dfrac{v_1}{3}$

So let $v_1 = 3$ units, then $v_2 = 1$ unit

(a) $\quad A_1 = \left(\dfrac{v_2 - v_1}{v_1 + v_2}\right) A_i$

and $\quad A_2 = \left(\dfrac{2v_2}{v_1 + v_2}\right) A_i$

$$\therefore \quad \left|\frac{A_1}{A_2}\right| = \left|\frac{v_2 - v_1}{2v_2}\right| = \left|\frac{3-1}{2}\right| = 1$$

(b) $\dfrac{v_1}{v_2} = 3$

(c) $I = \dfrac{1}{2}\rho\omega^2 A^2 v$

ρ is not given, so we cannot find I_1/I_2.

(d) $P = \dfrac{1}{2}\rho\omega^2 A^2 S v$

But, $\quad S\rho = \mu$

$\therefore \quad P = \dfrac{1}{2}\omega^2 A^2 \mu v$

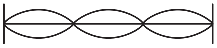
or $\quad P \propto A^2\mu v \qquad$ (as $\omega \to$ same)

$$\frac{P_1}{P_2} = \left(\frac{A_1}{A_2}\right)^2 \left(\frac{\mu_1}{\mu_2}\right)\left(\frac{v_1}{v_2}\right)$$

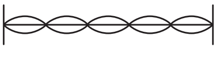
$$= (1)^2\left(\frac{1}{9}\right)(3) = \frac{1}{3}$$

or $\quad \dfrac{P_2}{P_1} = 3$

2.



(i)



(ii)

(a) Second overtone mode = $f_3 = 3f_1$

Fifth harmonic mode = $f_5 = 5f_1$

So, the ratio is 3/5,

(d) $\lambda = \dfrac{v}{f}$ or $\lambda \propto \dfrac{1}{f}$

$$\therefore \quad \frac{\lambda_3}{\lambda_5} = \frac{f_5}{f_3} = \frac{5}{3}$$

3. (d) $A_t = \left(\dfrac{2v_2}{v_1 + v_2}\right)A_i$

If $v_2 > v_1$, i.e. 2 is rarer then $A_t > A_i$.

4. $A = 2A_0 \cos\dfrac{\phi}{2}$ and $I = 4I_0 \cos^2\dfrac{\phi}{2}$

5. Second overtone frequency means

$$f_3 = 3f_1 = 210 \text{ Hz}$$

$$\therefore \quad f_1 = 70 \text{ Hz} = \text{ fundamental frequency}$$

Third overtone frequency = $f_4 = 4f_1$

Second harmonic frequency = $f_2 = 2f_1$

Subjective Questions

1. (a) $v_1 = \sqrt{T/\mu_1}$

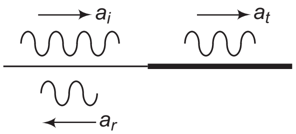
$$v_2 = \sqrt{F/4\mu_1} = \frac{1}{2}\sqrt{F/\mu_1},$$

$$v_3 = \sqrt{F/(\mu_1/4)} = 2\sqrt{F/\mu_1}$$

$$\therefore \quad t = t_1 + t_2 + t_3 = \frac{L}{v_1} + \frac{L}{v_2} + \frac{L}{v_3}$$

$$= \frac{7L}{2}\sqrt{\frac{\mu_1}{F}}$$ **Ans**

2. Let a_i and a_r be the amplitudes of incident and reflected waves.



Then,

$$\frac{a_i + a_r}{a_i - a_r} = 6 \quad \text{(given)}$$

Hence, $\dfrac{a_r}{a_i} = \dfrac{5}{7}$

Now, $\dfrac{E_r}{E_i} = \left(\dfrac{a_r}{a_i}\right)^2 = \left(\dfrac{5}{7}\right)^2$

$$= 0.51$$

or percentage of energy reflected is

$$100 \times \frac{E_r}{E_i} = 51\%.$$

So, percentage of energy transmitted will be

$$(100 - 51)\% \text{ or } 49\%.$$ **Ans.**

3. Amplitude at a distance x is $A = a \sin kx$

First node can be obtained at $x = 0$,

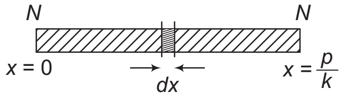
and the second at $x = \pi/k$

At position x, mass of the element PQ is

$$dm = (\rho S)dx$$

Its amplitude is $A = a \sin kx$

Hence mechanical energy stored in this element is



(energy of particle in SHM) $dE = \dfrac{1}{2}(dm)A^2\omega^2$

or $$dE = \frac{1}{2}(\rho SA^2\omega^2)\,dx$$

$$= \frac{1}{2}(\rho Sa^2\omega^2 \sin^2 kx)\,dx$$

Therefore, total energy stored between two adjacent nodes will be

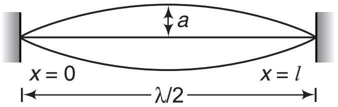
$$E = \int_{x=0}^{x=\pi/k} dE$$

Solving this, we get

$$E = \frac{\pi S\rho\omega^2 a^2}{4k}$$ **Ans.**

4. $l = \dfrac{\lambda}{2}$

or $$\lambda = 2l, k = \frac{2\pi}{\lambda} = \frac{\pi}{l}$$



The amplitude at a distance x from $x = 0$ is given by

$$A = a \sin kx$$

Total mechanical energy at x of length dx is

$$dE = \frac{1}{2}(dm)A^2\omega^2$$

$$= \frac{1}{2}(\mu dx)(a \sin kx)^2(2\pi f)^2$$

or $$dE = 2\pi^2 \mu f^2 a^2 \sin^2 kx\, dx \quad \text{...(i)}$$

Here, $f = \dfrac{v^2}{\lambda^2} = \dfrac{\left(\dfrac{T}{\mu}\right)}{(4l^2)}$ and $k = \dfrac{\pi}{l}$

Substituting these values in Eq. (i) and integrating it from $x = 0$ to $x = l$, we get total energy of string.

$$E = \frac{\pi^2 a^2 T}{4l} \qquad \textbf{Ans.}$$

5. Tension, $T = 80$ N

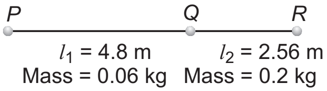
Amplitude of incident wave, $A_i = 3.5$ cm

Mass per unit length of wire PQ is

$$m_1 = \frac{0.06}{4.8} = \frac{1}{80} \text{ kg/m}$$

and mass per unit length of wire QR is

$$m_2 = \frac{0.2}{2.56} = \frac{1}{12.8} \text{ kg/m}$$



(a) Speed of wave in wire PQ is

$$v_1 = \sqrt{T/m_1} = \sqrt{\frac{80}{1/80}} = 80 \text{ m/s}$$

and speed of wave in wire QR is

$$v_2 = \sqrt{T/m_2}$$

$$= \sqrt{\frac{80}{1/12.8}} = 32 \text{ m/s}$$

∴ Time taken by the wave pulse to reach from P to R is

$$t = \frac{4.8}{v_1} + \frac{2.56}{v_2} = \left(\frac{4.8}{80} + \frac{2.56}{32}\right) \text{s}$$

$$= 0.14 \text{ s} \qquad \textbf{Ans.}$$

(b) The expressions for reflected and transmitted amplitudes (A_r and A_t) in terms of v_1, v_2 and A_i are as follows :

$$A_r = \frac{v_2 - v_1}{v_2 + v_1} A_i$$

and $$A_t = \frac{2v_2}{v_1 + v_2} A_i$$

Substituting the values, we get

$$A_r = \left(\frac{32 - 80}{32 + 80}\right)(3.5) = -1.5 \text{ cm}$$

i.e. the amplitude of reflected wave will be 1.5 cm. Negative sign of A_r indicates that there will be a phase change of π in reflected wave.

Similarly, $$A_t = \left(\frac{2 \times 32}{32 + 80}\right)(3.5) = 2.0 \text{ cm}$$

i.e. the amplitude of transmitted wave will be 2.0 cm.

The expressions of A_r and A_t are derived as below.

Derivation

Suppose the incident wave of amplitude A_i and angular frequency ω is travelling in positive x-direction with velocity v_1, then we can write

$$y_i = A_i \sin \omega[t - x/v_1] \qquad \ldots(\text{i})$$

In reflected as well as transmitted wave, ω will not change, therefore, we can write

$$y_r = A_r \sin \omega\,[t + x/v_1] \qquad \ldots(\text{ii})$$

and $$y_t = A_t \sin \omega\,[t - x/v_2] \qquad \ldots(\text{iii})$$

Now as wave is continuous, so at the boundary $(x = 0)$.

Continuity of displacement requires

$$y_i + y_r = y_i \quad \text{for} \quad x = 0$$

Substituting from Eqs. (i), (ii) and (iii) in the above, we get

$$A_i + A_r = A_t \qquad \ldots(\text{iv})$$

Also at the boundary, slope of wave will be continuous, i.e.

$$\frac{\partial y_i}{\delta x} + \frac{\partial y_r}{\delta x} = \frac{\partial y_t}{\partial x} \qquad [\text{for } x = 0]$$

which gives $$A_i - A_r = \left(\frac{v_1}{v_2}\right) A_t \qquad \ldots(\text{v})$$

Solving Eqs. (iv) and (v) for A_r and A_t, we get the required equations, i.e.

$$A_r = \frac{v_2 - v_1}{v_2 + v_1} A_i \quad \text{and} \quad A_t = \frac{2v_2}{v_2 + v_1} A_i$$

6. **When A is a node** Suppose n_1 and n_2 are the complete loops formed on left and right side of point A. Then,

$$f_1 = f_2$$

or $$n_1 \left(\frac{v_1}{2L}\right) = n_2 \left(\frac{v_2}{2L}\right)$$

or $$\frac{n_1}{n_2} = \left(\frac{v_2}{v_1}\right) = \sqrt{\frac{\mu_1}{\mu_2}} = \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots, \text{etc}$$

$$\left(\text{as } v \propto \frac{1}{\sqrt{\mu}}\right)$$

∴ Possible frequencies are

$$\frac{v_1}{2L}, 2\left(\frac{v_1}{2L}\right), \frac{3v_1}{2L}, \ldots, \text{etc.} \qquad \left(v_1 = \sqrt{\frac{T}{\mu}}\right)$$

or $\frac{1}{2L}\sqrt{\frac{T}{\mu}}, \frac{1}{L}\sqrt{\frac{T}{\mu}}, \frac{3}{2L}\sqrt{\frac{T}{2\mu}}, \ldots$, etc.

When A is an antinode Suppose n_1 and n_2 are complete loops on left and right side of point A,

$$n_1\frac{\lambda_1}{2}+\frac{\lambda_1}{4}=L$$

or $$f_1=\frac{v_1}{L}\left(\frac{n_1}{2}+\frac{1}{4}\right)$$

$$n_2\frac{\lambda_2}{2}+\frac{\lambda_2}{4}=L$$

or $$f_2=\frac{v_2}{L}\left(\frac{n_2}{2}+\frac{1}{4}\right)$$

Substituting $f_1=f_2$,

we get, $\frac{2n_1+1}{2n_2+1}=\frac{1}{3}$

For $n_1=1,\quad n_2=4$

$n_1=2,\quad n_2=7$

$n_1=3,\quad n_2=10,$ etc.

… … …

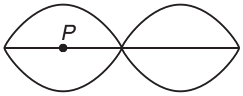
Therefore, the possible frequencies are

$\frac{v_1}{L}\left(\frac{1}{2}+\frac{1}{4}\right), \frac{v_1}{L}\left(\frac{2}{2}+\frac{1}{4}\right), \frac{v_1}{L}\left(\frac{3}{2}+\frac{1}{4}\right), \ldots$, etc.

or $\frac{3}{4L}\sqrt{\frac{T}{\mu}}, \frac{5}{4L}\sqrt{\frac{T}{\mu}}, \frac{7}{4L}\sqrt{\frac{T}{\mu}}, \ldots$, etc.

7. $$\frac{L}{4}=\frac{\lambda}{4}$$

$\therefore\quad L=\lambda$



In the next higher mode, there will be total 6 loops and the desired frequency is

$$\left(\frac{6}{2}\right)(100)=300\text{ Hz}\qquad\textbf{Ans.}$$

8. $\because\quad k=\frac{\omega}{v}=\frac{\pi}{3}$

$$y_1=0.06\,(\pi t-kx)=0.06\sin\left(\pi t-\frac{\pi}{3}\times 12\right)$$

$$=0.06\sin(\pi t-4\pi)$$

Similarly,

$$y_2=0.02\sin(\pi t-kx')=0.02\sin\left(\pi t-\frac{\pi}{3}\times 8\right)$$

$$=0.02\sin\left(\pi t-\frac{8\pi}{3}\right)$$

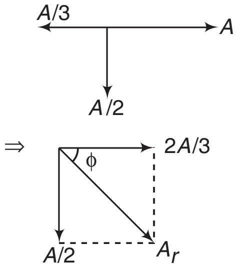
$$y=y_1+y_2$$

$$=0.06\sin\pi t\cos 4\pi-0.06\cos\pi t\sin 4\pi$$

$$+\,0.02\sin\pi t\cos\frac{8\pi}{3}-0.02\cos\pi t\sin\frac{8\pi}{3}$$

$$=0.05\sin\pi t-0.0173\cos\pi t\qquad\textbf{Ans.}$$

9. Resultant amplitude



$$A_r=\sqrt{\left(\frac{2A}{3}\right)^2+\left(\frac{A}{2}\right)^2}$$

$$=\frac{5}{6}A\qquad\textbf{Ans.}$$

$$\tan\phi=-\frac{A/2}{2A/3}=-\frac{3}{4}$$

or $$\phi=-\tan^{-1}(3/4)\qquad\textbf{Ans.}$$

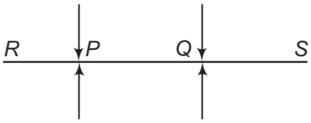
10. Speed of longitudinal waves in the rod,

$$v=\sqrt{\frac{Y}{\rho}}=\sqrt{\frac{1.6\times 10^{11}}{2500}}=8000\text{ m/s}$$

At the clamped position nodes will be formed. Between the clamps integer number of loops will be formed. Hence,

$$n_1\frac{\lambda}{2}=80$$

or $$n_1\lambda=160\qquad\ldots\text{(i)}$$



Between P and R, P is a fixed end and R is the free end. It means the number of loops between P and R will be odd multiple of $\frac{\lambda}{4}$. Then,

$$\frac{(2n_2-1)}{2}\frac{\lambda}{2}=5$$

or $$(2n_2-1)\lambda=20\qquad\ldots\text{(ii)}$$

Also between Q and S,

$$(2n_3 - 1)\lambda = 60 \qquad \text{...(iii)}$$

From Eqs. (i) and (ii), we get

$$\frac{n_1}{2n_2 - 1} = \frac{160}{20} = 8 \qquad \text{...(iv)}$$

and from Eqs. (i) and (iii).

$$\frac{n_1}{2n_3 - 1} = \frac{160}{60} = \frac{8}{3} \qquad \text{...(v)}$$

For minimum frequency n_1, n_2 and n_3 should be least from Eqs. (iv) and (v).

We get, $\quad n_1 = 8,\ n_2 = 1,\ n_3 = 2$

$$\lambda = \frac{20}{2n_2 - 1} = 20 \text{ cm} \qquad \text{[from Eq. (ii)]}$$

$$= 0.2 \text{ m}$$

$$\therefore \quad f_{\min} = \frac{v}{\lambda} = \frac{8000}{0.2}$$

$$= 40 \text{ kHz}$$ **Ans.**

Next higher frequency corresponds to

$$n_1 = 24,\ n_2 = 2$$

and $\quad n_3 = 5$

$$f = 120 \text{ kHz}$$ **Ans.**

11. (a) Distance between two nodes is $\lambda/2$ or π/k. The volume of string between two nodes is

$$V = \frac{\pi}{k}s \qquad \text{...(i)}$$

Energy density (energy per unit volume) of each wave will be

$$u_1 = \frac{1}{2}\rho\omega^2(8)^2 = 32\,\rho\omega^2$$

and $\quad u_2 = \frac{1}{2}\rho\omega^2(6)^2 = 18\,\rho\omega^2$

$\therefore$ Total mechanical energy between two consecutive nodes will be

$$E = (u_1 + u_2)V$$

$$= 50\frac{\pi}{k}\rho\omega^2 S$$

(b) $y = y_1 + y_2$

$$= 8 \sin(\omega t - kx) + 6 \sin(\omega t + kx)$$

$$= 2 \sin(\omega t - kx) + \{6 \sin(\omega t + kx) + 6 \sin(\omega t + kx)\}$$

$$= 2 \sin(\omega t - kx) + 12 \cos kx \sin \omega t$$

Thus, the resultant wave will be a sum of standing wave and a travelling wave.

Energy crossing through a node per second = power or travelling wave

$$\therefore \quad P = \frac{1}{2}\rho\omega^2(2)^2 Sv$$

$$= \frac{1}{2}\rho\omega^2(4)(S)\left(\frac{\omega}{k}\right)$$

$$= \frac{2\rho\omega^3 S}{k}$$

19. Sound Waves

INTRODUCTORY EXERCISE 19.1

1. $\Delta p_{\max} = BAk$

$$B = \frac{\Delta p_{\max}}{Ak} = \frac{\Delta p_{\max}}{A\,(2\pi/\lambda)}$$

$$= \frac{(\Delta p_{\max})\,\lambda}{2\pi A}$$

$$= \frac{(14)\,(0.35)}{(2\pi)\,(5.5\times 10^{-6})}$$

$$= 1.4\times 10^{5}\ \text{N/m}^2$$

2. $$\lambda_{\min} = \frac{v}{f_{\max}} = \frac{1450}{20000}$$

$$= 0.0725\ \text{m} = 7.25\ \text{cm}$$

$$\lambda_{\max} = \frac{v}{f_{\min}} = \frac{1450}{20} = 72.5\ \text{m}$$

3. (a) Displacement is zero when pressure is maximum.

(b) $\Delta p_{\max} = BAk$

$$= (\rho v^2)\,A\left(\frac{\omega}{v}\right) = 2\pi f A \rho v$$

$$\therefore\quad A = \frac{\Delta p_{\max}}{2\pi f \rho v}$$

$$= \frac{10}{(2\pi)\,(10^3)\,(1.29)\,(340)}$$

$$= 3.63\times 10^{-6}\ \text{m}$$

4. In the above problem, we have found that

$$A = \frac{\Delta p_{\max}}{2\pi f \rho v}$$

$$= \frac{(\Delta p_{\max})k}{\rho\omega^2}\qquad \left(\text{as } v = \frac{\omega}{k} \text{ and } 2\pi f = \omega\right)$$

Now substituting the value, we have

$$A = \frac{(12)\,(8.18)}{(1.29)\,(2700)^2} = 1.04\times 10^{-5}\ \text{m}$$

INTRODUCTORY EXERCISE 19.2

1. $v \propto \sqrt{T}$

$$\therefore\quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}\quad \text{or}\quad T_2 = \left(\frac{v_2}{v_1}\right)^2 T_1 = (2)^2\,(273)$$

$$= 1092\ \text{K}$$

$$= 819^\circ\text{C}$$

2. $\because\quad v \propto \sqrt{T}$

$$\therefore\quad v_2 = \left(\sqrt{\frac{T_2}{T_1}}\right) v_1$$

$$v_{-3^\circ\text{C}} = \left(\sqrt{\frac{270}{273}}\right)(332)$$

$$= 330.17\ \text{m/s}$$

$$v_{30^\circ\text{C}} = \left(\sqrt{\frac{273+30}{273}}\right)(332)$$

$$= 349.77\ \text{m/s}$$

The difference in these two speeds is approximately 19.6 m/s.

3. $$\because\quad v = f\lambda = \sqrt{\frac{B}{\rho}}$$

$$\therefore\quad B = \rho\ (f\lambda)^2$$

$$= (900)\,(250\times 8)^2$$

$$= 3.6\times 10^{9}\ \text{N/m}^2$$

4. $$v = \sqrt{\frac{\gamma RT}{M}}$$

$$= \sqrt{\frac{(7/5)\,(8.31)\,(273)}{(32\times 10^{-3})}}$$

$$= 315\ \text{m/s}$$

INTRODUCTORY EXERCISE 19.3

1. (a) $\Delta p_{\max} = BAk$

$$= (\rho v^2)\,(A)\left(\frac{\omega}{v}\right)$$

$$= (2\pi f A\,\rho v)$$

$$= (2\pi)\,(300)\,(6.0\times 10^{-3})(1.2)\,(344)$$

$$= 4.67\ \text{Pa}$$

(b) $$I = \frac{v\,(\Delta p)^2_{\max}}{2B}$$

$$= \frac{(\Delta p)^2_{\max}}{2\rho v}\qquad (\text{as } B = \rho v^2)$$

$$= \frac{(4.67)^2}{(2)\,(1.2)\,(344)}$$

$$= 0.0264\ \text{W/m}^2$$

$$= 2.64\times 10^{-2}\ \text{W/m}^2$$

(c) $$L = \log_{10}\left(\frac{I}{I_0}\right)$$

$$= 10\log_{10}\left(\frac{2.64 \times 10^{-2}}{10^{-12}}\right)$$

$$= 104 \text{ dB}$$

2. $L_2 - L_1 = 10\log_{10}\dfrac{I_1}{I_2}$

Given, $L_2 - L_1 = 9$ dB

Solving the equation, we get

$$\frac{I_1}{I_2} = 7.9$$

3. $I \propto \dfrac{1}{r^2}$

$$\therefore \quad \frac{I_1}{I_2} = \left(\frac{r_2}{r_1}\right)^2 = \left(\frac{300}{30}\right)^2 = 100$$

Now, $L_1 - L_2 = 10\log_{10}\dfrac{I_1}{I_2}$

Substituting $\dfrac{I_1}{I_2} = 100$

We get, $L_1 - L_2 = 20$ dB

4. (a) $$I = \frac{v\,(\Delta p)^2_{\max}}{2B} = \frac{(\Delta p)^2_{\max}}{2\rho v}$$

For finest sound,

$$I = \frac{(2 \times 10^{-5})^2}{2 \times 1.29 \times 345} = 4.49 \times 10^{-13}\ \text{W/m}^2$$

$$L = 10\log_{10}\left(\frac{I}{I_0}\right) = 10\log_{10}\left(\frac{4.49 \times 10^{-13}}{10^{-12}}\right) = -3.48 \text{ dB}$$

Same formulae can be applied for loudest sound.

(b) $$(\Delta p)_{\max} = BAk = (\rho v^2)(A)\left(\frac{\omega}{v}\right) = 2\pi f \rho v$$

$$\therefore \quad A = \frac{(\Delta p)_{\max}}{2\pi f \rho v}$$

For finest sound,

$$A = \frac{2 \times 10^{-5}}{(2\pi)(500)(1.29)(345)} = 1.43 \times 10^{-11} \text{ m}$$

INTRODUCTORY EXERCISE 19.4

1. $$\Delta x = \lambda/2 = \frac{v}{2f}$$

$$\therefore \quad f = \frac{v}{2\Delta x} = \frac{330}{2(0.12)} = 1375 \text{ Hz}$$

2. (a) $$\Delta\phi = \frac{2\pi}{\lambda}(\Delta x)$$

$$\therefore \quad \Delta x = \frac{\lambda \Delta\phi}{2\pi} = \frac{v\,(\Delta\phi)}{(f)(2\pi)} = \frac{(350)(\pi/3)}{(500)(2\pi)} = 0.1166 \text{ m} = 11.7 \text{ cm}$$

(b) $$\Delta\phi = \left(\frac{2\pi}{T}\right)\Delta t = (2\pi f)\,\Delta t = (2\pi)(500)(10^{-3}) = \pi \text{ or } 180^\circ$$

INTRODUCTORY EXERCISE 19.5

1. Length of the organ pipe is same in both the cases. Fundamental frequency of open pipe is $f_1 = \dfrac{v}{2l}$ and frequency of third harmonic of closed pipe will be

$$f_2 = 3\left(\frac{v}{4l}\right)$$

Given that, $f_2 = f_1 + 100$

or $f_2 - f_1 = 100$

or $$\frac{3}{4}\left(\frac{v}{l}\right) - \left(\frac{1}{2}\right)\left(\frac{v}{l}\right) = 100$$

$$\Rightarrow \quad \frac{v}{4l} = 100 \text{ Hz}$$

$$\therefore \quad \frac{v}{2l} \text{ or } f_1 = 200\,\text{Hz}$$

Therefore, fundamental frequency of the open pipe is 200 Hz.

2. First harmonic of closed pipe = Third harmonic of open pipe

$$\therefore \quad \frac{v}{4l_1} = 3\left(\frac{v}{2l_2}\right) \Rightarrow \therefore \quad \frac{l_1}{l_2} = \frac{1}{6}$$

3. $f_c = \dfrac{v}{4l} = 512\,\text{Hz}$

$$f_o = \frac{v}{2l} = 2f_c = 1024\,\text{Hz}$$

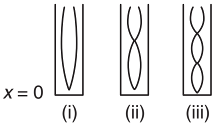
4. (a) $f_1 = \dfrac{v}{4l}$

$$\therefore \quad l = \frac{v}{4f_1} = \frac{345}{4 \times 220} = 0.392\text{ m}$$

(b) $5\left(\dfrac{v}{4l_c}\right) = 3\left(\dfrac{v}{2l_0}\right)$

$$\therefore \quad l_0 = \frac{6}{5}l_c = \left(\frac{6}{5}\right)(0.392) = 0.47\text{ m}$$

5.



(a) In second figure, $l = \dfrac{3\lambda}{4}$

$$\therefore \quad \frac{\lambda}{4} = \frac{l}{3} = \frac{0.8}{3} = 0.267\text{ m}$$

Displacement antinode is at

$$x = \frac{\lambda}{4} = 0.267\text{ m}$$

and $\quad x = 3\dfrac{\lambda}{4} = 0.8\text{ m}$

(b) Pressure antinode is displacement node:

In third figure,

Pressure antinode or displacement node is at

$$x = 0,\ x = \frac{\lambda}{2}$$

and $\quad x = \lambda$

6. (a) $\dfrac{400}{560} = \dfrac{5}{7}$

Since, these are odd harmonics (5 and 7). Hence, pipe is closed.

(c) Given frequencies are integer multiples of 80 Hz. Hence, fundamental frequency is 80 Hz.

$$\frac{v}{4l} = 80$$

$$\therefore \quad l = \frac{v}{320} = \frac{344}{320} = 1.075\text{ m}$$

INTRODUCTORY EXERCISE 19.6

1. Frequency of first will decrease by loading wax over it. Beat frequency is increasing. Hence,

$$f_2 > f_1$$

or $\quad f_2 - f_1 = 4$

$$\therefore \quad f_1 = f_2 - 4 = 256 - 4 = 252\text{ Hz}$$

2. By putting wax on first tuning fork, its frequency will decrease.

Beat frequency is also decreasing. Hence,

$$f_1 > f_2$$

$$\therefore \quad f_1 - f_2 = 3$$

or $\quad f_1 = 3 + f_2 = 3 + 384 = 387\text{ Hz}$

INTRODUCTORY EXERCISE 19.7

1. Source is moving towards the observer

$$f' = f\left(\frac{v}{v - v_s}\right) = 450\left(\frac{330}{330 - 33}\right)$$

$$f' = 500\text{ Hz}$$

2. $\quad f_1 = f\left(\dfrac{v}{v - v_s}\right)$

$$f_1 = f\left(\frac{340}{340 - 34}\right) = f\left(\frac{340}{306}\right)$$

and $\quad f_2 = f\left(\dfrac{340}{340 - 17}\right) = f\left(\dfrac{340}{323}\right)$

$$\therefore \quad \frac{f_1}{f_2} = \frac{323}{306} = \frac{19}{18}$$

3. Using the formula, $f' = f\left(\dfrac{v + v_0}{v}\right)$

We get, $\quad 5.5 = 5\left(\dfrac{v + v_A}{v}\right) \qquad \ldots\text{(i)}$

and $\quad 6.0 = 5\left(\dfrac{v + v_B}{v}\right) \qquad \ldots\text{(ii)}$

Here, v = speed of sound
v_A = speed of train A
v_B = speed of train B

Solving Eqs. (i) and (ii), we get $\dfrac{v_B}{v_A} = 2$

4. Observer is stationary and source is moving.

During approach, $f_1 = f\left(\dfrac{v}{v - v_s}\right)$

$$= 1000\left(\frac{320}{320-20}\right)$$

$$= 1066.67 \text{ Hz}$$

During recede, $f_2 = f\left(\frac{v}{v+v_s}\right)$

$$= 1000\left(\frac{320}{320+20}\right) = 941.18 \text{ Hz}$$

$$|\% \text{ change in frequency}| = \left(\frac{f_1 - f_2}{f_1}\right) \times 100$$

$$\approx 12\%$$

Exercise

LEVEL 1

Assertion and Reason

1. **Closed pipe**

Frequencies are, $\frac{v}{4l}, \frac{3v}{4l}, \frac{5v}{4l}$

Open pipe Frequencies are

$$\frac{v}{2l}, 2\left(\frac{v}{2l}\right), 3\left(\frac{v}{2l}\right)$$

They are never same.

2. $f' = f\left(\frac{v}{v-v_s}\right) \Rightarrow f' > f$ but $f' =$ constant.

3. Change in pressure is maximum at a point where displacement is zero.

5. $L = 10 \log_{10} \frac{I}{I_0}$

Increase in the value of L is not linear with I.

$$\Delta L = L_2 - L_1 = 10 \log \frac{I_2}{I_1}$$

$$\Delta L = 3 \text{dB} \quad \text{if} \quad I_2 = 2I_1$$

6. It is independent of pressure as long as temperature remains constant.

7. $f_A - f_B = 4$ Hz

When A is loaded with wax f_A will decrease. So, $f_A - f_B$ will be less than 4 Hz till $f_A > f_B$. But $f_B - f_A$ may be greater than 4 Hz when f_B becomes greater than f_A.

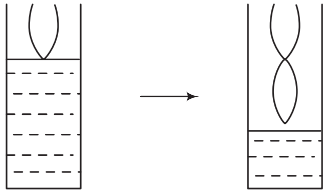
8. $\frac{450}{750} = \frac{3}{5}$

Successive harmonics are odd 3 and 5. Hence, it is a closed pipe.

9. $f_0 = \frac{v}{2(l + 1.2r)}$ of an open pipe and

$f_0 = \frac{v}{4(l + 0.6r)}$ for a closed pipe.

10. Wavelength remains same.



Objective Questions

1. Sound waves cannot travel in vacuum.

3. $f_0 = \frac{v}{2l}$ and $f_c = \frac{v}{4l}$

$\therefore \quad f_0 = 2f_c$

4. $v = \sqrt{\frac{\gamma RT}{M}} \propto \sqrt{\frac{T}{M}}$ ($\gamma = 1.4$ for both)

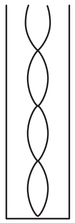
$$\therefore \quad \frac{T_{O_2}}{M_{O_2}} = \frac{T_{N_2}}{M_{N_2}}$$

$$\therefore \quad T_{O_2} = \left(\frac{M_{O_2}}{M_{N_2}}\right) T_{N_2}$$

$$= \left(\frac{32}{28}\right)(273 + 15)$$

$$= 329 \text{ K} = 56°\text{C}$$

5.



Third overtone

6. $f \propto v$ and $v \propto \sqrt{T}$

7. In air speed of sound is less, hence air is denser medium.

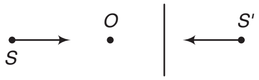
8. $\frac{v}{2l_0} = \frac{v}{4l_c} \Rightarrow l_c : l_0 = 1 : 2$

9. $f \propto v \propto \sqrt{T}$ $\qquad (T \rightarrow \text{tension})$

$$\therefore \quad \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}} \Rightarrow T_2 = T_1\left(\frac{f_2}{f_1}\right)^2$$

$$= (10)\left(\frac{256}{320}\right)^2 = 6.4 \text{ kg}$$

10.



Both S and S' are moving toward observer. Hence,

$$f_S = f_{S'} \quad \text{or} \quad f_b = 0$$

11. $f_b = \frac{10}{3} = f_1 - f_2 = \frac{v}{1} - \frac{v}{1.01}$

Solving we get, $v = 337$ m/s

12. $f' = f\left(\frac{v}{v + v_s}\right) = f\left(\frac{v}{v + v}\right) = 0.5f$

13. $I_R = I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$

If $\quad I_1 = I_2 = I_0$, then

$$I_R = 4I_0$$

14. $\frac{\lambda}{2} = 52 - 17 = 35$ cm

$$\therefore \quad \lambda = 70 \text{ cm} = 0.7 \text{ m}$$

$$v = f\lambda = 500 \times 0.7$$

$$= 350 \text{ m/s}$$

15. $f' = f\left(\frac{v}{v \pm v_s \cos\theta}\right)$

At $\quad \theta = 90°;\ f' = f$

$\therefore \quad n_1 = 0$

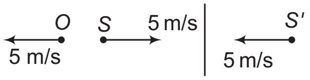
16. Fundamental frequency,

$$f_0 = \frac{v}{4l} = \frac{340}{4 \times 1} = 85 \text{ Hz}$$

Six frequencies can be produced below 1 kHz. Those six frequencies are, $f_0, 3f_0, 5f_0, 7f_0, 9f_0$ and $11f_0$.

As $13f_0 = 1105 \text{ Hz} > 1000$ Hz or 1 kHz.

17. There is no relative motion between O and S'.



$$f_b = f_{S'} - f_S$$

$$= f - f_S = 180 - 180\left(\frac{355 - 5}{355 + 5}\right)$$

$$= 5 \text{ Hz}$$

18. $$n = n_1 - n_2 = \frac{v}{\lambda_1} - \frac{v}{\lambda_2}$$

$$\therefore \quad v = \frac{n\lambda_1\lambda_2}{\lambda_2 - \lambda_1}$$

19. By putting wax on A. Its frequency will decrease. But beat frequency between A and B is also decreasing.

$$\therefore \quad f_A > f_B$$

$$\text{or} \quad f_A - f_B = 5 \text{ Hz} \qquad \text{...(i)}$$

$$\therefore \quad f_B = f_A - 5 = 345 \text{ Hz}$$

Now, $\quad f_B \sim f_C = 4$ Hz

f_C is either 341 Hz or 349 Hz.

If it is 341 Hz, then beat frequency with A will be 9 Hz.

If it is 349 Hz, then beat frequency will be 1 Hz.

If wax is loaded on A, its frequency will decrease.

To produce 6 beats/s with C it should become either 347 Hz (if $f_C = 341$ Hz) or it should become 343 Hz (if $f_C = 349$ Hz).

If it becomes 347 Hz, then only it produces 2 beats/s with B, which is given in the question.

$\therefore \quad f_C = 341$ Hz **Ans.**

20. $\frac{\lambda}{2} = 122 - 40 = 82$ cm

$\therefore$ Next resonance length $= 122$ cm $+$ 82 cm

$= 204$ cm

21. $f \propto \sqrt{T}$

$$\frac{f'}{f} = \sqrt{\frac{T'}{T}} = \sqrt{\frac{101}{100}} = 1.0049$$

$$f' = (1.0049)(200)$$

$$\approx 201 \text{ Hz}$$

$$\therefore \quad f_b = f' - f = 1 \text{ Hz}$$

22. $\lambda = \frac{v}{f} = \frac{340}{340} = 1 \text{ m} = 100$ cm

$\frac{\lambda}{4} = 25$ cm

Air column lengths required are,

$$\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \text{ etc.}$$

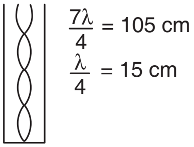
or 25 cm, 75 cm, 125 cm etc.

Maximum we can take 75 cm.

$\therefore$ Minimum water length

$= 120 - 75 = 45$ cm

23. Pressure node means displacement antinode



Displacement antinodes are at a distance $\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ and $\frac{7\lambda}{4}$ from closed end or at a distance of 15 cm, 45 cm, 75 cm and 105 cm.

24. Number of moles ∝ Volume

$$M = \frac{n_1 M_{O_2} + n_2 M_{H_2}}{n_1 + n_2}$$

$$= \frac{(1)(32) + (1)(2)}{2} = 17$$

Now, $v = \sqrt{\frac{\gamma RT}{M}} \propto \frac{1}{\sqrt{M}}$

$$\frac{v_1}{v_2} = \sqrt{\frac{M_2}{M_1}} = \sqrt{\frac{2}{17}}$$

25. $f_1 = f\left(\frac{v}{v - v_s}\right)$ = constant. But $f_1 > f$

$f_2 = f\left(\frac{v}{v + v_s}\right)$ = constant, but $f_2 < f$

26. $f \propto \frac{1}{l}$

Both frequencies will becomes half. Hence, $f_b = f_1 - f_2$ will also become half.

27. $f_b = 243\left(\frac{320}{320 - 4}\right) - 243\left(\frac{320}{320 + 4}\right) = 6$ Hz

28. **Closed pipe**

Fundamental frequency is

$$f_1 = \frac{v}{4l} = \frac{320}{4 \times 1} = 80 \text{ Hz}$$

Other frequencies are

$3f_1 : 5f_1$ etc. or 240 Hz, 400 Hz etc.

Open pipe Fundamental frequency is

$$f_1 = \frac{v}{2l} = \frac{320}{2 \times 1.6} = 100 \text{ Hz}$$

Other frequencies are $2f_1 : 3f_1 : 4f_1$ etc. or 200 Hz, 300 Hz and 400 Hz etc. So, then resonate at 400 Hz.

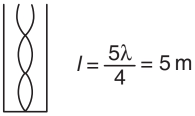
29. Resultant amplitude will become 4 time. Therefore, resultant intensity is 16 times

$$L_2 - L_1 = 10 \log_{10} \frac{I_1}{I_2}$$

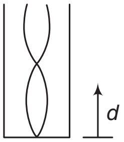
or $L_2 - 10 = 10 \log_{10}(16)$

or $L_2 = 22$ dB

30. $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/2} = 4$ m



31. $\lambda = \frac{v}{f} = \frac{330}{600} = 0.55$ m = 55 cm



The desired distance, $d = \frac{\lambda}{4} = 13.75$ cm

32. $f_b = f_1 - f_2 = \frac{v}{\lambda_1} - \frac{v}{\lambda_2}$

$$= \frac{332}{0.49} - \frac{332}{0.5} \approx 13 \text{ Hz}$$

33. $(f_b)_A = f_B - f_A$

$$= 300\left(\frac{300 + 30}{300}\right) - 300 = 30 \text{ Hz}$$

$(f_b)_B = f_A - f_B$

$$= 300\left(\frac{300}{300 - 30}\right) - 300 = 33.33 \text{ Hz}$$

34. $f_a = f\left(\frac{v + v_0}{v}\right)$

∴ $\frac{v_0}{v} = \frac{f_a}{f} - 1$ …(i)

$f_r = f\left(\frac{v - v_0}{v}\right)$

∴ $\frac{v_0}{v} = 1 - \frac{f_r}{f}$ …(ii)

From Eqs. (i) and (ii), we get

$$f = \frac{f_a + f_r}{2}$$

Subjective Questions

1. Speed of sound wave,

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{(2 \times 10^9)}{10^3}} = 1414 \text{ m/s}$$

Wavelength, $\lambda = \frac{v}{f} = 5.84$ m **Ans.**

2. $f' = f\left(\dfrac{v \pm v_m}{v \pm v_m}\right) = f$

3. $$v = \sqrt{\frac{\gamma RT}{M}}$$

$$= \sqrt{\frac{1.40 \times 8.31 \times 300}{2 \times 10^{-3}}} = 1321 \text{ m/s}$$

4. $$L_1 = 10 \log_{10}\left(\frac{I_1}{I_0}\right)$$

$$= 10 \log_{10}\left(\frac{10^{-6}}{10^{-12}}\right) = 60 \text{ dB}$$

$$L_2 = 10 \log_{10}\left(\frac{I_2}{I_0}\right)$$

$$= 10 \log_{10}\left(\frac{10^{-9}}{10^{-12}}\right) = 30 \text{ dB}$$

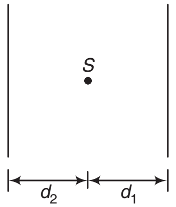
$$L_1 = 2L_2$$

5. $$f' = f\left(\frac{v - v_w - v_0}{v - v_w + v_S}\right)$$

$$= 300\left[\frac{340 - 5 - 20}{340 - 5 + 10}\right]$$

$$= 274 \text{ Hz}$$

6.



$$2d_1 = vt_1$$

$$\therefore \quad d_1 = \frac{vt_1}{2} = \frac{332 \times 3/2}{2}$$

$$= 249 \text{ m}$$

Similarly, $$d_2 = \frac{vt_2}{2} = \frac{332 \times 5/2}{2}$$

$$= 415 \text{ m}$$

$\therefore$ Total distance $= d_1 + d_2 = 664$ m

Next echo he will hear after time,

$$t = t_2 + t_1 = \frac{5}{2} + \frac{3}{2} = 4 \text{ s}$$

7. $$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{(5/3) \times 0.76 \times 13.6 \times 10^3 \times 9.8}{0.179}}$$

where, $p = h\rho g$

$\therefore$ $v = 972$ m/s

8. (a) $$v = f\lambda = \sqrt{\frac{B}{\rho}}$$

$$\therefore \quad B = \rho\,(f\lambda)^2$$

$$= (1300)\,(400 \times 8)^2$$

$$= 1.33 \times 10^{10} \text{ N/m}^2$$

(b) $$v = \frac{l}{t} = \sqrt{\frac{Y}{\rho}}$$

$$\therefore \quad Y = \rho\left(\frac{l}{t}\right)^2 = (6400)\left(\frac{1.5}{3.9 \times 10^{-4}}\right)^2$$

$$= 9.47 \times 10^{10} \text{ N/m}^2$$

9. $$\sqrt{\frac{Y}{\rho}} = 30\sqrt{\frac{T}{\mu}} = 30\sqrt{\frac{F}{\rho A}} \qquad (\text{as } T = F \text{ and } \mu = \rho A)$$

$$\therefore \quad \frac{F}{A} = \frac{Y}{900}$$

10. Equal volume of different gases contains equal number of moles at STP.

$$\therefore \quad n \propto V$$

$$M = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$$

$$= \frac{(2)\,(2) + (1)\,(28)}{2 + 1} = 10.67$$

$$v = \sqrt{\frac{\gamma RT}{M}}$$

$$\therefore \quad \frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1} \cdot \frac{M_1}{M_2}}$$

or $$v_2 = \left(\sqrt{\frac{T_2}{T_1} \cdot \frac{M_1}{M_2}}\right) v_1$$

$$= \left(\sqrt{\frac{300}{273} \times \frac{2}{10.67}}\right)(1300)$$

$$\approx 591 \text{ m/s}$$

11. $$L_1 = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

$$100 = 10 \log_{10}\left(\frac{I}{10^{-12}}\right)$$

Solving we get, $I = 10^{-2}$ W/m^2

Now, $$I = \frac{P}{4\pi r^2}$$

$$\therefore \quad P = I\,(4\pi r^2)$$

$$= (10^{-2})(4\pi)(40)^2$$

$$= 201 \text{ W}$$

12. (a) $$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

$$60 = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$$

Solving we get,

$$I = 10^{-6}\ \text{W/m}^2$$

(b) $I = \frac{P}{S}$

$$\therefore\quad P = (I)(S)$$
$$= (10^{-6})(120\times 10^{-4})$$
$$= 1.2\times 10^{-8}\ \text{W}$$

13. (a) $L_2 - L_1 = 10\log_{10}\left(\frac{I_2}{I_1}\right)$

$$\therefore\quad 13 = 10\log_{10}\left(\frac{I_2}{I_1}\right)$$

Solving we get, $\frac{I_2}{I_1} = 20$

(b) From the above equation we can see that we do not need L_1 and L_2 separately.

14. (a) $I = \frac{P}{4\pi r^2} = \frac{5}{4\pi\,(20)^2}$

$$= 9.95\times 10^{-4}\ \text{W/m}^2$$

(b) $I = \frac{1}{2}\rho\omega^2 A^2 v$

$$\therefore\quad A = \sqrt{\frac{2I}{\rho\,(2\pi f)^2\, v}}$$
$$= \sqrt{\frac{2\times 9.95\times 10^{-4}}{1.29\,(2\pi\times 300)^2\,(330)}}$$
$$= 1.15\times 10^{-6}\text{m}$$

15. $L = 10\log_{10}\left(\frac{I}{I_0}\right)$

$$60 = 10\log_{10}(I/10^{-12})$$

Solving, we get $I = 10^{-6}\ \text{W/m}^2$

Now, using the result derived in above problem.

$$A = \sqrt{\frac{2I}{\rho\,(2\pi f)^2\, v}}$$
$$= \sqrt{\frac{2\times 10^{-6}}{1.29\,(2\pi\times 800)^2\,(330)}}$$
$$= 1.36\times 10^{-8}\text{m}$$
$$= 13.6\ \text{nm}$$

16. $$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

$$\therefore\quad 10^2 = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$$

Solving we get,

$$I = 1.59\times 10^{-2}\ \text{W/m}^2$$
$$I = \frac{P}{4\pi r^2}$$
$$\therefore\quad P = (I)(4\pi r^2)$$
$$= (1.59\times 10^{-2})(4\pi)(20)^2$$
$$= 80\ \text{W}$$

17. $I = \frac{1}{2}\rho\omega^2 A^2 v$

$$= 2\rho\pi^2 f^2 A^2 v \qquad (\text{as } \omega = 2\pi f)$$
$$= (2)(1.29)(\pi)^2(300)^2\,(0.2\times 10^{-3})^2\,(330)$$
$$= 30.27\ \text{W/m}^2$$

$$L = 10\log_{10}(I/I_0)$$

where, $I_0 = 10^{-12}\ \text{W/m}^2$

Substituting the value we get,

$$L = 134.4\ \text{dB}$$

18. (a) $v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.18\times 10^9}{1000}}$

$$= 1476\ \text{m/s}$$
$$\lambda = \frac{v}{f} = \frac{1476}{3400} = 0.43\ \text{m}$$
$$I = \frac{1}{2}\rho\omega^2 A^2 v$$
$$\therefore\quad A = \sqrt{\frac{2I}{\rho\,(2\pi f)^2 v}}$$
$$= \sqrt{\frac{2\times 3\times 10^{-6}}{(1000)\;(2\pi\times 3400)^2\,(1476)}}$$
$$= 9.44\times 10^{-11}\text{m}$$

(b) $v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{(1.4)(10^5)}{(1.2)}}$

$$= 341.56\ \text{m/s}$$
$$\lambda = \frac{v}{f} = \frac{341.56}{3400} = 0.1\ \text{m}$$
$$A = \sqrt{\frac{2I}{\rho\,(2\pi f)^2\, v}}$$

$$= \sqrt{\frac{2\times 3.0\times 10^{-6}}{1.2\times(2\pi\times 3400)^2\;(341.56)}}$$

$$= 5.66\times 10^{-9}\text{ m}$$

(c) $\dfrac{A_{\text{air}}}{A_{\text{water}}} = \dfrac{5.66\times 10^{-9}}{9.44\times 10^{-11}} = 60$

$$I = \frac{1}{2}\rho\omega^2A^2v$$

ρ and v are less in air. So, for same intensity A should be large.

19. $I = \dfrac{v\,(\Delta p)^2_{\max}}{2B} \quad\Rightarrow\quad v = \sqrt{\dfrac{B}{\rho}}$

$$\therefore \qquad B = \rho v^2$$

$$\therefore \qquad I = \frac{(\Delta p)^2_{\max}}{2\rho v}$$

$$= \frac{(6\times 10^{-5})^2}{2\times 1.29\times 330}$$

$$= 4.2\times 10^{-12}\text{ W/m}^2$$

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

$$= 10\log_{10}\left(\frac{4.2\times 10^{-12}}{10^{-12}}\right)$$

$$= 6.23\text{ dB}$$

20. (a) Fundamental, $f_1 = \dfrac{v}{2l} = \dfrac{344}{2\times 0.45} = 382.2\text{ Hz}$

First two overtones are $2f_1$ and $3f_1$.

(b) Fundamental, $f_1 = \dfrac{v}{4l} = \dfrac{344}{4\times 0.45} = 191.1\text{ Hz}$

First two overtones are $3f_1$ and $5f_1$.

21. $\dfrac{\lambda}{2} = (45-15)\text{ cm}$

$$\therefore \qquad \lambda = 60\text{ cm}$$

$$v = f\lambda = (500)\,(0.6)$$

$$= 300\text{ m/s}$$

Lowest frequency when open from both ends is $\dfrac{v}{2l}$.

$$= \frac{300}{2\times 0.6} = 250\text{ Hz}$$

22. Fundamental frequency $= \dfrac{v}{2l}$

$$f = \frac{330}{2\times 0.8}$$

$$f = 206.25\text{ Hz}$$

$$\omega = 2\pi f = 1297\text{ rad/s}$$

$$k = \frac{\omega}{v} = \frac{1297}{330} = 3.93\text{ rad/s}$$

Now, $x = 0$ (the open end) is an antinode. Hence, we can write the equation,

$$y = A\cos kx\sin\omega t$$

23. $\dfrac{\lambda}{2} = (84-50)\text{ cm} = 34\text{ cm}$

Next length $= 84 + \dfrac{\lambda}{2} = 118\text{ cm}$

$$\lambda = 68\text{ cm} \quad\text{or}\quad 0.68\text{ m}$$

$$v = f\lambda = (512)\,(0.68)$$

$$= 348.16\text{ m/s}$$

24. $2\left(\dfrac{v}{2l_1}\right) = \left(\dfrac{v_2}{4l_2}\right)$

or $\dfrac{\sqrt{T/\mu}}{l_1} = \left(\dfrac{v_2}{4l_2}\right)$

$$\therefore \qquad T = \mu\left(\frac{v_2\,l_1}{4l_2}\right)^2$$

$$= \left(\frac{4\times 10^{-3}}{0.4}\right)\left(\frac{340\times 40}{4\times 100}\right)^2$$

$$= 11.56\text{ N}$$

25. $f = \dfrac{v}{4\,(l+0.6\,\pi)}$

$$\therefore \qquad v = 4f\,(l+0.6\,\pi)$$

$$= 4\times 480\,(0.16+0.6\times 0.025)$$

$$= 336\text{ m/s}$$

26. (a) $f = \dfrac{v}{4l}$

$$\therefore \qquad l = \frac{v}{4f} = \frac{345}{4\times 220}$$

$$= 0.392\text{ m}$$

(b) $5\left(\dfrac{v}{4l_c}\right) = 3\left(\dfrac{v}{2l_0}\right)$

$$\therefore \qquad l_0 = \frac{6}{5}\,l_c = \frac{6}{5}\,(0.392)$$

$$= 0.470\text{ m}$$

27. $\dfrac{v_1}{2\,(0.8l)} = \dfrac{v_2}{4l} \quad\Rightarrow\quad \dfrac{v_1}{v_2} = 0.4$

28. (a) $\lambda = \dfrac{v}{f} = \dfrac{340}{300} = 1.13\text{ m}$

(b) **Ahead**

$$f' = f\left(\frac{v}{v-v_s}\right) = 300\left(\frac{340}{340-30}\right)$$

$$= 329\text{ Hz}$$

$$\lambda' = \frac{v}{f'} = \frac{340}{329} = 1.03 \text{ m}$$

Behind

$$f' = f\left(\frac{v}{v+v_s}\right) = 300\left(\frac{340}{340+30}\right)$$

$$= 275.67 \text{ Hz}$$

$$\lambda' = \frac{v}{f'} = \frac{340}{275.67}$$

$$= 1.23 \text{ m}$$

29. (a) Possible fundamental frequencies of retuned string is (440 ± 1.5) Hz.

(b) $f \propto \sqrt{T}$

$$\therefore \quad \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$$

$$\therefore \quad T_2 = \left(\frac{f_2}{f_1}\right)^2 T_1$$

(i) $T_2 = \left(\dfrac{441.5}{440}\right)^2$

$$T_1 = 1.0068\, T_1$$

$$\% \text{ change} = \frac{T_2 - T_1}{T_1} \times 100$$

$$= +\,0.68\%$$

(ii) $T_2 = \left(\dfrac{438.5}{440}\right)^2 T_1 = 0.9932\, T_1$

$$\% \text{ change} = \frac{T_2 - T_1}{T_1} \times 100$$

$$= -\,0.68\%$$

30. (a) $\lambda = 0.12$

Surface wave speed, $v = 0.32$ m/s

$$f' = \frac{v}{\lambda} = \frac{0.32}{0.12} = 2.66 \text{ Hz}$$

$$f = \frac{1}{T} = \frac{1}{1.6} = 0.625 \text{ Hz}$$

Now, using $f' = f\left(\dfrac{v}{v - v_s}\right)$

Here, $v_s =$ velocity of source or velocity of duck

$$\therefore \quad 2.66 = 0.625\left(\frac{0.32}{0.32 - v_s}\right)$$

Solving we get, $v_s = 0.245$ m/s

(b) **Behind the duck**

$$f' = f\left(\frac{v}{v+v_s}\right)$$

$$= 0.625\left(\frac{0.32}{0.32 + 0.245}\right)$$

$$= 0.354 \text{ Hz}$$

$$\lambda = \frac{v}{f'} = \frac{0.32}{0.354} = 0.904 \text{ m}$$

31. (a) $f' = f\left(\dfrac{v + v_0}{v - v_s}\right)$

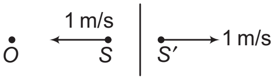
$$= 262\left(\frac{340+18}{340-30}\right)$$

$$= 302 \text{ Hz}$$

(b) $f' = f\left(\dfrac{v - v_0}{v + v_s}\right)$

$$= 262\left(\frac{340-18}{340+30}\right) = 228 \text{ Hz}$$

32.



$$f_b = f_S - f_{S'}$$

$$4 = f\left(\frac{340}{340-1}\right) - f\left(\frac{340}{340+1}\right)$$

or $\quad 4 = f\left(1 - \dfrac{1}{340}\right)^{-1} - f\left(1 + \dfrac{1}{340}\right)^{-1}$

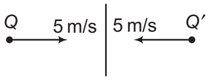
Applying Binomial, we get

$$4 = f\left(1 + \frac{1}{340}\right) - f\left(1 - \frac{1}{340}\right)$$

$$= \frac{f}{170} \quad \text{or} \quad f = 680 \text{ Hz}$$

33. Beat frequency between P and Q is $\dfrac{7}{2}$ or 3.5 Hz.

On loading P with wax, its frequency will decrease and beat frequency is increasing.



$\therefore \quad f_Q > f_P$

or $\quad f_Q - f_P = 3.5 \quad \ldots\text{(i)}$

Given, $\quad f'_Q - f_Q = 5$

or $\quad f_Q\left(\dfrac{332+5}{332-5}\right) - f_Q = 5$

Solving this equation, we get

$$f_Q = 163.5 \text{ Hz}$$

Now, from Eq. (i) we have

$$f_P = 160 \text{ Hz}$$

34. $f_b = f_1 - f_2$

$$2 = 680\left(\frac{340}{340 - v_s}\right) - 680\left(\frac{340}{340 + v_s}\right)$$

$$= 680\left(1 - \frac{v_s}{340}\right)^{-1} - 680\left(1 + \frac{v_s}{340}\right)^{-1}$$

Using Binomial expansion, we have

$$2 = 680\left[\frac{2v_s}{340}\right] \Rightarrow v_s = 0.5\text{ m/s}$$

35. $\dfrac{\lambda}{2} = 11.5\text{ cm}$ or $\lambda = 23\text{ cm}$ or 0.23 m

$$f = \frac{v}{\lambda} = \frac{331.2}{0.23} = 1440\text{ Hz}$$

36. $\lambda = \dfrac{v}{f} = \dfrac{330}{220} = 1.5\text{ m}$

$$\Delta\phi_1 = \left(\frac{2\pi}{\lambda}\right)(\Delta x_1) = \left(\frac{2\pi}{1.5}\right)(0.75) = \pi$$

$$\Delta\phi_2 = \left(\frac{2\pi}{\lambda}\right)(\Delta x_2) = \left(\frac{2\pi}{1.5}\right)(3) = 4\pi$$

Since, $\phi = \Delta\phi_2 - \Delta\phi_1 = 3\pi$

They will interfere destructively.

$\therefore$ $$I_R = (\sqrt{I_1} - \sqrt{I_2})^2 \qquad \text{...(i)}$$

where,

$$I_1 = \frac{P_1}{4\pi r_1^2} = \frac{1.2 \times 10^{-3}}{(4\pi)(0.75)^2}$$

$$= 16.97 \times 10^{-5}\text{ W/m}^2$$

$$I_2 = \frac{P_2}{4\pi r_2^2} = \frac{1.8 \times 10^{-3}}{(4\pi)(3)^2}$$

$$= 1.59 \times 10^{-5}\text{ W/m}^2$$

Substituting in Eq. (i), we get

$$I_R = 8.2 \times 10^{-5}\text{ W/m}^2$$

37. $SOD - SD = \lambda$

where, $\lambda = \dfrac{v}{f} = \dfrac{360}{360} = 1\text{ m}$

$\therefore$ $$2\sqrt{\frac{x^2}{4} + (2)^2} - x = \lambda = 1$$

Solving this equation, we get

$$x = 7.5\text{ m}$$

38. $\because$ $v = f\lambda = (1000)(2 \times 6.77 \times 10^{-2})$

$$= 135.4\text{ m/s} = \sqrt{\frac{\gamma RT}{M}}$$

$\therefore$ $$\gamma = (135.4)^2\left(\frac{M}{RT}\right)$$

$$= \frac{(135.4)^2(0.127)}{(8.31 \times 400)}$$

$$= 0.7$$

Since, $\gamma \nless 1$

So, we will have to substitute,

$$M = 2 \times 0.127$$

The γ comes out to be 1.4 which is the γ of a diatomic gas.

39. $\dfrac{\lambda}{2} = (100 - 60) \times 10^{-2}\text{ m}$

$\therefore$ $\lambda = 0.8\text{ m}$

Now, $v = f\lambda = 440 \times 0.8$

$$= 352\text{ m/s}$$

40. When T_A is increased, f_A will increase. So, it will become $(600 + 6) = 606$ Hz

Now, $f \propto \sqrt{T}$

$\therefore$ $$\frac{T_A}{T_B} = \left(\frac{f_A}{f_B}\right)^2 = \left(\frac{606}{600}\right)^2$$

$$= 1.02$$

41. $f \propto \dfrac{1}{l}$

By decreasing the length, frequency of wire will increase. But beat frequency is decreasing. Hence, its original frequency was $(256 - 4)$ Hz or 252 Hz. Now, we have to make it 256 Hz for no beats.

$$\frac{f_1}{f_2} = \frac{l_2}{l_1}$$

$\therefore$ $$l_2 = \left(\frac{f_1}{f_2}\right)l_1$$

$$= \left(\frac{252}{256}\right)(25)$$

$$= 24.6\text{ cm}$$

$\therefore$ $\Delta l = l_1 - l_2 = 0.4\text{ cm}$

42. If observer is moving, then

$$f' = f\left(\frac{v \pm v_0}{v}\right) = f\left(1 \pm \frac{v_0}{v}\right)$$

If source is moving, then

$$f' = f\left(\frac{v}{v \mp v_s}\right)$$

$$= f\left(1 \mp \frac{v_s}{v}\right)^{-1} = f\left(1 \pm \frac{v_s}{v}\right)$$

If $v_s << v$

Both expressions come out to be of similar type.

43. $f' = f\left(\dfrac{v}{v - v_s}\right) = 200\left(\dfrac{340}{340 - 80}\right)$

$$= 261.53 \text{ Hz} \approx 262 \text{ Hz}$$

$$\lambda' = \frac{v}{f'} = \frac{340}{261.53} = 1.3 \text{ m}$$

44. In front of locomotive,

$$f' = f\left(\frac{v}{v - v_s}\right)$$

$$= 500\left(\frac{344}{344 - 30}\right) = 548 \text{ Hz}$$

$$\therefore \quad \lambda' = \frac{v}{f'} = \frac{344}{548} = 0.628 \text{ m}$$

Behind the locomotive,

$$f' = f\left(\frac{v}{v + v_s}\right)$$

$$= 500\left(\frac{344}{344 + 30}\right) = 460 \text{ Hz}$$

$$\therefore \quad \lambda' = \frac{v}{f'} = \frac{344}{460} = 0.748 \text{ cm}$$

45. (a) The given frequencies are in the ratio 5 : 7 : 9. As the frequencies are odd multiple of 85 Hz, the pipe must be closed at one end.

(b) Now, the fundamental frequency is the lowest, i.e. 85 Hz.

$$\therefore \quad 85 = \frac{v}{4l}$$

$$\Rightarrow \quad l = \frac{340}{4 \times 85} = 1 \text{ m} \qquad \textbf{Ans.}$$

46. Let the frequency of the first fork be f_1 and that of second be f_2.

We then have,

$$f_1 = \frac{v}{4 \times 32}$$

and $$f_2 = \frac{v}{4 \times 33}$$

We also see that $f_1 > f_2$

$$\therefore \quad f_1 - f_2 = 8 \qquad \ldots\text{(i)}$$

and $$\frac{f_1}{f_2} = \frac{33}{32} \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$f_1 = 264 \text{ Hz}$$

and $$f_2 = 256 \text{ Hz} \qquad \textbf{Ans.}$$

LEVEL 2

Single Correct Option

1. $\sin\theta_C = \dfrac{330}{1400} = 0.2357$

$\therefore \quad \theta_C = 13.6°$

Since, $i > \theta_C$, therefore only reflection will take place.

2.

$$\Delta p_{\max} = BAk \qquad \ldots\text{(i)}$$

$$v = \sqrt{\frac{B}{\rho}}$$

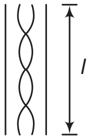
$$\therefore \quad B = \rho v^2$$

$$l = \frac{3\lambda}{2}$$

$$\therefore \quad \lambda = \frac{2l}{3}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2l/3} = \frac{3\pi}{l}$$

$$= \frac{3\pi}{3.9\,\pi} = \frac{1}{1.3} \text{ m}^{-1}$$



Substituting in Eq. (i), we have

$$A = \frac{\Delta p_{\max}}{Bk} = \frac{\Delta p_{\max}}{\rho v^2 k}$$

$$= \frac{(0.01 \times 10^5)}{1.3 \times (200)^2 \times (1/1.3)}$$

$$= 0.025 \text{ m} = 2.5 \text{ cm}$$

3.

$$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i$$

$$\therefore \quad \frac{A_t}{A_i} = \frac{2 \times 100}{200 + 100} = \frac{2}{3}$$

4. $f_b = f_1 - f_2$

or $$10 = 1700\left(\frac{340}{340 + v_2}\right) - 1700\left(\frac{340}{340 + v_1}\right)$$

$$= 1700\left(1 + \frac{v_2}{340}\right)^{-1} - 1700\left(1 + \frac{v_1}{340}\right)^{-1}$$

Using Binomial therefore, we get

$$10 = 1700\left(1 - \frac{v_2}{340}\right) - 1700\left(1 - \frac{v_1}{340}\right)$$

$$= (v_1 - v_2)\left(\frac{1700}{340}\right)$$

$$\therefore \quad v_1 - v_2 = 2 \text{ m/s}$$

5. Velocity of source after 1 s,

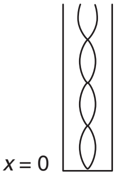
$$v_s = gt = 10 \times 1 = 10 \text{ m/s}$$

$$\Delta f = f\left(\frac{v+v_0}{v-v_s}\right) - f\left(\frac{v-v_0}{v+v_s}\right)$$

$$= 150\left(\frac{300+2}{300-10}\right) - 150\left(\frac{300-2}{300+10}\right)$$

$$= 12 \text{ Hz}$$

6. $L = \frac{7\lambda}{4}$



$$\therefore \qquad \lambda = \frac{4L}{7}$$

$$k = \frac{2\pi}{\lambda}$$

$$= \frac{2\pi}{4L/7} = \frac{7\pi}{2L}$$

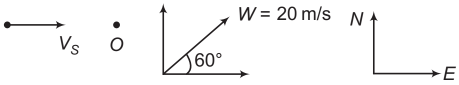
$x = 0$ is a node

$$\therefore \quad A_x = a \sin kx = a \sin\left(\frac{7\pi}{2L}\right)\left(\frac{L}{7}\right) = a$$

7. $\Delta x_{\max} = 3\text{m} = \lambda$

$\therefore$ $f = \frac{v}{\lambda} = \frac{330}{3} = 110$ Hz. For frequencies less than 100 Hz, λ will be more than 3m and no maximum will be obtained.

8.

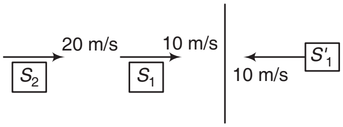


$$f' = f\left(\frac{v + w\cos 60°}{v + w\cos 60° - v_s}\right)$$

$$= 500\left(\frac{300+10}{300+10-20}\right)$$

$$= 534 \text{ Hz}$$

9.



$$f_b = f_{S'} - f_{S_1}$$

$$= 500\left[\frac{340+20}{340-10}\right] - 500\left[\frac{340+20}{340+10}\right]$$

$$\approx 31 \text{ Hz}$$

10. $f_1 = \frac{\omega_1}{2\pi} = \frac{(400\pi)}{2\pi} = 200$ Hz

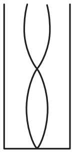
$$f_2 = \frac{\omega_2}{2\pi} = \frac{404\,\pi}{2\pi} = 202 \text{ Hz}$$

$$\therefore \quad f_b = f_2 - f_1 = 2 \text{ Hz}$$

$A_1 = 2$ and $A_2 = 1$

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2 = \frac{9}{1}$$

11. $l = \frac{3\lambda}{4}$



$$\therefore \qquad \lambda = \frac{4l}{3} = \frac{4\,(34)}{3} = 45.33 \text{ cm}$$

$$\lambda = \frac{v}{f}$$

$\lambda \propto v$ (as f = constant)

and $v \propto \sqrt{T}$ (T = temperature)

$$\therefore \qquad \lambda \propto \sqrt{T}$$

$$\therefore \qquad \frac{\lambda_2}{\lambda_1} = \sqrt{\frac{T_2}{T_1}}$$

$$\therefore \qquad \lambda_2 = \left(\sqrt{\frac{T_2}{T_1}}\right)\lambda_1$$

$$= \left(\sqrt{\frac{16+273}{51+273}}\right)(45.33)$$

$$= 42.8 \text{ cm}$$

12. $\because \quad f_1 = f_2$

$$\therefore \qquad 176\left(\frac{330-v}{330-22}\right) = 165\left(\frac{330+v}{330}\right)$$

Solving this equation we get,

$$v = 22 \text{ m/s}$$

13. $M = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$

$$= \frac{(2)\,(32) + (3)\,(48)}{5} = 41.6$$

$$f = \frac{v}{4l} \quad \text{or} \quad f \propto v$$

But, $v = \sqrt{\frac{\gamma RT}{M}}$

or $v \propto \frac{1}{\sqrt{M}}$

$$\therefore \qquad f \propto \frac{1}{\sqrt{M}}$$

$$\therefore \qquad \frac{f_2}{f_1} = \sqrt{\frac{M_1}{M_2}}$$

or $f_2 = \left(\sqrt{\frac{M_1}{M_2}}\right) f_1$

$$= \left(\sqrt{\frac{32}{48}}\right)(200) = 175.4 \text{ Hz}$$

14. $$f = f_0\left(\frac{v + v_0}{v}\right)$$

$$= 10^3\left(1 + \frac{10t}{v}\right) \qquad (\text{as } v_0 = gt)$$

Hence, f *versus* t graph is a straight line of slope $\frac{10^4}{v}$.

$$\therefore \qquad \frac{10^4}{v} = \text{slope} = \frac{100}{3}$$

$$\therefore \qquad v = 300 \text{ m/s}$$

15. $\because \quad v_p = -v\left(\frac{\partial y}{\partial x}\right)$, where $v = +$ ve.

At E, $\frac{\partial y}{\partial x}$ or slope is positive.

Hence, v_p is negative.

At D, $\frac{\partial y}{\partial x}$ or slope is zero. Hence, v_p is zero.

More than One Correct Options

1. $\because \quad f = n\left(\frac{v}{4l}\right) \qquad (n = 1, 3, 5 \dots)$

$$\therefore \qquad l = \frac{nv}{4f} = n\left(\frac{330}{4 \times 264}\right)$$

$$= (0.3125\, n) \text{ m}$$

$$= (31.25\, n) \text{ cm}$$

Now, keep on substituting $n = 1, 3$, etc.

2. (a) Velocity of sound wave in air independent of pressure if T = constant

(b) $v = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{\gamma R\,(t + 273)}{M}}$

or $\quad v^2 \propto (t + 273)$

(c) $v = \sqrt{T/\mu}$

$\therefore \quad v \propto \sqrt{T}$

or $\quad v^2 \propto T$

(d) $f_0 = \frac{v}{2l}$ or $\frac{v}{4l}$

$\Rightarrow \quad f_0 \propto \frac{1}{l}$

4. Let f_1 = fundamental frequency of closed pipe and

f_2 = fundamental frequency of open pipe.

Given, $\quad 5 f_1 = 4 f_2$

or $\quad f_2 = \frac{5}{4} f_1$

$\therefore \quad f_2 > f_1$

(b) First overtone frequency of closed pipe = $3 f_1$ and first overtone frequency of open pipe

$$= 2 f_2 = 2\left(\frac{5}{4} f_1\right) = 2.5\; f_1$$

(c) Fifteenth harmonic of closed pipe = $15 f_1$

Twelfth harmonic of open pipe

$$= 12 f_2 = 12\left(\frac{5}{4} f_1\right) = 15\; f_1$$

(d) Tenth harmonic of closed pipe = $10\; f_1$

Eighth harmonic of open pipe

$$= 8 f_2 = 8\left(\frac{5}{4} f_1\right) = 10 f_1$$

5. $f = \frac{v}{4\,(l + 0.6\text{r})} = \frac{\sqrt{\gamma RT/M}}{4\,(l + 0.6\, r)}$

(d) v does not depend on pressure if temperature is kept constant.

6. During approach,

$$f = f_0\left(\frac{v}{v - v_s}\right)$$

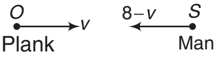
$> f_0$ but f is constant.

During receding.

$$f = f_0\left(\frac{v}{v - v_s}\right)$$

$< f_0$ but f is again constant.

Comprehension Based Questions

1. 

From conservation of linear momentum,

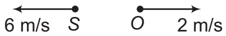
$$50\,(8 - v) = 150 v$$

$$\therefore \qquad v = 2 \text{ m/s}$$

$$8 - v = 6 \text{ m/s}$$

$$f_1 = f_0\left(\frac{v + v_0}{v - v_s}\right)$$

$$= f_0\left(\frac{330 + 2}{330 - 6}\right) = \frac{332}{324}\, f_0$$

2. 

$$f_2 = f_0\left(\frac{v - v_0}{v + v_s}\right)$$

$$= f_0\left(\frac{330 - 2}{330 + 6}\right) = \left(\frac{328}{336}\right) f_0$$

3. $f_1 > f_0$ but f_1 = constant.

Similarly, $f_2 < f_0$ but f_2 = constant

Match the Columns

1. (a) $f = \dfrac{v}{2l}$

$$f_c = \frac{v}{4(2l)} = \frac{v}{8l} = \frac{f}{4}$$

(b) Second overtone frequency

$$= 5f_c = 5(f/4) = 1.25f$$

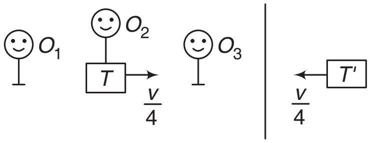
(c) Third harmonic frequency $= 3f_c$

$$= 3(f/4) = 0.75\ f$$

(d) First overtone frequency $= 3f_c$

$$= 3(f/4) = 0.75f$$

2.



(a) $f_b = f_{T'} - f_T$

$$= f\left(\frac{v}{v - v/4}\right) - f\left(\frac{v}{v + v/4}\right)$$

$$= \frac{8}{15}f$$

(b) $f_b = f_{T'} - f_T = f\left(\dfrac{v + v/4}{v - v/4}\right) - f$

$$= \frac{2}{3}f$$

(c) T and T' both are approaching towards O_3.

$\therefore \qquad f_T = f_{T'} \quad \text{or} \quad f_b = 0$

(d) Not T and T' both are receding from O_3.

Hence, again

$$f_T = f_{T'} \quad \text{or} \quad f_b = 0$$

3. Let $f_1 =$ frequency of tuning fork and

$f_2 =$ frequency of streamed wire

Given, $\qquad f_1 > f_2$

$\therefore \qquad f_b = f_1 - f_2$

(a) By loading wax on turning fork f_1 will decrease. Hence, $f_1 - f_2$ may be less than f_b or $f_2 - f_1$ may be greater than f_b.

(b) If prongs are filed, f_1 will increase. Hence,

$$f_1 - f_2 > f_b$$

(c) If tension in stretched wire is increased, f_2 will increase. Hence, $f_1 - f_2$ may be less than f_b or $f_2 - f_1$ may be greater than f_b.

(d) If tension in stretched wire is decreased, f_2 will decrease. Hence,

$$f_1 - f_2 > f_b$$

4. $I \propto A^2 \quad \text{or} \quad A \propto \sqrt{I}$

Due to a point source,

$$I = \frac{P}{4\pi r^2} \qquad (P = \text{power of source})$$

or $\qquad I \propto \dfrac{1}{r^2} \quad \text{and} \quad A \propto \dfrac{1}{r}$

Due to a line source

$$I = \frac{P}{2\pi l} \quad \text{or} \quad I \propto \frac{1}{r}$$

$$\therefore \qquad A \propto \frac{1}{\sqrt{r}}$$

5. $\because \quad k = \pi$

$$\lambda = \frac{2\pi}{k} = 2\text{ m}$$

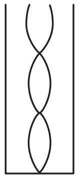
$$l = \frac{5\lambda}{4} = \frac{5}{4}(2)$$

$$= 2.5\text{ m}$$

Distance of displacement node from the closed end

$$= \frac{\lambda}{2} \quad \text{or} \quad \lambda$$

$$= 1\text{ m} \quad \text{or} \quad 2\text{m}$$



Pressure node means displacement antinode. Its distance from closed end

$$= \frac{\lambda}{4} \quad \text{or} \quad \frac{3\lambda}{4} \quad \text{or} \quad \frac{5\lambda}{4}$$

$$= 0.5\text{ m}, 1.5\text{ m and } 2.5\text{ m}$$

Subjective Questions

1. By definition sound level $= 10\log\dfrac{I}{I_0} = 60$

or $\qquad \dfrac{I}{I_0} = 10^6$

$\Rightarrow \qquad I = 10^{-12} \times 10^6 = 1\,\mu\text{W/m}^2$

Power entering the room

$$= 1 \times 10^{-6} \times 2 = 2\,\mu\text{W}$$

Energy collected in a day $= 2 \times 10^{-6} \times 86400$

$$= 0.173\text{ J}$$ **Ans.**

2. (a) At a distance r from a point source of power P, the intensity of the sound is

$$I = \frac{P}{4\pi r^2} = \frac{0.8}{(4\pi)(1.5)^2}$$

or $I = 2.83 \times 10^{-2}$ W/m^2 ...(i)

Further, the intensity of sound in terms of $(\Delta p)_m$, ρ and v is given by

$$I = \frac{(\Delta p)_m^2}{2\rho v} \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$(\Delta p)_m = \sqrt{2 \times 2.83 \times 10^{-2} \times 1.29 \times 340}$$

$$= 4.98 \text{ N/m}^2$$ **Ans.**

(b) Pressure oscillation amplitude $(\Delta p)_m$ and displacement oscillation amplitude A are related by the equation

$$(\Delta p)_m = BAk$$

Substituting $B = \rho v^2$, $k = \frac{\omega}{v}$

and $\omega = 2\pi f$

We get, $(\Delta p)_m = 2\pi A \rho v f$

$$\therefore \quad A = \frac{(\Delta p)_m}{2\pi\rho v f} = \frac{4.98}{(2\pi)(1.29)(340)(600)}$$

$$= 3.0 \times 10^{-6} \text{ m}$$ **Ans.**

3. (a) Fundamental frequency when the pipe is open at both ends is

$$f_1 = \frac{v}{2l} = \frac{340}{2 \times 0.6}$$

$$= 283.33 \text{ Hz}$$ **Ans.**

(b) Suppose the hole is uncovered at a length l from the mouthpiece, the fundamental frequency will be

$$f_1 = \frac{v}{2l}$$

$$\therefore \quad l = \frac{v}{2f_1} = \frac{340}{2 \times 330}$$

$$= 0.515 \text{ m}$$

$$= 51.5 \text{ cm}$$ **Ans.**

Note *Opening holes in the side effectively shortens the length of the resonance column, thus increasing the frequency.*

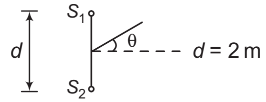
4. $2\sqrt{H^2 + d^2/4} - d = n\lambda$ ...(i)

Now, $2\sqrt{(H+h)^2 + d^2/4} - d = n\lambda + \frac{\lambda}{2}$ ...(ii)

Solving these two equations, we get

$$\lambda = 2\sqrt{4(H+h)^2 + d^2} - 2\sqrt{4H^2 + d^2}$$

5. (a) $\lambda = \frac{v}{f} = \frac{340}{600} = 0.57$ m



$$d \sin\theta = \frac{\lambda}{2}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{\lambda}{2d}\right)$$

$$= \sin^{-1}\left(\frac{0.57}{4}\right)$$

$$= 8.14°$$

(b) $d \sin\theta = \lambda$ or $\theta = \sin^{-1}\left(\frac{\lambda}{d}\right)$

$$= \sin^{-1}\left(\frac{0.57}{2}\right)$$

$$= 16.5°$$

(c) $d \sin\theta = n\lambda$

$$\therefore \quad n_{\max} = \frac{d}{\lambda} \quad (\text{When } \sin\theta = 1)$$

$$= \frac{2}{0.57} = 3.5$$

∴ Maximum 3 maximas can be heard corresponding to $n = 1, 2$ and 3 (beyond $\theta = 0°$)

6. $\Delta\phi_1 = \left(\frac{2\pi}{\lambda}\right)\Delta x = \left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{2} = \pi$

(a) $\Delta\phi_{\text{net}} = \Delta\phi_1 + \Delta\phi_2$

$$= \pi + 0$$

$$= \pi$$

So, they will interfere destructively and

$$I_{\text{net}} = 0$$

(b) No interference will take place. Hence,

$$I_{\text{net}} = I_1 + I_2 = 2I_0$$

(c) $\Delta\phi_{\text{net}} = \Delta\phi_1 \pm \Delta\phi_2$

$$= 2\pi \text{ or } 0 \quad (\text{as } \Delta\phi_2 = \pi)$$

In both conditions, they interfere constructively. Hence,

$$I = I_{\max} = 4I_0$$

7. (a) $\Delta\phi = \frac{2\pi}{\lambda}(\Delta x) = \left(\frac{2\pi f}{v}\right)(\Delta x)$

$$= \frac{(2\pi)(170)}{340}(11 - 8)$$

$$= 3\pi = \phi \text{ (say)}$$

$$I = 4I_0 \cos^2\frac{\theta}{2} \quad \text{...(i)}$$

Substituting value of ϕ, we get

$$I = 0$$

(b) Now, $\phi = 3\pi \pm \pi = 4\pi$ or 2π

Hence, from Eq. (i)

$$I = 4I_0$$

Now, $L_1 - L_2 = 10 \log_{10} \dfrac{I_1}{I_2}$

$$\therefore \quad L_1 - 60 = 10 \log_{10} \left(\frac{4I_0}{I_0}\right) = 6$$

$$\therefore \quad L_1 = 66 \text{ dB}$$

(c) Sources are incoherent.

Hence,

$$I = I_0 + I_0 = 2I_0$$

$$L_1 - L_2 = 10\log_{10} \frac{I_1}{I_2}$$

$$= 10\log_{10} \frac{2I_0}{I_0}$$

$$\therefore \quad L_1 - 60 = 10 \log_{10} 2 = 3$$

or $\quad L_1 = 63$ dB

8. (a) $I = \dfrac{P}{4\pi r^2}$

$$\therefore \quad I_1 = \frac{1 \times 10^{-3}}{4\pi\,(2)^2} = 19.9 \times 10^{-6} \text{ W/m}^2$$

$$= 19.9\,\mu\text{W/m}^2$$

$$I_2 = \frac{1 \times 10^{-3}}{4\pi\,(3)^2} = 8.84 \times 10^{-6} \text{ W/m}^2$$

$$= 8.84\,\mu\text{W/m}^2$$

(b) $I_p = I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$

$$= 55.3\,\mu\text{W/m}^2$$

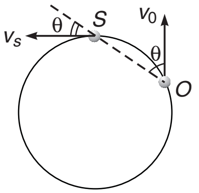
(c) $I_p = I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2$

$$= 2.2\,\mu\text{W/m}^2$$

(d) $I_p = I_1 + I_2$

$$= 28.7\,\mu\text{W/m}^2$$

9. $v_s = v_0 = v$



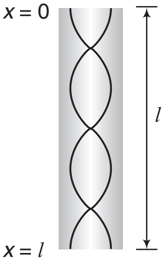
Let u be the speed of sound. Then,

$$f' = f\left(\frac{u + v_0 \cos\theta}{u + v_s \cos\theta}\right)$$

$$= f\left(\frac{u + v \cos\theta}{u + v \cos\theta}\right)$$

or $\quad f' = f$ **Ans.**

10. Given length of pipe, $l = 3$ m



Third harmonic implies that

$$3\left(\frac{\lambda}{2}\right) = l$$

or $\quad \lambda = \dfrac{2l}{3} = \dfrac{2 \times 3}{3} = 2$ m

The angular frequency is

$$\omega = 2\pi f.$$

$$= \frac{2\pi v}{\lambda} = \frac{(2\pi)(332)}{2} \text{ or}$$

$$\omega = 332\pi \text{ rad/s}$$

The particle displacement $y(x, t)$ can be written as

$$y(x, t) = A \cos kx \sin \omega t$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{(2l/3)} = \frac{3\pi}{l}$$

and $\quad \omega = kv = \dfrac{3\pi v}{l} \qquad \left(v = \dfrac{\omega}{k}\right)$

$$\therefore \quad y(x, t) = A \cos\left(\frac{3\pi x}{l}\right) \cdot \sin\left(\frac{3\pi v}{l}\right) t$$

The longitudinal oscillations of an air column can be viewed as oscillations of particle displacement or pressure wave or density wave. Pressure variation is related to particle displacement as

$$p\,(x, t) = -B\frac{\partial y}{\partial x} \qquad (B = \text{Bulk modulus})$$

$$= \left(\frac{3BA\pi}{l}\right) \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi v}{l}\right) t$$

The amplitude of pressure variation is

$$p_{\max} = \frac{3BA\pi}{l} \Rightarrow v = \sqrt{\frac{B}{\rho}} \text{ or } B = \rho v^2$$

$$\therefore \quad p_{\max} = \frac{3\rho v^2 A\pi}{l} \quad \text{or} \quad A = \frac{p_{\max} l}{3\rho v^2 \pi}$$

Here, $p_{\max} = 1\%$ or $p_0 = 10^3$ N/m^2

Substituting the values,

$$A = \frac{(10^3)(3)}{(3)(1.03)(332)^2\pi} = 0.0028 \text{ m}$$

or $\quad A = 0.28$ cm **Ans.**

According to definition of Bulk modulus (B)

$$B = \frac{-dP}{(dV/V)} \qquad \text{...(i)}$$

$$\text{Volume} = \frac{\text{Mass}}{\text{Density}} \quad \text{or} \quad V = \frac{m}{\rho}$$

or $\quad dV = -\dfrac{m}{\rho^2}d\rho = -\dfrac{Vd\rho}{\rho}$

or $\quad \dfrac{dV}{V} = -\dfrac{d\rho}{\rho}$

Substituting in Eq. (i), we get

$$d\rho = \frac{\rho\,(dP)}{B}$$

or amplitude of density oscillation is

$$d\rho_{\max} = \frac{\rho}{B}p_{\max} = \frac{p_{\max}}{v^2} \qquad \left(\frac{\rho}{B} = \frac{1}{v^2}\right)$$

$$= \frac{10^3}{(332)^2} = 9\times 10^{-3}\ \text{kg/m}^3$$

11. Sound level (in dB)

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where, $I_0 = 10^{-12}\ \text{W/m}^2$

$L = 60\ \text{dB}$

Hence, $I = (10^6)I_0 = 10^{-6}\ \text{W/m}^2$

Intensity, $I = \dfrac{\text{Power}}{\text{Area}} = \dfrac{P}{4\pi r^2}$

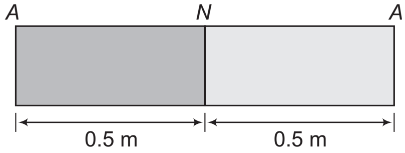
$\Rightarrow \quad P = I(4\pi r^2)$

$P = (10^{-6})(4\pi)(500)^2 = 3.14\ \text{W}$

$$\therefore \quad \text{Time, } t = \frac{(1.0\times 10^3)\left(\dfrac{30}{100}\right)}{3.14}$$

$t = 95.5\ \text{s}$

12. Let n_1 harmonic of the chamber containing H_2 is equal to n_2 harmonic of the chamber containing O_2. Then,



$$n_1\left(\frac{v_{H_2}}{4l}\right) = n_2\left(\frac{v_{O_2}}{4l}\right)$$

$$\therefore \quad \frac{n_1}{n_2} = \frac{v_{O_2}}{v_{H_2}} = \frac{300}{1100} = \frac{3}{11}$$

$$\therefore \quad f_{\min} = 3\left(\frac{v_{H_2}}{4l}\right) = 3\left(\frac{1100}{4\times 0.5}\right)$$

$= 1650\ \text{Hz}$ **Ans.**

13. This problem is a Doppler's effect analogy.

(a) Here, $f = 20\ \text{min}^{-1}$

$v = 300\ \text{m/min}$

$v_s = 0$ and $v_0 = 0$

Spacing between the pies $= \dfrac{300}{20} = 15\ \text{m}$ **Ans.**

and $f' = f = 20\ \text{min}^{-1}$

(b) $v_s = 30\ \text{m/min}$

Spacing between the pies will be

$= \dfrac{300-30}{20}$ or $13.5\ \text{m}$

and $f' = f\left(\dfrac{v}{v-v_s}\right) = (20)\left(\dfrac{300}{300-30}\right)$

$= 22.22\ \text{min}^{-1}$ **Ans.**

14. (a) Comparing with the equation of a travelling wave

$y = a\sin(kx-\omega t)$

$k = 15\pi$ and $\omega = 6000\pi$

$\therefore$ Velocity of the sound, $v = \dfrac{\omega}{k}$

$= \dfrac{6000\pi}{15\pi} = 400\ \text{ms}^{-1}$ as $v = \sqrt{\dfrac{B}{\rho}}$

Hence, $\rho = \dfrac{B}{v^2} = \dfrac{1.6\times 10^5}{(400)^2} = 1\ \text{kg/m}^3$ **Ans.**

(b) Pressure amplitude, $p_0 = BAk$

Hence, $A = \dfrac{p_0}{Bk} = \dfrac{24\pi}{1.6\times 10^5\times 15\pi}$

$= 10^{-5}\ \text{m} = 10\,\mu\text{m}$ **Ans.**

(c) Intensity received by the person,

$$I = \frac{W}{4\pi R^2} = \frac{W}{4\pi(10)^2}$$

Also, $I = \dfrac{p_0^2}{2\rho v} \Rightarrow \dfrac{W}{4\pi(10)^2} = \dfrac{(24\pi)^2}{2\times 1\times 400}$

$W = 288\pi^3\ \text{W}$ **Ans.**

15. (a) Path difference and hence phase difference at P from both the sources is 0°, whether $\theta = 45°$, or $\theta = 60°$. So, both the wave will interfere constructively. Or maximum intensity will be obtained. From

$I_{\max} = (\sqrt{I_1}+\sqrt{I_2})^2$ we have

$I_0 = (\sqrt{I}+\sqrt{I})^2 = 4I \quad (I_1 = I_2 = I, \text{ say})$

$\therefore \quad I = I_0/4$

When one source is switched off, no interference will be obtained. Intensity will be due to a single source or $I_0/4$.

(b) At $\theta = 60°$, also maximum intensity or I_0 will be observed.

16. (a) Frequency of second harmonic in pipe A = frequency of third harmonic in pipe B

$$\therefore \quad 2\left(\frac{v_A}{2l_A}\right) = 3\left(\frac{v_B}{4l_B}\right)$$

or $$\frac{v_A}{v_B} = \frac{3}{4} \Rightarrow \frac{\sqrt{\frac{\gamma_A R T_A}{M_A}}}{\sqrt{\frac{\gamma_B R T_B}{M_B}}} = \frac{3}{4}$$

or $$\sqrt{\frac{\gamma_A}{\gamma_B}}\sqrt{\frac{M_B}{M_A}} = \frac{3}{4} \qquad (\text{as } T_A = T_B)$$

$$\therefore \quad \frac{M_A}{M_B} = \frac{\gamma_A}{\gamma_B}\left(\frac{16}{9}\right) = \left(\frac{5/3}{7/5}\right)\left(\frac{16}{9}\right)$$

or $$\frac{M_A}{M_B} = \left(\frac{25}{21}\right)\left(\frac{16}{9}\right) = \frac{400}{189}$$

(b) Ratio of fundamental frequency in pipe A and in pipe B is

$$\frac{f_A}{f_B} = \frac{v_A/2l_A}{v_B/2l_B} = \frac{v_A}{v_B} \qquad (\text{as } l_A = l_B)$$

$$= \sqrt{\frac{\frac{\gamma_A R T_A}{M_A}}{\frac{\gamma_B R T_B}{M_B}}} = \sqrt{\frac{\gamma_A}{\gamma_B} \cdot \frac{M_B}{M_A}} \qquad (\text{as } T_A = T_B)$$

Substituting $\frac{M_B}{M_A} = \frac{189}{400}$ from part (a), we get

$$\frac{f_A}{f_B} = \sqrt{\frac{25}{21} \times \frac{189}{400}} = \frac{3}{4}$$ **Ans.**

17. Velocity of sound in water is

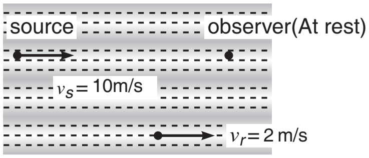
$$v_w = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.088 \times 10^9}{10^3}} = 1445 \text{ m/s}$$

Frequency of sound in water will be

$$f_0 = \frac{v_w}{\lambda_w} = \frac{1445}{14.45 \times 10^{-3}} \text{ Hz}$$

$$f_0 = 10^5 \text{ Hz}$$

(a) Frequency of sound detected by receiver (observer) at rest would be

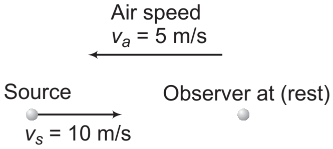


$$f_1 = f_0\left(\frac{v_w + v_r}{v_w + v_r - v_s}\right)$$

$$= 10^5\left(\frac{1445 + 2}{1445 + 2 - 10}\right) \text{ Hz}$$

$$f_1 = 1.0069 \times 10^5 \text{ Hz}$$ **Ans.**

(b) Velocity of sound in air is



$$v_a = \sqrt{\frac{\gamma R T}{M}} = \sqrt{\frac{(1.4)(8.31)(20 + 273)}{28.8 \times 10^{-3}}}$$

$$= 344 \text{ m/s}$$

Frequency does not depend on the medium. Therefore, frequency in air is also $f_0 = 10^5$ Hz.

$\therefore$ Frequency of sound detected by receiver (observer) in air would be

$$f_2 = f_0\left(\frac{v_a - v_w}{v_a - v_w - v_s}\right)$$

$$= 10^5\left[\frac{344 - 5}{344 - 5 - 10}\right] \text{Hz}$$

$$f_2 = 1.0304 \times 10^5 \text{ Hz}$$ **Ans.**

18. Frequency of fundamental mode of closed pipe,

$$f_1 = \frac{v}{4l} = 200 \text{ Hz}$$

Decreasing the tension in the string decrease the beat frequency.

Hence, the first overtone frequency of the string should be 208 Hz (not 192 Hz)

$$\therefore \qquad 208 = \frac{1}{l}\sqrt{\frac{T}{\mu}}$$

$$\therefore \qquad T = \mu \cdot l^2 (208)$$

$$= \left(\frac{2.5 \times 10^{-3}}{0.25}\right)(0.25)^2 (208)^2$$

$$= 27.04 \text{ N}$$ **Ans.**

19. (a) Wavelength of sound ahead of the source is

$$\lambda' = \frac{v - v_s}{f} = \frac{332 - 32}{1000} = 0.3 \text{ m}$$ **Ans.**

(b) $$f' = f\left(\frac{v + v_0}{v - v_s}\right) = 1000\left(\frac{332 + 64}{332 - 32}\right)$$

$$= 1320 \text{ Hz}$$ **Ans.**

(c) Speed of reflected wave will remain 332 m/s. **Ans.**

(d) Wavelength of reflected wave.

$$\lambda'' = \frac{v - v_0}{f'} = \frac{332 - 64}{1320}$$

$$= 0.2 \text{ m}$$ **Ans.**

20. Thermometry, Thermal Expansion and Kinetic Theory of Gases

INTRODUCTORY EXERCISE 20.1

1. (a) $\frac{T_C - 0}{100} = \frac{T_F - 32}{180}$

Putting $T_F = 0$, we get $T_C = -17.8°\text{C}$

(b) $\frac{T_C - 0}{100} = \frac{T_F - 32}{180}$ or $\frac{T_K - 273}{100} = \frac{T_F - 32}{180}$

Putting $T_K = 0$, we get $T_F = -459.67°\text{F}$

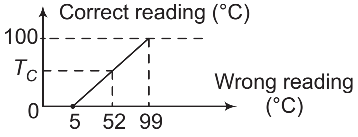
2. (a) $\frac{T_C - 0}{100} = \frac{T_F - 32}{180}$

Putting $T_F = 2T_C$, we get $T_C = 160°\text{C}$

(b) Putting $T_F = \frac{T_C}{2}$, we get $T_C = -24.6°\text{C}$

3. $\frac{T_C}{52 - 5} = \frac{100}{99 - 5}$

$\therefore \ T_C = 50°\text{C}$



Now, using the equation, $\frac{T_C - 0}{100} = \frac{T_F - 32}{180}$

Putting $T_C = 50°\text{C}$, we get $T_F = 122°\text{F}$

4. $\frac{T_C - 0}{100} = \frac{T_F - 32}{180}$ or $\frac{T_K - 273}{100} = \frac{T_F - 32}{180}$

Putting $T_F = T_K$,

We get, T_F or $T_K = 574.25$

5. $\frac{T_C - 0}{100} = \frac{T_F - 32}{180}$

Putting $T_C = T_F$, we get

T_C or $T_F = -40°\text{C}$ or $-40°\text{F}$

INTRODUCTORY EXERCISE 20.2

1. $\Delta T = \frac{1}{2} T \alpha \Delta\theta$

Temperature is decreased. Hence, l or T will decrease. So, it will gain time.

Time gained $= \frac{\Delta T}{T'} t \approx \frac{\Delta T}{T} t$ (as $T' \approx T$)

$= \frac{1}{2} \alpha \Delta\theta t = \frac{1}{2} (1.2 \times 10^{-5}) (30) (24 \times 60 \times 60)$

$= 15.55$ s

2. Density of water will increase by increasing the temperature from 0 to 4°C, then it will decrease. Fraction of volume immersed is given by

$$f_i = \frac{\rho_s}{\rho_l}$$

ρ_l is first increased the decreased first decreased the increased. Or percentage of volume above water level will first increased then decreased.

3. $f_i = \frac{\rho_s}{\rho_l} \Rightarrow f_i = \frac{\rho_s'}{\rho_l'}$

$$\therefore \quad \frac{f_i'}{f_i} = \left(\frac{\rho_s'}{\rho_s}\right)\left(\frac{\rho_l}{\rho_l'}\right) = \left(\frac{1 + \gamma_l \Delta T}{1 + \gamma_s \Delta T}\right) = \left(\frac{1 + \gamma_2 \Delta T}{1 + \gamma_1 \Delta T}\right)$$

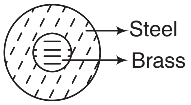
Now, $\frac{1}{1 + \gamma_1 \Delta T} \approx (1 - \gamma_1 \Delta T)$

$$\therefore \quad \frac{f_i'}{f_i} = (1 + \gamma_2 \Delta T)(1 - \gamma_1 \Delta T)$$

Neglecting $\gamma_1 \gamma_2 (\Delta T)^2$ term, we get

$$\frac{f_i'}{f_i} \approx (\gamma_2 - \gamma_1) \Delta T$$

4.



Since, $\alpha_B > \alpha_{Fe}$

On cooling brass will contract more.

5. $\Delta d = d (\alpha_B - \alpha_{Fe}) \Delta\theta$

$$\therefore \quad \Delta\theta = \frac{\Delta d}{d (\alpha_B - \alpha_{Fe})}$$

$$= \frac{0.01}{60 \times 0.8 \times 10^{-5}} = 20.83°\text{C}$$

$\therefore$ New temperature $= (30 + 20.83) = 50.83°\text{C}$

6. (a) $\Delta l = l \alpha \Delta\theta$

$= (88.42)(2.4 \times 10^{-5})(35 - 5)$

$= 0.064$ cm

(b) At higher temperature, it measures less. Hence,

$l = l' + \Delta l$

$= (88.42) + (0.064) = 88.484$ cm

7. $\frac{\Delta l}{l} \times 100 = \frac{-l \alpha \Delta\theta}{l} \times 100$

$= -(\alpha \Delta\theta) \times 100$

$= -(1.2 \times 10^{-5})(35) \times 100$

$= -0.042\%$

INTRODUCTORY EXERCISE 20.3

1. $$pV = nRT = \left(\frac{m}{M}\right) RT$$

$$\therefore \qquad T = \left(\frac{VM}{mR}\right) p$$

At constant volume T - p graph is a straight line of

$$\text{slope}, \qquad = \frac{VM}{mR} \propto \frac{1}{m}$$

Slope of m_1 is less. Hence, m_1 is greater.

2. $p = \dfrac{nRT}{V}$ or $p \propto nT$

$$\therefore \qquad \frac{p_2}{p_1} = \frac{n_2 T_2}{n_1 T_1}$$

$$\text{or} \qquad p_2 = \left(\frac{n_2}{n_1}\right)\left(\frac{T_2}{T_1}\right) p_1$$

$$= \left(\frac{1}{2}\right)\left(\frac{87 + 273}{27 + 273}\right) (20 \text{ atm})$$

$$= 12 \text{ atm}$$

3. $n_1 =$ number of moles of nitrogen $= \dfrac{7}{28} = \dfrac{1}{4}$

$n_2 =$ number of moles of $CO_2 = \dfrac{11}{44} = \dfrac{1}{4}$

$$M = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$$

$$= \frac{\left(\frac{1}{4}\right)(28) + \left(\frac{1}{4}\right)(44)}{\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)} = 36$$

$$\text{Now,} \qquad \rho = \frac{RM}{RT}$$

$$= \frac{(1.01 \times 10^5)(36 \times 10^{-3})}{(8.31)(290)}$$

$$= 1.5 \text{ kg/m}^3$$

4. $$pV = nRT$$

$$\therefore \qquad n = \frac{pV}{RT} = \frac{(\rho g h) V}{RT}$$

$$= \frac{(13.6 \times 10^3)(9.8)(10^{-6})(250 \times 10^{-6})}{8.31 \times 300}$$

$$= 1.33 \times 10^{-8}$$

$\therefore$ Number of molecules $= (n) N_A$

$$= (1.33 \times 10^{-8})(6.02 \times 10^{23})$$

$$= 8 \times 10^{15}$$

5. $$V = \frac{nRT}{p} = \left(\frac{nR}{p}\right) T$$

For given mass, V - T graph is a straight line passing through origin having

$$\text{slope} = \frac{nR}{p} \propto \frac{1}{p}$$

Slope of p_2 is more. Hence, $p_2 < p_1$.

6. $$p = (nRT)\frac{1}{V}$$

i.e. p *versus* $\dfrac{1}{V}$ graph is a straight line passing through origin of slope nRT.

7. pV and T both are constants.

INTRODUCTORY EXERCISE 20.4

1. Internal energy of n moles of an ideal gas at temperature T is given by

$$U = n\left(\frac{f}{2} RT\right)$$

where, $f =$ degrees of freedom

$= 5$ for O_2 and 3 for Ar

$$\text{Hence,} \quad U = U_{O_2} + U_{\text{Ar}}$$

$$= 2\left(\frac{5}{2} RT\right) + 4\left(\frac{3}{2} RT\right)$$

$$= 11 RT$$

2. Average translational kinetic energy of an ideal gas molecule is $\dfrac{3}{2} kT$ which depends on temperature only. Therefore, if temperature is same, translational kinetic energy of O_2 and N_2 both will be equal.

3. $$\frac{f_T}{f_R} = \frac{3}{2} = \frac{K_T}{K_R} = \frac{K_T}{K_O}$$

$$\therefore \qquad K_T = \frac{3}{2} K_O$$

INTRODUCTORY EXERCISE 20.5

1. $$v_{\text{rms}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3 \times 8.31 \times 373.15}{2 \times 10^{-3}}}$$

$$= 2156 \text{ m/s} = 2.15 \text{ km/s}$$

2. $$v_{\text{av}} = \frac{500 + 600 + 700 + 800 + 900}{5} = 700 \text{ m/s}$$

$$v_{\text{rms}} = \sqrt{\frac{(500)^2 + (600)^2 + (700)^2 + (800)^2 + (900)^2}{5}} = 714 \text{ m/s}$$

4. Helium gas is monoatomic. So, its degree of freedom $f = 3$. Average kinetic energy of 1 molecule of gas

$$= \frac{f}{2} kT = \frac{3}{2} kT$$

$$= \frac{3}{2}(1.38 \times 10^{-23})(300)$$

$$= 6.21 \times 10^{-21} \text{ J}$$

5. (a) $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$

For He gas : $v_{\text{rms}} = \sqrt{\frac{3 \times 8.31 \times 300}{4 \times 10^{-3}}}$

$$= 1368 \text{ m/s}$$

For Ne gas : $v_{\text{rms}} = \sqrt{\frac{3 \times 8.31 \times 300}{20.2 \times 10^{-3}}}$

$$= 609 \text{ m/s}$$

(b) Each gas is monoatomic for which degree of freedom $f = 3$. Hence, average kinetic energy of one atom

$$= \frac{f}{2} kT = \frac{3}{2} kT$$

$$= \left(\frac{3}{2}\right)(1.38 \times 10^{-23})(300)$$

$$= 6.21 \times 10^{-21} \text{ J}$$

6. $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$

$$\therefore \quad T = \frac{mv^2_{\text{rms}}}{3R} = \frac{(4 \times 10^{-3})(10^3)^2}{3 \times 8.31}$$

$$= 160 \text{ K}$$

7. Suppose n_1 molecules have v_1 velocity and n_2 molecules have v_2 velocity. Then,

$$v_{\text{av}} = \frac{n_1 v_1 + n_2 v_2}{n_2 + n_2}$$

But, $v_{\text{rms}} = \sqrt{\frac{n_1 v_1^2 + n_2 v_2^2}{n_1 + n_2}}$

Now, $v_{\text{rms}} \geq v_{\text{av}}$ because v_1 and v_2 may be in opposite directions also.

Exercises

LEVEL 1

Assertion and Reason

1. Straight line passing through origin represents isochoric process.

3. In monoatomic gas, there is no rotational degree of freedom.

4. Temperature of the gas is associated with random or disordered motion of the gas molecules. By the motion of container total kinetic energy of gas molecules will increase. But temperature will not increase.

5. Internal energy is equally distributed in all degrees of freedom. For example

$$\frac{K_T}{K_R} = \frac{3}{2} \text{ in diatomic gas}$$

As $\quad f_T = 3$ and $f_R = 2$

7. Density of water is maximum at 4°C. Hence, volume is minimum at 4°C.

8. $U \propto T \propto pV$

9. $p = \frac{1}{3}\left(\frac{mN}{V}\right) v^2_{\text{rms}}$

$$\frac{mN}{V} = \frac{\text{Total mass of gas}}{\text{Volume of gas}} = \text{density of gas}$$

Objective Questions

2. $v_{\text{rms}} = \sqrt{\frac{(1)^2 + (0)^2 + (2)^2 + (3)^4}{4}}$

$$= \sqrt{3.5} \text{ m/s}$$

3. $T_1 = 273 + 27 = 300$ K

$T_2 = 273 + 927 = 1200$ K

$v_{\text{rms}} \propto \sqrt{T}$

T has become four times.

Therefore, v_{rms} will become two times.

4. Hydrogen and oxygen both are diatomic gases. Therefore, average kinetic energy per molecule is

$$\frac{f}{2} kT$$

or $$\frac{5}{2} kT \qquad (\text{as } f = 5)$$

5. $E \propto T$

$$\therefore \quad \frac{T_2}{T_1} = \frac{E_2}{E_1} = \frac{2E}{E} = 2$$

$$\therefore \quad T_2 = 2T_1 = 2(273 + 10)$$

$$= 566 \text{ K} = 293°\text{C}$$

7. If $p =$ constant, then $V \propto T$

8. It means rod is compressed from its natural length by Δl.

$$\therefore \quad \text{Strain} = -\frac{\Delta l}{l} = -\frac{l\Delta\alpha\Delta\theta}{l}$$

$$= (-\alpha d\theta) = -(12\times10^{-6})(50)$$

$$= -6\times10^{-4}$$

9. $\Delta l_1 = \Delta l_2$

$$\therefore \quad l_1\alpha_1\Delta\theta = l_2\alpha_2\Delta\theta$$

$$\text{or} \quad \frac{l_1}{l_2} = \frac{\alpha_2}{\alpha_1} = \frac{9}{11}$$

In option (a), ratio is $\frac{19}{11}$.

10. $V = \frac{nRT}{p} = \frac{mR(t+273)}{Mp}$

$\frac{m}{p}$ ratio is same. Therefore, slope and intercept of straight line between V and t should also remain same.

Subjective Questions

1. **For Q.Nos. 1 and 2**

Apply the formula,

$$\frac{T_C - 0}{100} = \frac{T_K - 273}{100} = \frac{T_F - 32}{180} = \frac{T_R - 0}{80}$$

3. In the above equation, putting $T_F = T_C$,

We get $T_F = -40°\text{F}$ or $T_C = -40°\text{C}$

4. **Initially** $T_C = \left(\frac{T_F - 32}{180}\right)(100)$

$$= \left(\frac{68.2 - 32}{180}\right)(100)$$

$$= 20.11°\text{C}$$

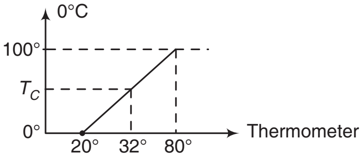
Finally $T_C = 20.11 + 40 = 60.11°\text{C}$

$$\therefore \quad T_F = 32 + \left(\frac{180}{100}\right)T_C$$

$$= 32 + \left(\frac{180}{100}\right)(60.11)$$

$$= 140.2°\text{F}$$

5.



$$\frac{T_C}{32-20} = \frac{100}{80-20} \quad \text{or} \quad T_C = 20°\text{C}$$

6. The temperature on the platinum scale is

$$t = \frac{R_t - R_0}{R_{100} - R_0}\times100°$$

$$= \frac{86-80}{90-80}\times100°\text{C}$$

$$= 60°\text{C}$$ **Ans.**

7. Let the relation between the thermometer reading and centigrade be $y = ax + b$

Given, at $x = 100$, $y = 80$ and at $x = 0$, $y = 10$

$\therefore \quad 80 = 100a + b, 10 = b$

$\Rightarrow \quad a = 0.7$

Now, we have to find x when $y = 59$

$\therefore \quad 59 = 0.7x + b \quad \Rightarrow \quad x = 70$

$\therefore$ The answer is 70°C. **Ans.**

8. If T be the corresponding temperature,

$$\sqrt{\frac{3RT}{M_O}} = \sqrt{\frac{3R(300)}{M_H}}$$

$$\therefore \quad T = (300)\left(\frac{M_O}{M_H}\right)$$

$$= 4800\text{ K}$$ **Ans.**

9. Mass of 6.02×10^{23} molecules is 17 g or 0.017 kg.

$\therefore$ Mass of one NH_3 molecule

$$= \frac{0.017}{6.02\times10^{23}}\text{ kg}$$

$$= 2.82\times10^{-26}\text{ kg}$$

10. $\frac{n_1 + n_2}{\gamma - 1} = \frac{n_1}{\gamma_1 - 1} + \frac{n_2}{\gamma_2 - 1}$

$$\therefore \quad \frac{3+2}{\gamma-1} = \frac{3}{1.67-1} + \frac{2}{1.4-1}$$

$$\frac{5}{\gamma-1} = 9.48$$

Solving, we get $\gamma = 1.53$

11. $v = \sqrt{\frac{\gamma p}{\rho}}$

$$\therefore \quad \gamma = \frac{\rho v^2}{p} = \frac{1.3\times(330)^2}{1.013\times10^5} = 1.4$$

$$\gamma = 1 + \frac{2}{f} \qquad (f = \text{degree of freedom})$$

$$1.4 = 1 + \frac{2}{f}$$

Solving, we get

$$f = 5$$

12. 4 g hydrogen = 2 moles hydrogen

11.2 litre He at STP = $\frac{1}{2}$ mole of He

$$p = p_H + p_{He} = (n_H + n_{He})\frac{RT}{V}$$

$$= \left(2 + \frac{1}{2}\right)\frac{8.31 \times (300\text{K})}{(20 \times 10^{-3})\text{m}^3}$$

$$= 3.12 \times 10^5 \text{ N/m}^2$$ **Ans.**

13. Since, pressure is constant.

$\therefore \quad V \propto T$

$\therefore \quad \frac{V_i}{T_i} = \frac{V_f}{T_f}$

$\therefore \quad T_f = \frac{V_f}{V_i} T_i$

$\Rightarrow \quad T_f = 2T_i = 600\text{K}$

$\therefore \quad \Delta U = \frac{f}{2} nR\Delta T$

$= \frac{3}{2} R(600 - 300) = 450R$ **Ans.**

14. From energy conservation principle,

$$E_i = E_f$$

or $$E_1 + E_2 = E$$

$$\therefore \; n_1\left(\frac{3}{2}RT_1\right) + n_2\left(\frac{3}{2}RT_2\right) = (n_1 + n_2)\left(\frac{3}{2}RT\right)$$

$$\therefore \quad T = \frac{n_1T_1 + n_2T_2}{n_1 + n_2}$$ **Ans.**

15. Number of gram moles $= \frac{1}{18} = n$

Avogadro number, $N_A = 6.02 \times 10^{23}$/g-mol

$\therefore$ Total number of molecules,

$$N = nN_A = 3.34 \times 10^{22}$$

$\therefore$ Number of molecules per cm^2

$$= \frac{N}{4\pi R^2} \qquad (R \to \text{in cm})$$

$$= \frac{3.34 \times 10^{22}}{(4\pi)(6400 \times 10^5)^2} = 6500$$

16. $pV = nRT$

$\therefore \quad nRT = pV$

$U = \frac{f}{2}(nRT) \quad (f = \text{degree of freedom})$

$\therefore \quad U = \frac{f}{2}(pV)$

or $\quad U \propto fpV$

f has become $\frac{5}{3}$ time (3 for He and 5 for O_2). p has become $\frac{3}{2}$ times and V has become 3 times.

$\therefore \quad U$ will become $\frac{5}{3} \times \frac{3}{2} \times 3$ or 7.5 times.

17. $v_{\text{rms}} = \sqrt{\frac{3RT}{M}} \propto \sqrt{\frac{T}{M}}$

v_{rms} is same. Hence,

$$\left(\frac{T}{M}\right)_{H_2} = \left(\frac{T}{M}\right)_{O_2}$$

or $$T_{H_2} = \left(\frac{M_{H_2}}{M_{O_2}}\right)T_{O_2}$$

$$= \left(\frac{2}{32}\right)(273 + 47)$$

$$= 20 \text{ K} = -253°\text{C}$$

18. (i) $\sqrt{\frac{3RT}{M}} = 11.2 \times 10^3$ m/s

$$\therefore \quad T = \frac{(11.2 \times 10^3)^2 M}{3R} \qquad \text{…(i)}$$

$$= \frac{(11.2 \times 10^3)^2 \times 2 \times 10^{-3}}{3 \times 8.31}$$

$$= 10059 \text{ K}$$

(ii) Escape velocity from moon

$$= \sqrt{2g_m R_m}$$

$$= \sqrt{1.6 \times 2 \times 1750 \times 10^3}$$

$$= 2366 \text{ m/s}$$

Substituting in Eq. (i), we have

$$T = \frac{(2366)^2 \times 2 \times 10^{-3}}{3 \times 8.31}$$

$$= 449 \text{ K}$$

19. At constant volume,

$$p \propto T$$

$$\therefore \quad \frac{p_1}{T_1} = \frac{p_2}{T_2}$$

$$\therefore \quad T_2 = \left(\frac{p_2}{p_1}\right)T_1$$

$$= \left(\frac{160}{80}\right)(273) = 546 \text{ K}$$

20. $t = \left(\frac{R_t - T_0}{R_{100} - R_0}\right) \times 100 = \left(\frac{6.5 - 2.5}{3.5 - 2.5}\right) \times 100$

$$= 400°$$

21. At constant volume,

$$p \propto T$$

$$\therefore \qquad \frac{p_2}{p_1} = \frac{T_2}{T_1} \quad \text{or} \quad T_2 = \left(\frac{p_2}{p_1}\right) T_1$$

$$= \left(\frac{45 + 75}{5 + 75}\right)(273 + 30)$$

$$= 450 \text{ K} = 450 - 273$$

$$= 177°\text{C}$$

22. Strain $= \frac{\Delta l}{l} = (\alpha \Delta \theta)$

Stress $= Y \times$ strain $= \gamma \alpha \Delta \theta$

$\therefore$ Tension $= (A)$ (stress) $= YA\alpha\Delta\theta$

Substituting the value, we get

$$T = YA\alpha\Delta\theta$$

$$= (2.0 \times 10^{11})(2 \times 10^{-6})(1.2 \times 10^{-5})(40)$$

$$= 192 \text{ N}$$

23. Change in weight = upthrust on 100% volume

$$\therefore \qquad \frac{\Delta W'}{\Delta W} = \frac{F'}{F} = \left(\frac{1 + \gamma_s \, \Delta\theta}{1 + \gamma_l \, \Delta\theta}\right)$$

$$\text{or} \qquad \frac{(50 - 45.1)}{(50 - 45)} = \frac{1 + (12 \times 10^{-6})(75)}{1 + \gamma_l \, (75)}$$

Solving this equation, we get

$$\gamma_l = 3.1 \times 10^{-4} \text{ per°C}$$

24. (a) $n = \frac{pV}{RT} = \frac{(1.52 \times 10^6)(10^{-2})}{(8.31)(298.15)}$

$$= 6.135$$

(b) $\rho = \frac{pM}{RT} = \frac{(1.52 \times 10^6)(2 \times 10^{-3})}{8.31 \times 298.15}$

$$= 1.24 \text{ kg/m}^3$$

(c) $\rho \propto M$ (if P and T are same)

$$\therefore \qquad \frac{\rho_{O_2}}{\rho_{H_2}} = \frac{M_{O_2}}{M_{H_2}}$$

$$\therefore \qquad \rho_{O_2} = \left(\frac{M_{O_2}}{M_{H_2}}\right)\rho_{H_2}$$

$$= \left(\frac{32}{2}\right)(1.23)$$

$$= 19.68 \text{ kg/m}^3$$

25. If T = constant, then

$$p_1 V_1 = p_2 V_2$$

$$\therefore \qquad p_2 = p_1 \left(\frac{V_1}{V_2}\right) = (1 \text{ atm}) = \left(\frac{70}{6}\right)$$

$$= 11.7 \text{ atm}$$

26. $n_1 = n_2$

$$\therefore \qquad \frac{p_1 V_1}{RT_1} = \frac{p_2 V_2}{RT_2}$$

$$\therefore \qquad V_2 = \left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)V_1 = \left(\frac{1}{0.5}\right)\left(\frac{270}{300}\right)(500)$$

$$= 900 \text{ m}^3$$

27. p = constant

$$\therefore \qquad V \propto T$$

$$\text{or} \qquad h \propto T$$

$$\therefore \qquad \frac{h_f}{T_f} = \frac{h_i}{T_i}$$

$$\text{or} \qquad h_f = \left(\frac{T_f}{T_i}\right) h_i$$

$$= \left(\frac{273 + 100}{273 + 20}\right)(4) = 5.09 \text{ cm}$$

28. $n_1 = \frac{0.025}{28} = 8.93 \times 10^{-4}$

$$n_2 = \frac{0.04}{4} = 0.01$$

$$\therefore \qquad \frac{n_1}{n_2} = 0.089$$

In equilibrium p and T are same.

$$\therefore \qquad pV = nRT$$

$$\therefore \qquad V \propto n \quad \text{or} \quad L \propto n$$

$$\therefore \qquad \frac{L_1}{L_2} = \frac{n_1}{n_2} = 0.089$$

29. $n = n_1 + n_2$

$$\frac{pV}{RT} = \frac{p_1 V_1}{RT} + \frac{p_2 V_2}{RT}$$

$$\therefore \qquad p = \frac{p_1 V_1 + p_2 V_2}{V}$$

$$= \frac{(0.11)(1.38) + (0.16)(0.69)}{0.11 + 0.16}$$

$$= 0.97 \text{ MPa}$$

30. $(n_1 + n_2)_1 = (n_1 + n_2) f$

$$\therefore \qquad \frac{p_i V_1}{RT_i} + \frac{p_i V_2}{RT_i} = \frac{pV_1}{RT_1} + \frac{pV_2}{RT_2}$$

$$\therefore \qquad P = \frac{P\,(V_1 + V_2)}{T_i\,(V_1/T_1 + V_2/T_2)}$$

$$= \frac{(1 \text{ atm})(600)}{(293)\left(\frac{400}{373} + \frac{200}{273}\right)}$$

$$= 1.13 \text{ atm}$$

31. $V = \dfrac{nRT}{p} = \dfrac{(1)\,(8.31)\,(273.15)}{1.013 \times 10^5}$

$= 0.0224 \text{ m}^3 = 22.4 \text{ L}$

32. $n_1 = n_2$

$\therefore \quad \dfrac{p_1V_1}{RT_1} = \dfrac{p_2V_2}{RT_2} \quad \text{or} \quad p_2 = \left(\dfrac{V_1}{V_2}\right)\left(\dfrac{T_2}{T_1}\right)p_1$

$= \left(\dfrac{0.75}{0.48}\right)\left(\dfrac{273 + 157}{273 + 27}\right)(1.5 \times 10^5)$

$= 3.36 \times 10^5 \text{ Pa}$

33. $n_1 =$ number of moles of $N_2 = \dfrac{1.4}{28} = 0.05$

30% of the gas dissociates into atoms.

$\therefore \quad n_1' = 2\,(0.3)\,(0.05) + (0.7)\,(0.05) = 0.065$

$n_2 =$ number of moles of He $= \dfrac{0.4}{4} = 0.1$

$\therefore \quad n = n_1' + n_2 = 0.165$

Now, $P = \dfrac{nRT}{V} = \dfrac{(0.165)\,(8.31)\,(1500)}{(5 \times 10^{-3})}$

$= 4.1 \times 10^5 \text{ N/m}^2$

34. In diatomic gas rotational degree of freedom is two.

$\therefore$ Rotational KE of one molecule

$\dfrac{1}{2} I\,\omega^2 = \dfrac{f}{2} kT = kT \qquad (\text{as } f = 2)$

$\therefore \quad \omega = \sqrt{\dfrac{2kT}{I}}$

$= \sqrt{\dfrac{2 \times 1.38 \times 10^{-23} \times 300}{8.28 \times 10^{-38} \times 10^{-9}}}$

$= 10^{13} \text{ rad/s}$

35. (a) $C_p = (1 + f/2)\,R$

$29 = (1 + f/2)\,8.31$

Solving, we get

$f = 5$

(b) $pT =$ constant

$\therefore \quad p\,(pV) =$ constant

or $\quad pV^{1/2} =$ constant

C in the process $pV^x =$ constant is given by

$$C = C_V + \frac{R}{1 - x} = \frac{f}{2}R + \frac{R}{1 - x}$$

$\therefore \quad 29 = 8.31\left(\dfrac{f}{2} + \dfrac{1}{1 - 1/2}\right)$

Solving, we get $\quad f = 3$

36. (a) Average translational kinetic energy of any type of molecules is

$\dfrac{3}{2} kT = \dfrac{3}{2}(1.38 \times 10^{-23})\,(300)$

$= 6.21 \times 10^{-21}\text{J}$

(b) $\dfrac{2}{5}$ energy is associated with rotation. Hence, it is diatomic gas.

$Q = nC_V\Delta T = (1)\left(\dfrac{5}{2}R\right)(1)$

$= \dfrac{5}{3} \times 8.31 = 20.8 \text{ J}$

37. $C_p = \dfrac{n_1C_{p_1} + n_2C_{p_2}}{n_1 + n_2}$

$= \dfrac{2.5R + 3.5R}{1 + 1} = 3R$

$C_V = \dfrac{n_1C_{V_1} + n_2C_{V_2}}{n_1 + n_2}$

$= \dfrac{1.5R + 2.5R}{2} = 2R$

$\gamma = \dfrac{C_p}{C_V} = \dfrac{3}{2} = 1.5$

38. $pV^{-b} =$ constant

$C = \dfrac{R}{\gamma - 1} + \dfrac{R}{1 - (-b)}$

$C = 0$, if $\gamma - 1 = -(1 + b)$

$\therefore \quad b = -\gamma$

39. $pV^{-1} =$ constant

$\therefore \quad C = C_V + \dfrac{R}{1 - x}$

Here $\quad x = -1$

$\therefore \quad C = C_V + \dfrac{R}{2}$

40. $K = \dfrac{3}{2}kT \quad \Rightarrow \quad T = \dfrac{2K}{3k}$

Now, $n = \dfrac{pV}{RT} = \dfrac{3kpV}{2KR}$

$= \dfrac{3\,(200 \times 10^3)\,(0.1 \times 10^{-3})\,(1.38 \times 10^{-23})}{2 \times 6 \times 10^{-21} \times 8.31}$

$= 8.3 \times 10^{-3}$ mol

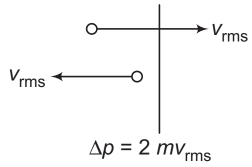
Now, $N = nN_A$

$= (8.3 \times 10^{-3})\,(6.02 \times 10^{23})$

$= 5 \times 10^{21}$

41. $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times (273 + 57)}{46 \times 10^{-3}}}$

$= 423 \text{ m/s}$



Mass of one gas molecule,

$$m = \frac{46}{6.02 \times 10^{23}} \text{ g}$$

$$= 7.64 \times 10^{-26} \text{ kg}$$

Let n molecules strike per second per unit area. Then,

$$\text{Pressure} = \frac{F}{A} = \frac{\Delta P/\Delta t}{A} = 2mn\, v_{\text{rms}}$$

$$\therefore \quad n = \frac{\text{Pressure}}{(2m)\, v_{\text{rms}}} = \frac{2 \times 1.013 \times 10^5}{2 \times 7.64 \times 10^{-26} \times 423}$$

$$= 3.1 \times 10^{27}$$

42. $K_T = \dfrac{3}{2} kT = \dfrac{3}{2} k \left(\dfrac{pV}{nR}\right)$

$$= \frac{3}{2} k \frac{pV}{(M/mN_A)\, R}$$

$$= \frac{3kPV\ m\, N_A}{2MR}$$

$$= \frac{(3 \times 1.38 \times 10^{-23})\,(100 \times 10^3)\,(2 \times 10^{-6})\,(8.0 \times 10^{-23})\,(6.02 \times 10^{23})}{2 \times (50 \times 10^{-3})\,(8.31)}$$

$$= 4.8 \times 10^{-22} \text{ J}$$

43. (a) $v_{\text{rms}} \propto \sqrt{T}$

$$\therefore \quad v_0' = \left(\sqrt{\frac{T'}{T}}\right) v_0 = \left(\sqrt{\frac{573}{293}}\right) v_0$$

$$= 1.40\, v_0$$

(b) v_{rms} does not depend on pressure, as long as temperature remains constant.

(c) $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}}$ or $v_{\text{rms}} \propto \dfrac{1}{\sqrt{M}}$

44. (a) $K_T = \dfrac{3}{2} kT$

$$= \frac{3}{2}\,(1.38 \times 10^{-23})\,(300)$$

$$= 6.21 \times 10^{-21} \text{ J}$$

(b) K_T in one mole $= \dfrac{3}{2} RT$

$$= \frac{3}{2} \times 8.31 \times 300$$

$$= 3740 \text{ J}$$

(c) $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times 300}{32 \times 10^{-3}}}$

$= 484 \text{ m/s}$

45. Let mass of nitrogen $= (m)$ g. Then, mass of oxygen $= (100 - m)$ g. Number of moles of nitrogen, $n_1 = \dfrac{m}{28}$ and number of moles of oxygen

$$n_2 = \left(\frac{100 - m}{32}\right)$$

For air

$$\rho = \frac{pM}{RT} = \frac{p}{RT}\left(\frac{m_1 + m_2}{n_1 + n_2}\right)$$

$$\therefore \quad 1.284 = \frac{1.01 \times 10^5}{8.31 \times 273}\left[\frac{100 \times 10^{-3}}{(m/28) + (100 - m)/32}\right]$$

Solving this equation, we get

$$m = 76.5 \text{ g}$$

This is also percentage of N_2 by mass on air as total mass we have taken is 100 g.

46. $n_1 = n_2$

$$\therefore \quad \frac{p_1 V_1}{RT_1} = \frac{p_2 V_2}{RT_2}$$

$$\therefore \quad V_2 = \frac{p_1 V_1 T_2}{p_2 T_1} = \frac{(p_0 + \rho g h)\, V_1\, T_2}{p_0 T_1}$$

$$= \frac{(1.01 \times 10^5 + 10^3 \times 10 \times 40)\,(20)\,(273 + 20)}{1.01 \times 10^5\,(273 + 4)}$$

$$\approx 105 \text{ cm}^3$$

47. (a) Molar heat capacity $C_p = \dfrac{Q}{n\Delta T}$

and heat capacity $C_p = \dfrac{Q}{\Delta T}$

$\therefore \quad C_p = nC_P$

Similarly, $\quad C_v = nC_V$

Now, $\quad C_p - C_v = n\,(C_P - C_V) - nR$

$$\therefore \quad n = \frac{C_p - C_v}{R} = \frac{29.1}{8.31} = 3.5$$

(b) $C_p = nC_P = n\left(\dfrac{5}{2}R\right)$

$= (3.5)\,(2.5)\,(8.31)$

$= 72.75 \text{ J/K}$

$$C_v = nC_V = n\left(\frac{3}{2}R\right)$$
$$= (3.5)\,(1.5)\,(8.31)$$
$$= 43.65 \text{ J/K}$$

(c) $C_p = nC_P = n\left(\frac{7}{2}R\right)$
$$= (3.5)\,(3.5)\,(8.31)$$
$$= 101.8 \text{ J/K}$$
$$C_v = nC_V = n\left(\frac{5}{2}R\right)$$
$$= (3.5)\,(2.5)\,(8.31)$$
$$= 72.75 \text{ J/K}$$

48. (a) As discussed in the above problem,
$$C_v = nC_V$$
$$\therefore \quad n = \frac{C_v}{C_V} = \frac{C_v}{(3/2)R}$$
$$= \frac{35}{1.5 \times 8.31} = 2.81$$

(b) Internal energy,
$$U = n\left(\frac{f}{2}RT\right)$$
$$f = \text{degree of freedom} = 3$$
$$\therefore \quad U = (2.81)\left(\frac{3}{2}\right)(8.31)\,(273)$$
$$= 9562 \text{ J} = 9.56 \text{ kJ}$$

(c) $C_P = \frac{5}{2}R = \left(\frac{5}{2}\right)(8.31)$
$$= 20.8 \text{ J/mol-K}$$

LEVEL 2

Single Correct Option

1. $(n_1 + n_2)_i = (n_1 + n_2)_f$
$$\therefore \quad \frac{p_1V_1}{RT_1} + \frac{p_2V_2}{RT_2} = \frac{p\,(V_1 + V_2)}{RT}$$

Solving, we get $\quad T = \dfrac{T_1T_2p\,(V_1 + V_2)}{p_1V_1T_2 + p_2V_2T_1}$

2. $\Delta l = l\alpha\Delta\theta$
$$\therefore \quad \alpha = \frac{\Delta l}{l\,\Delta\theta} = \frac{0.008}{(10)\,(100)} = 8 \times 10^{-6} \text{ per °C}$$
$$\gamma = 3\alpha = 2.4 \times 10^{-5} \text{ per°C}$$
$$\Delta V = V\gamma\,\Delta\theta = (100)\,(2.4 \times 10^{-5})\,(100)$$
$$= 0.24 \text{ cc}$$
$$\therefore \quad V' = V + \Delta V = 100.24 \text{ cc}$$

3. $T = mV + C$
$$\therefore \quad \frac{pV}{nR} = mV + C$$
$$\text{or} \quad p = nR\left(m + \frac{C}{V}\right)$$

$\therefore$ p-V graph is as shown in graph (a).

4. $p^2V = \text{constant}$
$$\therefore \quad \left(\frac{nRT}{V}\right)^2 V = \text{constant}$$
$$\therefore \quad T^2 \propto V$$
$$\text{or} \quad T \propto \sqrt{V}$$

V is made three times. So, T will become $\sqrt{3}$ times.

5. $n = \dfrac{pV}{RT}$
$$\therefore \quad n \propto \frac{1}{T} \quad \text{or} \quad m \propto \frac{1}{T}$$
$$\frac{m_1}{m_2} = \frac{T_2}{T_1} = \frac{273 + 37}{273 + 7}$$
$$= \frac{310}{280} = 1.1$$

6. $v_{\text{av}} = \sqrt{\dfrac{8RT}{\pi M}}$

T and M are same. Hence, average speed of O_2 molecules in both vessels will remain same or v_1.

7. $n = \dfrac{pV}{RT}$
$$\therefore \quad N = nN_A = \frac{pVN_A}{RT}$$
$$= \frac{(1.013 \times 10^5 \times 10^{-13})\,(10^{-6})\,(6.02 \times 10^{23})}{(8.31)\,(300)}$$
$$= 2.4 \times 10^6$$

8. $v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times 273}{28 \times 10^{-3}}}$
$$= 493 \text{ m/s}$$

Now, increase in gravitation PE

= decrease in kinetic energy
$$\therefore \quad \frac{(mg\,h)}{1 + \dfrac{h}{R}} = \frac{1}{2}m\,v_{\text{rms}}^2 \quad \text{or} \quad \frac{2gh}{1 + (h/R)} = v_{\text{rms}}^2$$
$$\text{or} \quad \frac{2 \times 9.81 \times h}{1 + (h/6400 \times 10^3)} = (493)^2$$

Solving this equation, we get
$$h \approx 12000 \text{ m}$$
$$\text{or} \quad 12 \text{ km}$$

9. $U \propto T$

According to given graph,

$$p \propto V$$

$$\therefore \qquad p \propto T$$

Hence, $\qquad V =$ constant

10. 28 g of N_2 mean 1 g-mol

Now, $\quad n = \dfrac{pV}{RT} \quad$ or $\quad n \propto \dfrac{p}{T}$

$$\frac{n_f}{n_i} = \left(\frac{p_f}{p_i}\right)\left(\frac{T_i}{T_f}\right) = \left(\frac{1}{2}\right)\left(\frac{273+57}{273+27}\right) = \frac{11}{20}$$

or $\quad n_f = \left(\dfrac{11}{20}\right) n_i = \dfrac{11}{20}$ mol

$$\Delta n = n_i - n_f = 1 - \frac{11}{20} = \frac{9}{20} \text{ g - mol}$$

$$\Delta m = \frac{9}{20} \times 28 \text{ g} = \frac{63}{5} \text{ g}$$

11. 4 g H_2 means 2 g-moles and 8 g He means 2 g-moles.

Now, $\quad M = \dfrac{n_1 M_1 + n_2 M_2}{n_1 + n_2} = \dfrac{(2)(2) + (2)(4)}{2+2}$

$$= 3 \text{ g/mol}$$

$$\rho = \frac{PM}{RT} = \frac{(1.013 \times 10^5)(3 \times 10^{-3})}{(8.31)(273)} = 0.13 \text{ kg/m}^3$$

12. $\rho = \dfrac{1}{V}$

Given, $\qquad p \propto K\rho$

$$\therefore \qquad \rho \propto \frac{1}{V}$$

or $\qquad pV =$ constant

$\therefore \qquad T =$ constant

More than One Correct Options

1. $p \propto \sqrt{\rho}$ as per process. If ρ becomes half, then p will becomes $\sqrt{2}$ times.

Further, $\qquad \rho \propto \dfrac{1}{V}$

$$\therefore \qquad \frac{p^2}{(1/V)} = \text{constant}$$

or $\qquad (pV)\,p =$ constant

Hence, $\qquad Tp =$ constant $\qquad$ (as $pV \propto T$)

$$\therefore \qquad p \propto \frac{1}{T}$$

or P - T graph is a rectangular hyperbola.

2. $p \propto \dfrac{1}{\sqrt{V}}$

If volume becomes 4 times than p will remain half

$$V \propto \frac{T}{p}$$

$\therefore \qquad Vp^2 =$ constant

$\therefore \qquad \left(\dfrac{T}{P}\right) p^2 =$ constant

or $\qquad pT =$ constant

or $\qquad p \propto \dfrac{1}{T}$

i.e. p - T graph is a rectangular hyperbola.

If p is halved than T will becomes two times.

4. $\rho \propto \dfrac{1}{V}$

V is doubled so ρ will remain half.

$$U \propto T \propto pV$$

According to given graph, $p \propto V$

$\therefore \qquad U \propto T \propto p^2$ or V^2

V is doubled, so U and T both will become four times.

$$p \propto V$$

$\therefore \qquad \dfrac{T}{V} \propto V \qquad \left(\text{as } p \propto \dfrac{T}{V}\right)$

$\therefore \qquad T \propto V^2$

or T - V graph is a parabola passing through origin.

6. $V = \dfrac{nRT}{p} = \dfrac{mRT}{Mp} = \left(\dfrac{mR}{Mp}\right) T$

i.e. V - T graph is a straight line passing through origin of slope

$$= \left(\frac{mR}{Mp}\right) \quad \Rightarrow \quad \text{Slope} \propto \frac{m}{p}$$

Hence, slope depends on both m and p.

7. (a) $p = \dfrac{nRT}{V} = \dfrac{(m/V)RT}{M}$

So, given m in the question is mass of gas per unit volume.

(b) $m =$ total mass of gas

(c) $m =$ mass of one molecule of gas.

Match the Columns

1. (a) $K_T = \dfrac{n f_T}{2} RT = \dfrac{(2)(3)\,RT}{2} = 3RT$

(b) $K_R = \dfrac{n f_R RT}{2} = \dfrac{(2)(2)\,RT}{2} = 2RT$

(c) Potential energy = 0

(d) $U = K_R + K_T = 5RT$

2. $U \propto T$ and $\rho \propto \frac{1}{V}$

According to given graph,

$$U \propto \rho \Rightarrow T \propto \frac{1}{V}$$

$$\therefore \quad pV \propto \frac{1}{V} \quad \text{or} \quad p \propto \frac{1}{V^2}$$

Density is increasing. So, volume is decreasing. Hence, pressure will increase.

U is increasing. So, temperature will also increase.

Now, $\frac{T}{V} \propto \frac{1}{V^2}$

Volume is decreasing. Hence, $\frac{T}{V}$ will increase.

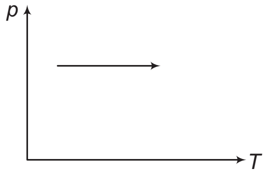
3. Speed of sound $= \sqrt{\frac{\gamma RT}{M}}$

4. (a) Density of water is maximum at 4°C. From 0°C to 4°C, density will first increase, then decrease.
(b) and (c) : Same logic as given in option (a).

(d) $T = 2\pi\sqrt{\frac{l}{g}}$ or $T \propto \sqrt{l}$

As temperature is increased, length will increase. So, time period will also increase.

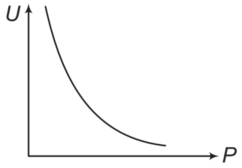
5. (a) $p =$ constant. Hence p - T graph is as shown below.



(b) $\rho = \frac{pM}{RT}$

If $p =$ constant, then

$$\rho \propto \frac{1}{T} \quad \text{or} \quad \rho \propto \frac{1}{U} \qquad (\text{as } U \propto T)$$



(c) If $p =$ constant, then

$$T \propto V$$

i.e. T - V graph is a straight line passing through origin.

(d) $\rho = \frac{pM}{RT}$

If $p =$ constant, then $\rho \propto \frac{1}{T}$

i.e. ρ - T graph is rectangular hyperbola.

Subjective Questions

1. $pV = nRT$

$$pdV = nRdT$$

$$\frac{dV}{dT} = \frac{nR}{P}, \; dV = V\gamma dT$$

$$\therefore \quad \gamma = \frac{dV/dT}{V} = \frac{nR}{PV} = \frac{1}{T}$$

2. In the process $pV^x =$ constant $C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$,

Given, $C = R$ and $\gamma = 7/5$

Substituting these values, we get $x = \frac{5}{3}$

Now, $pV^{5/3} =$ constant or $p \propto \frac{1}{(V)^{5/3}}$

By increasing volume to two times pressure will decrease $(2)^{5/3}$ times.

$$v_{\text{rms}} \propto \sqrt{T} \quad \text{or} \quad v_{\text{rms}} \propto \sqrt{pV}$$

or v_{rms} will become $\sqrt{\frac{(2)}{(2)^{5/3}}}$ times

or v_{rms} will become $(2)^{-1/3}$ times

or $\frac{1}{(2)^{1/3}}$ times.

Now, $p \propto$ (no. of collisions) (v_{rms})

$$\frac{1}{(2)^{5/3}} \propto (\text{no. of collisions}) \frac{1}{(2)^{1/3}}$$

or number of collisions will decrease $(2)^{4/3}$ times.

Ans.

3. $p_1V = n_1RT$

$$p_2V = n_2RT \qquad (T = 273 \text{ K})$$

$$\therefore \quad (p_1 - p_2)V = (n_1 - n_2)RT = \left(\frac{m_1 - m_2}{M}\right)RT$$

$$(\Delta p)V = \frac{\Delta m}{M}RT \qquad \ldots\text{(i)}$$

In the initial condition (at STP)

$$\frac{RT}{M} = \frac{p_0}{\rho} \qquad \ldots\text{(ii)}$$

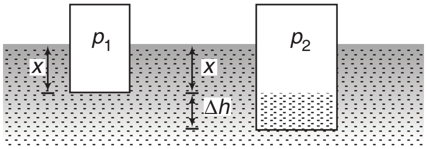
$$(T = 273 \text{ K})$$

From Eqs. (i) and (ii), we get

$$\Delta m = \frac{\rho V \Delta p}{P_0} = \frac{1.2 \times 0.4 \times 0.24}{1.0}$$

$$= 0.1152 \text{ kg} = 115.2 \text{ g} \qquad \textbf{Ans.}$$

4. $p_2 =$ pressure at depth x



$$p_1 V_1 = p_2 V_2$$

or $$p_1(Ah) = p_2(A)(h - \Delta h)$$

$\therefore$ $$p_1 = \frac{p_2(h - \Delta h)}{h} \qquad \ldots\text{(i)}$$

Initially, $mg = \rho g x A$

Now, $p_2 = p_0 + \rho g x = p_0 + \dfrac{mg}{A}$

Substituting in Eq. (i), we have

$$p_1 = \left(p_0 + \frac{mg}{A}\right)\left(1 - \frac{\Delta h}{h}\right) \qquad \textbf{Ans.}$$

5. $p = \dfrac{RT}{V}$ $\qquad (n = 1)$

or $$p = \frac{R}{V}(T_0 + \alpha V^2) \qquad \ldots\text{(i)}$$

For minimum attainable pressure

$$\frac{dp}{dV} = 0 \quad \text{or} \quad \frac{-RT_0}{V^2} + \alpha R = 0$$

or $$V = \sqrt{\frac{T_0}{\alpha}}$$

At this volume we can see that $\dfrac{d^2p}{dV^2}$ is positive or p is minimum.

From Eq. (i)

$$p_{\min} = \frac{RT_0}{\sqrt{T_0/\alpha}} + \alpha R \sqrt{T_0/\alpha}$$

$$= 2R\sqrt{\alpha T_0} \qquad \textbf{Ans.}$$

6. At 50°C, density of solid = density of liquid.

At 0°C, fraction submerged $\left(\dfrac{\rho_s}{\rho_l}\right)_{0^\circ\text{C}} \times 100 \qquad \ldots\text{(i)}$

$$(\rho)_{50} = \frac{\rho_0}{1 + 50\gamma}$$

$\therefore$ $$\rho_0 = \rho_{50}(1 + 50\gamma)$$

Substituting in Eq. (i), we have

% of fraction submerged

$$= \left(\frac{1 + 50\gamma_s}{1 + 50\gamma_l}\right) \times 100 = \left(\frac{1 + 0.3 \times 10^{-5}}{1 + 8 \times 10^{-5}}\right) \times 100$$

$$= 99.99\ \% \qquad \textbf{Ans.}$$

7. $\rho_{100} g h_{100} = \rho_0 g h_0$ or $\dfrac{\rho_{100}}{\rho_0} = \dfrac{h_0}{h_{100}}$

$$\left(\frac{1}{1 + 100\gamma}\right) = \frac{39.2}{40}$$

Solving this equation we get

$$\gamma = 2.0 \times 10^{-4} \text{ per °C} \qquad \textbf{Ans.}$$

8. Let Δl be the change in length. (let $\Delta l_s > \Delta l > \Delta l_a$)

$$\text{Strain in steel} = \frac{\Delta l_s - \Delta l}{l_0}$$

$$\text{and strain in aluminium} = \frac{\Delta l - \Delta l_a}{l_0}$$

In equilibrium, $2F_s = F_a$

$$2\left[\frac{\Delta l_s - \Delta l}{l_0}\right] Y_s A = \left[\frac{\Delta l - \Delta l_a}{l_0}\right] Y_a A$$

or $$2(l_0 \alpha_s \theta - \Delta l) Y_s = (\Delta l - l_0 \alpha_a \theta) Ya$$

Solving this equation, we get

$$\Delta l = \left(\frac{2\alpha_s Y_s + \alpha_a Y_a}{2Ys + Y_a}\right) l_0 \theta$$

$\therefore$ Total length

$$= l_0 + \Delta l = l_0 \left[1 + \left(\frac{\alpha_a Y_a + 2\alpha_s Y_s}{2Y_s + Y_a}\right)\theta\right] \qquad \textbf{Ans.}$$

9. From $\Delta l = l\alpha\Delta\theta$ we have,

$$0.05 = 25\alpha_A(100)$$

$\therefore$ $$\alpha_A = 0.00002 \text{ per°C}$$

$$0.04 = 40\,\alpha_B(100)$$

$$\alpha_B = 0.00001 \text{ per°C}$$

In third case let l is the length of rod A. Then length of rod B will be $(50 - l)$ cm.

$$\Delta l = \Delta l_1 + \Delta l_2$$

or $0.03 = l(0.00002)(50) + (50 - l)(0.00001)(50)$

Solving we get, $l = 10$ cm and $50 - l = 40$ cm

Ans.

21. Laws of Thermodynamics

INTRODUCTORY EXERCISE 21.1

1. ΔU will be same along all paths. Then, we can apply $Q = W + \Delta U$ for all.

2. $\Delta U = Q - W$

$$= 254 - (-73) = 327 \text{ J}$$

INTRODUCTORY EXERCISE 21.2

1. (a) Work done by gas under constant pressure is

$$W = p(V_f + V_i)$$
$$= (1.7 \times 10^5)(0.8 - 1.2)$$
$$= -6.8 \times 10^4 \text{ J}$$

(b) $$Q = W + \Delta U$$
$$= -6.8 \times 10^4 - 1.1 \times 10^5$$

or $|Q| = 1.78 \times 10^5$ J

As Q is negative. So, net heat flow is out of the gas.

2. (a) $W_1 = +$ ve, as cycle is clockwise

$W_2 = -$ ve as cycle is anti-clockwise

Since, $|W_1| > |W_2|$

$\therefore$ Net work done by the system is positive.

(b) In cyclic process, $Q_{\text{net}} = W_{\text{net}}$

Since, W_{net} is positive. So, Q_{net} is also positive. So, heat flows into the system.

(c) $W_1 = Q_1 = +$ ve, so into the system.

$W_2 = Q_2 = -$ve, so out of the system.

3. Helium is monoatomic. So, degree of freedom $f = 3$.

$$U = \frac{nf}{2}RT = \frac{3n}{2}RT$$

or $$n = \frac{2U}{3RT} = \frac{2 \times 100}{3 \times 8.31 \times 300}$$

$$= 2.67 \times 10^{-2} \text{mol}$$ **Ans.**

4. In isochoric process,

$$W = 0$$
$$Q = \Delta U = nC_V\Delta T = 4\left(\frac{3}{2}R\right)(300)$$
$$= 1800\, R$$

5. W = Area under p-V diagram

Further $p \propto V$ along AB.

Therefore, $p_B = 2p_A = 2p_0$

6. VT = constant

or $V(pV)$ = constant (as $T \propto pV$)

$\therefore$ pV^2 = constant

Comparing with pV^x = constant, we have

$$x = 2$$
$$W = \frac{nR\Delta T}{1 - x} \quad \text{(Table 21.3)}$$
$$= \frac{(2)(R)(300)}{1 - 2}$$
$$= -600\, R$$

7. $pV^2 = K = p_0V_0^2$

$\therefore$ $$p = \frac{K}{V^2}$$

$$W = \int p dV = \int_{V_0}^{2V_0} \frac{K}{V^2}\, dV$$
$$= \left[-\frac{K}{V}\right]_{V_0}^{2V_0}$$
$$= \frac{K}{V_0} - \frac{K}{2V_0}$$
$$= \frac{K}{2V_0}$$

Substituting, $K = p_0V_0^2$, we have

$$W = \frac{1}{2}p_0V_0$$

INTRODUCTORY EXERCISE 21.3

1. (a) $Q = nC_V\,\Delta T$

$$200 = (1)\left(\frac{3}{2}R\right)(T_f - 300)$$

or $200 = 1.5 \times 8.31(T_f - 300)$

Solving we get,

$$T_f = 316 \text{ K}$$

(b) $Q = nC_p\,\Delta T$

$$200 = (1)\left(\frac{5}{2}R\right)(T_f - 300)$$

or $200 = 2.5 \times 8.31\,(T_f - 300)$

Solving we get,

$$T_f = 309.6 \text{ K}$$

2. p = constant

$\therefore$ $$T \propto V$$

As the gas expands V increases. So, T also increases. Hence, ΔT is positive. Therefore, in the expression,

$$Q = nC_p \Delta T$$

Q is positive.

3. ***AB***

V is increasing, so $W = +\text{ ve}$. Product of pV and therefore T and therefore U is increasing.

So, ΔU is also + ve.

$$Q = W + \Delta U$$

$\therefore$ Q is also positive.

BC $\qquad V = \text{constant}$

$\therefore \qquad W = 0$

$\therefore \qquad Q = \Delta U = -\text{ ve}$

CA

V is decreasing.

$\therefore$ $W = -\text{ ve}$

ΔU is also negative.

$\therefore$ $Q = W + \Delta U$ is also negative.

4. $Q = 0$ as the box is insulated.

$W_{\text{gas}} = 0$ as on the right hand side there is vacuum.

$\therefore \qquad \Delta U = Q - W = 0$

$\therefore$ ΔT is also zero,

as $\qquad U \propto T$

5. In an isobaric process $p =$ constant.

Therefore, $C = C_p$.

Now, $$\frac{\Delta Q}{\Delta U} = \frac{nC_p \Delta T}{nC_V \Delta T} = \frac{C_p}{C_V} = \gamma$$

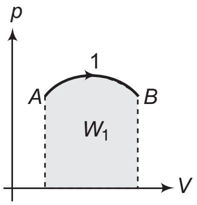
and $$\frac{\Delta Q}{\Delta W} = \frac{\Delta Q}{\Delta Q - \Delta U}$$

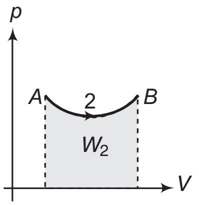
$$= \frac{nC_p \Delta T}{nC_p \Delta T - nC_V \Delta T}$$

$$= \frac{C_p}{C_p - C_V} = \frac{C_p/C_V}{C_p/C_V - 1}$$

$$= \frac{\gamma}{\gamma - 1}$$

6. $Q_1 = W_1 + \Delta U_1$ and $Q_2 = W_2 + \Delta U_2$





U is a state function. Hence, ΔU depends only on the initial and final positions. Therefore,

$$\Delta U_1 = \Delta U_2$$

But, $\qquad W_1 > W_2$

as the area under 1 is greater than area under 2.

Hence,

$$Q_1 > Q_2$$

7. $$W = \int_{V_i}^{V_f} p\,dV = \int_1^2 (\alpha V^2)\,dV$$

$$= \int_1^2 (5 \times 1.01 \times 10^5)\,V^2 dV$$

Solving we get,

$$W = 1.18 \times 10^6 \text{ J}$$

$$= 1.18 \text{ MJ}$$

8. $W = nR\,\Delta T \qquad$ (in isobaric process)

$$= (1)(8.31)(72)$$

$$= 600 \text{ J} = 0.6 \text{ kJ}$$

$$\Delta U = Q - W = 1.0 \text{ kJ}$$

$$\gamma = \frac{C_p}{C_V} = \frac{n\,C_p\,\Delta T}{n\,C_V\,\Delta T} = \frac{Q}{\Delta U}$$

$$= \frac{1.6}{1.0} = 1.6$$

INTRODUCTORY EXERCISE 21.4

1. Given, $\quad Q_1 = 10^6$ cal

$$T_1 = (827 + 273) = 1100 \text{ K}$$

and $\qquad T_2 = (27 + 273) = 300 \text{ K}$

as, $\qquad \dfrac{Q_2}{Q_1} = \dfrac{T_2}{T_1} \qquad$ (in Carnot engine)

$\therefore \qquad Q_2 = \dfrac{T_2}{T_1} \cdot Q_1 = \left(\dfrac{300}{1100}\right)(10^6)$

$= 2.72 \times 10^5$ cal $\qquad$ **Ans.**

Efficiency of the cycle,

$$\eta = \left(1 - \frac{T_2}{T_1}\right) \times 100$$

or $\qquad \eta = \left(1 - \dfrac{300}{1100}\right) \times 100$

$= 72.72\%$ $\qquad$ **Ans.**

2. $T_1 = 2100$ K, $T_2 = 700$ K, $\eta_{\text{actual}} = 40\%$

Maximum possible efficiency of Carnot engine is

$$\eta_{\max} = 1 - \frac{T_2}{T_1} = 1 - \frac{700}{2100}$$

$$= 0.666 = 66.6\%$$

Actual efficiency as the percentage of the maximum possible efficiency is

$$\frac{\eta_{\text{actual}}}{\eta_{\max}}\times 100=\frac{40}{66.6}\times 100$$

$$=60\% \qquad \textbf{Ans.}$$

3. (a) $\eta=1-\frac{T_2}{T_1}=1-\frac{375}{500}=0.25=25\%$

(b) $W=\eta Q_1=0.25\times 25\times 10^5$

$$=6.25\times 10^5\ \text{J}$$

(c) $Q_2=Q_1-W=(25-6.25)\times 10^5$

$$=18.75\times 10^5\text{J}$$

4. $T_1=627+273=900$ K

$T_2=27+273=300$ K

$$Q_1=3\times 10^6\ \text{cal}$$

$$\eta=\left(1-\frac{T_2}{T_1}\right)=\left(1-\frac{Q_2}{Q_1}\right)$$

$$\therefore \qquad \frac{Q_2}{Q_1}=\frac{T_2}{T_1}$$

$$\therefore \qquad Q_2=\frac{T_2}{T_1}\times Q_1$$

$$=\frac{3\times 10^6\times 300}{900}=10^6\,\text{cal}$$

Work done by the engine,

$$W=Q_1-Q_2=3\times 10^6-10^6\ \text{cal}$$

$$=2\times 10^6\ \text{cal}$$

5. $\eta_1=1-\frac{T_2}{T_1}$ or $\frac{1}{6}=1-\frac{T_2}{T_1}$ ...(i)

In the second case, the temperature of the sink is reduced by 65°C. Hence,

$$\eta_2=1-\frac{T_2-65}{T_1}$$

or $\frac{1}{3}=1-\frac{T_2-65}{T_1}$ ...(ii)

Solving Eqs. (i) and (ii), we get

$$T_1=390\ \text{K}=117^\circ\text{C},$$

$$T_2=325\ \text{K}$$

$$=52^\circ\text{C}$$

6. Coefficient of performance is given by

$$\beta=\frac{T_2}{T_1-T_2}$$

$$\beta_A=\frac{(-10+273)}{(27+273)-(-10+273)}$$

$$=7.1$$

$$\beta_B=\frac{(-27+273)}{(17+273)-(-27+273)}=5.6$$

$$\beta_A>\beta_B$$

Therefore, refrigerator A is better.

7. Here, $T_1=25+273=298$ K

$$T_2=-10+273=263\ \text{K}$$

$$Q_2=263\ \text{J/s}$$

Coefficient of performance is given by

$$\beta=\frac{Q_2}{Q_1-Q_2}=\frac{T_2}{T_1-T_2}$$

$$\therefore \qquad \frac{Q_1}{Q_2}=\frac{T_1}{T_2}$$

$$\therefore \qquad Q_1=\frac{T_1}{T_2}\times Q_2=\frac{298}{263}\times 263$$

$$=298\ \text{Js}^{-1}$$

Average power consumed,

$$=298-263=35\ \text{J s}^{-1}=35\ \text{W}$$

8. $\eta=\frac{W_{\text{net}}}{\Sigma Q_{+\text{ve}}}\times 100$

$$W_{\text{net}}=\text{Area under the cycle}$$

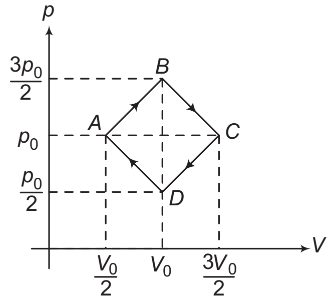
$$=\frac{p_0V_0}{2}$$

$$\Sigma Q_{+\text{ve}}=Q_{ABC}$$

$$=W_{ABC}+\Delta U_{ABC}$$

$$=(\text{Area under } ABC)+nC_V\,\Delta T$$

$$=\left(\frac{5}{4}p_0V_0\right)+n\left(\frac{3}{2}R\right)(T_C-T_A)$$



$$=\frac{5}{4}p_0V_0+\frac{3}{2}(p_CV_C-p_AV_A)$$

$$=\frac{5}{4}\ p_0V_0+\frac{3}{2}\left(\frac{3p_0V_0}{2}-\frac{p_0V_0}{2}\right)$$

$$=\frac{11}{4}p_0V_0$$

$$\therefore \qquad \eta=\frac{(p_0V_0/2)}{(11p_0V_0/4)}\times 100=18.18\ \%$$

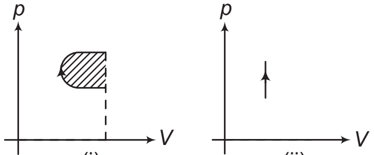
Exercises

LEVEL 1

Assertion and Reason

1. $Q = 0, \Delta U = -W$

   In expansion, W is positive. So, ΔU is negative. Hence, T also decreases.

2. 

   (i) (ii)

   In both figures, $V_i = V_f$

   but $W_1 =$ Hatched area

   $$W_2 = 0$$

3. First law is just energy conservation law, which can be applied for any system.

5. $Q - W = \Delta U$ is state function.

   So, it will remain constant for any two thermodynamic states.

6. For given temperatures T_1 and T_2 efficiency of a heat engine can't be greater than efficiency of Carnot engine.

7. $pT =$ constant

   $$\therefore \quad \left(\frac{T}{V}\right) T = \text{constant}$$

   $$\text{or} \quad V \propto T^2$$

   If temperature is increased, then volume will also increase. So, work done is positive.

8. $Q = 0$ as chamber is adiabatic

   $W = 0$ as expansion is taking place in vacuum.

   $$\therefore \quad \Delta U = Q - W = 0$$

   $$\therefore \quad U \text{ or } T = \text{constant}$$

   $$\text{or} \quad pV = \text{constant}$$

   $$\therefore \quad p \propto \frac{1}{V}$$

Objective Questions

1. $$p \propto V^2$$

   $$\therefore \quad \frac{T}{V} \propto V \quad \text{or} \quad T \propto V^3 \Rightarrow V \propto T^{1/3}$$

   T is increasing. So, V will also increase. Hence, work done will be positive.

2. $$W_{\text{gas}} = nRT \ln\left(\frac{V_f}{V_i}\right) = nRT \ln\left(\frac{1}{2}\right)$$

   $$= -nRT \ln 2$$

   $\therefore$ Work done on the gas $= + nRT \ln 2$

3. ΔU is same p.

   $$Q = W + \Delta U$$

   $$W_1 = +\text{ ve}, W_2 = 0$$

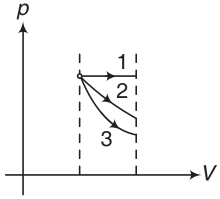
   $$\text{and} \quad W_3 = -\text{ ve}$$

4. In adiabatic compression temperature increases. Hence, pV increases, as $T \propto pV$

5. $$W = -\pi(\text{Radius})^2$$

   $$= -\pi\,(\text{Radius})\,(\text{Radius})$$

   $$= -\pi\left(\frac{p_2 - p_1}{2}\right)\left(\frac{V_2 - V_1}{2}\right)$$

6. 

   1 → Isobaric

   2 → Isothermal

   3 → Adiabatic

   Minimum area is under graph-3.

7. $W_1 > W_2$ as area under graph-a is more. Hence,

   $$\Delta Q_1 > \Delta Q_2, \text{ as}$$

   $$\Delta Q = W + \Delta U \text{ and } \Delta U_1 = \Delta U_2$$

8. $$\eta = \left(1 - \frac{T_2}{T_1}\right) \times 100$$

   $$T_2 = 300 \text{ K} \quad \text{and } T_1 = 600 \text{ K}$$

   $$\therefore \quad \eta = 50\%$$

9. $$p = \frac{nRT}{V}$$

   $V =$ constant, but temperature will decrease with time. So, p will decrease.

10. In adiabatic compression, internal energy of gas increases. So, temperature increases.

11. $$pT^{\frac{\gamma}{1-\gamma}} = \text{constant} \quad \text{or} \quad p \propto T^{\frac{\gamma}{\gamma-1}}$$

   $$\therefore \quad \alpha = \frac{\gamma}{\gamma - 1} = \frac{1.4}{1.4 - 1} = 3.5$$

12. $$C = C_V + \frac{R}{1-x}$$

$$= \frac{5}{2}R + \frac{R}{1-x} = \frac{5}{2}R - \frac{R}{x-1}$$

C is negative if

$$x - 1 < \frac{2}{5}$$

$$\therefore \qquad x < 1.4$$

If x lies between 1 and 1.4, then also C is negative.

13. $$C_V = \frac{n_1C_{V_1} + n_2C_{V_2}}{n_1 + n_2}$$

In option (c),

$$C_V = \frac{(2)\left(\frac{3}{2}R\right) + (4)\left(\frac{5}{2}R\right)}{6}$$

$$= \frac{13R}{6}$$

14. $$Q_{\text{net}} = W_{\text{net}} = \text{ area under the graph}$$

$$= \pi(\text{Radius})(\text{Radius})$$

$$= \pi(10\times10^3 \text{ Pa})(10\times10^{-3} \text{ m}^3)$$

$$= 10^2\pi \text{ J}$$

15. Area is same, signs are opposite.

Subjective Questions

1. Helium is monoatomic. So, degree of freedom $f = 3$.

$$U = \frac{nf}{2}RT = \frac{3n}{2}RT$$

or $$n = \frac{2U}{3RT}$$

$$= \frac{2\times100}{3\times8.31\times300}$$

$$= 2.67\times10^{-2} \text{ mol}$$

2. For all process,

$$\Delta U = nC_V\Delta T$$

$$= \frac{nR\Delta T}{\gamma - 1} \qquad \left(\text{as } C_V = \frac{R}{\gamma-1}\right)$$

3. (a) In a cyclic process,

$$Q_{\text{net}} = W_{\text{net}}$$

cyclic is clockwise. Hence, work done is positive. So, Q_{net} is also positive.

(b) $W_{\text{net}} = Q_{\text{net}} = 7200$ J

(c) In counter-clockwise direction,

$$W_{\text{net}} = Q_{\text{net}} = -\text{ ve}$$

$$|Q_{\text{net}}| = 7200 \text{ J}$$

4. $$Q_{\text{net}} = W_{\text{net}} = \text{area under the cycle}$$

$$= (2\times10^5)(2.5\times10^{-6})$$

$$= 0.5 \text{ J}$$

5. (a) $W_{\text{I}} = +$ ve (as cycle is clockwise)

$W_{\text{II}} = -$ ve (as cycle is anti-clockwise)

But,

$$|W_{\text{I}}| > |W_{\text{II}}|$$

$$\therefore \qquad W_{\text{net}} = +\text{ ve}$$

(c) $$Q_{\text{net}} = W_{\text{net}} = +\text{ ve}$$

(d) In loop-I,

$$Q_{\text{I}} = W_{\text{I}} = +\text{ ve}$$

In loop-II,

$$Q_{\text{II}} = W_{\text{II}} = -\text{ ve}$$

6. Work done per cycle = area under the cycle

$$= \frac{1}{2}(3-2)(10^{-3})(30-10)(1.01\times10^5)$$

$$= 1010 \text{ J}$$

Total work done per second

$$= 1010\times\frac{100}{60}$$

$$= 1683 \text{ J/s}$$

7. (a) $$Q = nC_V\,\Delta T = (1)\left(\frac{3}{2}R\right)(T_f - T_i)$$

$$\therefore \qquad 200 = 1.5\times8.31(T_f - 300)$$

Solving we get,

$$T_f = 316 \text{ K}$$

(b) $$Q = nC_p\,\Delta T = (1)\left(\frac{5}{2}R\right)(T_f - T_i)$$

$$\therefore \qquad 200 = 2.5\times8.31\,(T_f - 300)$$

$$\therefore \qquad T_f \approx 310 \text{ K}$$

8. $V =$ constant

From $pV = nRT$,

$$V\Delta p = nR(\Delta T)$$

$$Q = nC_V\,\Delta T$$

$$= n\left(\frac{5}{2}R\right)\Delta T = \frac{5}{2}(V\Delta p)$$

$$= \frac{5}{2}\times(10\times10^{-3})(4\times10^5)$$

$$= 10^4 \text{ J}$$ **Ans.**

9. (a) $W = +$ (Area of the cycle)

$$= +\frac{1}{2}(2V_0 - V_0)(3p_0 - p_0)$$

$$= p_0V_0$$

(b) $Q_{CA} = nC_p\Delta T$

$$= n\left(\frac{5}{2}R\right)(T_A - T_C)$$

$$= \frac{5}{2}(p_AV_A - p_CV_C)$$

$$= \frac{5}{2}(p_0V_0 - 2p_0V_0) = -\frac{5}{2}p_0V_0$$

$$Q_{AB} = nC_V\Delta T$$

$$= n\left(\frac{3}{2}R\right)(T_B - T_A)$$

$$= \frac{3}{2}(p_BV_B - p_AV_A)$$

$$= 3p_0V_0$$

(c) $W_{BC} = +$ (Area under the graph)

$$= 2p_0V_0$$

$$\Delta U_{BC} = nC_V\,\Delta T$$

$$= n\left(\frac{3}{2}R\right)(T_C - T_B)$$

$$= \frac{3}{2}(p_CV_C - p_BV_B) = -\frac{3}{2}p_0V_0$$

$$\therefore \quad Q_{BC} = W_{BC} + \Delta U_{BC} = \frac{p_0V_0}{2}$$

(d) p - V equation along path BC is

$$p = -\left(\frac{2p_0}{V_0}\right)V + 5p_0$$

or
$$pV = -\left(\frac{2p_0}{V_0}\right)V^2 + 5p_0V$$

or
$$RT = -\left(\frac{2p_0}{V_0}\right)V^2 + 5p_0V$$

$$\therefore \quad T = \frac{1}{R}\left[5p_0V - \frac{2p_0}{V_0}\cdot V^2\right] \qquad \ldots\text{(i)}$$

For T to be maximum,

$$\frac{dT}{dV} = 0$$

$$\therefore \quad 5p_0 - \frac{4p_0}{V_0}V = 0$$

$$\therefore \quad V = \frac{5}{4}V_0$$

So at this volume, temperature is maximum. Substituting this value of V in Eq. (i), we get

$$T_{\max} = \frac{25p_0V_0}{8R}$$

10. **First process**

$$V = \text{constant}$$

$$\therefore \quad p \propto T$$

Pressure becomes three times. Therefore, temperature also becomes three times.

$$(T_f = 3T_i)$$

Second process

$$p = \text{constant}$$

$$\therefore \quad V \propto T$$

Volume is doubled. So, temperature also becomes two times. ($T_f' = 6T_i$)

Now,
$$C = \frac{Q}{n\,\Delta T} = \frac{Q_1 + Q_2}{n\,\Delta T}$$

$$= \frac{nC_V\,\Delta T_1 + nC_p\,\Delta T_2}{n\,\Delta T}$$

$$= \frac{\left(\frac{5}{2}R\right)(3T_i - T_i) + \left(\frac{7}{2}R\right)(6T_i - 3T_i)}{(6T_i - T_i)}$$

$$= 3.1\,R$$

11. **First Process**

$$V = \text{constant}$$

$$\therefore \quad T \propto p$$

Pressure becomes half. So, temperature also becomes half.

$$Q_1 = n_1C_V\Delta T = 2C_V(150 - 300)$$

$$= -300\,C_V$$

Second Process

$$p = \text{constant}$$

$$Q_2 = nC_p\,\Delta T$$

$$= 2\,C_p(300 - 150)$$

$$= 300\,C_p$$

$$Q = Q_1 + Q_2$$

$$= 300(C_p - C_V) = 300\,R$$

$$= 300 \times 8.31 = 2493\text{ J}$$

12. $\Delta U = Q - W = nC_V\,\Delta T = n\left(\frac{3}{2}R\right)(T_f - T_i)$

$$\therefore \quad 1200 - 2100 = 5\left(\frac{3}{2}\times 8.31\right)(T_f - 400)$$

$$\therefore \quad T_f = 385\text{ K}$$

$$\approx 113\,°\text{C}$$

13. $V_i = \dfrac{m}{\rho_i} = \dfrac{2}{999.9} = 2.0002 \times 10^{-3}\text{ m}^3$

$$V_f = \frac{m}{\rho_f} = \frac{2}{1000} = 2 \times 10^{-3}\text{m}^3$$

$$\Delta U = Q - W$$

$$= ms\,\Delta\theta - p_a\Delta V$$

$$= (2)(4200)(4) - (10)^5(-0.002 \times 10^{-3})$$

$$= 33600.2\text{ J}$$

14. $$V_i = \frac{m}{\rho_i} = \frac{0.01}{1000} = 10^{-5}\,\text{m}^3$$

$$V_f = \frac{m}{\rho_f} = \frac{0.01}{0.6} = 0.0167\ \text{m}^3$$

$$\begin{aligned}\Delta U &= Q - W\\ &= ms\Delta\theta + mL - p_0\Delta V\\ &= (0.01\times 4200\times 100) + (0.01)(2.5\times 10)^6\\ &\qquad - 10^5(0.0167 - 10^{-5})\\ &= 27530\ \text{J}\end{aligned}$$

15. (a) $$\begin{aligned}W &= p_0\Delta V\\ &= (1.013\times 10^5)(1671-1)\times 10^{-6}\\ &= 169\ \text{J}\end{aligned}$$

(b) $$\begin{aligned}Q &= mL = (10^{-3})(2.256\times 10^6)\\ &= 2256\ \text{J}\\ \therefore\quad \Delta U &= Q - W\\ &= 2087\ \text{J}\end{aligned}$$

16. (a) $$\begin{aligned}W &= p\Delta V = (2.30\times 10^5)(1.2-1.7)\\ &= -1.15\times 10^5\ \text{J}\end{aligned}$$

(b) $$\begin{aligned}Q &= W + \Delta U\\ &= (-1.15\times 10^5) + (-1.40\times 10^5)\\ &= -2.55\times 10^5\,\text{J}\end{aligned}$$

17. (a) and (b) $(pV)_A = (pV)_C = 3p_0V_0$

$\therefore\qquad T_A = T_C$ or $\Delta T = 0$

$\therefore\qquad \Delta U_{ABC} = 0$

$Q_{ABC} = W_{ABC} =$ Area under the graph

$= -2p_0V_0$ **Ans.**

i.e. Heat is released during the process.

(c) **Process AB**

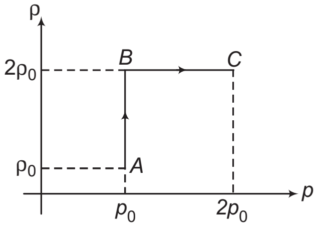
ρ - p graph will be a straight line parallel to ρ - axis because pressure is constant. Further, V is decreasing. Therefore, ρ will increase.

Process BC

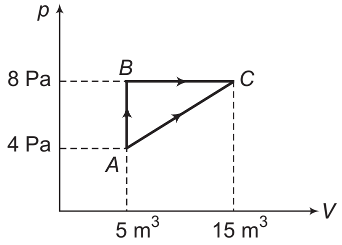
V = constant $\Rightarrow$ ρ = constant

Therefore, ρ-p graph is a straight line parallel to p - axis.

ρ - p graph is as shown below.



18. (a) W_{AC} is less than W_{ABC} as area under graph is less.



(b) For process A to C,

$$Q = 200\ \text{J}$$

Work done

$$\begin{aligned}W_{AC} &= \text{area under } AC\\ &= \frac{1}{2}(8+4)\times 10\\ &= 60\ \text{J}\end{aligned}$$

From first law of thermodynamics,

$$\begin{aligned}\Delta U &= Q - W_{AC}\\ U_C - U_A &= 200 - 60\\ \therefore\quad U_C &= U_A + 140\\ &= 10 + 140 = 150\ \text{J}\end{aligned}$$ **Ans.**

(c) From A to B,

$$\begin{aligned}U_B &= 20\ \text{J}\\ \therefore\quad \Delta U &= Q - W_{AB}\\ U_B - U_A &= Q - 0\\ 20 - 10 &= Q\\ \therefore\quad Q &= 10\ \text{J}\end{aligned}$$ **Ans.**

19. (a) $U_C - U_A$ along two paths should be same.

$\therefore\quad (Q_{AB} + Q_{BC}) - (W_{AB} + W_{BC}) = Q_{AC} - W_{AC}$

$\therefore\quad (250 + Q_{BC}) - (500 + 0) = 300 - 700$

Solving, we get

$$Q_{BC} = -150\ \text{J}$$

(b) $$\begin{aligned}Q_{CDA} &= (W_{CD} + W_{DA}) + (U_A - U_C)\\ &= (-800 + 0) - (U_C - U_A)\\ &= -800 - (300 - 700)\\ &= -400\ \text{J}\end{aligned}$$

20. $$\begin{aligned}V &= \frac{m}{\rho} = \frac{1.0}{8.92\times 10^3}\\ &= 1.12\times 10^{-4}\text{m}^3\end{aligned}$$

$$\gamma = 3\alpha = 2.1\times 10^{-5}\text{per }^\circ\text{C}$$

$$\begin{aligned}\Delta V &= V\gamma\,\Delta\theta\\ &= (1.12\times 10^{-4})(2.1\times 10^{-5})(30)\\ &= 7.056\times 10^{-8}\,\text{m}^3\end{aligned}$$

$$\begin{aligned} W &= p\Delta V \\ &= (1.01\times 10^{5})(7.056\times 10^{-8}) \\ &= 7.13\times 10^{-3}\,\text{J} \end{aligned}$$

$$\begin{aligned} Q &= mc\,\Delta Q \\ &= (1)(387)(30) \end{aligned}$$

$$\begin{aligned} \Delta U &= Q - W \\ &= 11609.99287\ \text{J} \end{aligned}$$

21. (a) $T = \dfrac{pV}{nR} = \dfrac{(2\times 10^{5})(1.0\times 10^{-2})}{(1)(8.31)}$

$$= 240.7\ \text{K}$$

(b) In adiabatic process,

$$TV^{\gamma-1} = \text{constant}$$

$$\therefore\quad T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$

$$\therefore\quad T_f = T_i\left(\frac{V_i}{V_f}\right)^{\gamma-1}$$

$$= (240.7)\left[\frac{1.0\times 10^{-2}}{5.0\times 10^{-3}}\right]^{5/3-1}$$

$$\approx 383\ \text{K}$$

(c) Work done on the gas $= \Delta U$

$$\begin{aligned} &= nC_V\,\Delta T \\ &= (1)\left(\frac{3}{2}R\right)(T_f - T_i) \\ &= 1.5\times 8.31(383-240.7) \\ &\approx 1770\ \text{J} \end{aligned}$$

(d) $\Delta U \approx 1770$ J

22. (a) From conservation of linear momentum,

$$0.01\times 200 = (2+0.01)\,v$$

$$\therefore\quad v \approx 1\ \text{m/s}$$

Mechanical energy dissipated in collision,

$$\begin{aligned} &= K_i - K_f \\ &= \frac{1}{2}\times(0.01)(200)^2 - \frac{1}{2}(2.01)(1)^2 \\ &\approx 199\ \text{J} \end{aligned}$$

(b) $n = \dfrac{m}{M} = \dfrac{2000+10}{200} = 10.05$

Using,

$$Q = nC_V\,\Delta T$$

$$\therefore\quad \Delta T = \frac{Q}{nC_V} = \frac{199}{10.05\times 3\times 8.31}$$

$$= 0.80^\circ\text{C}$$

23. Along the process CD apply,

pV = constant and find V_C

Similarly along path AB, again apply,

pV = constant and find V_B.

Now, $W_{\text{net}} = W_{AB} + W_{BC} + W_{CD} + W_{DA}$

$$= nRT_A \ln\left(\frac{V_B}{V_A}\right) + p_B(V_C - V_B) + nRT_C \ln\left(\frac{V_D}{V_C}\right) + p_D(V_A - V_D)$$

Further, $nRT_A = p_A V_A$

and $nRT_C = p_C V_C$

24. $pV = nRT$

$$\therefore\quad p\Delta V = nR\Delta T$$

Work done under constant pressure is

$$\begin{aligned} W &= p\Delta V = nR\Delta T \\ &= (0.2)(8.31)(100) = 166.2\ \text{J} \end{aligned}$$

LEVEL 2

Single Correct Option

1. $W = \int_V^{2V} p\,dV$

$$\begin{aligned} &= \int_V^{2V}\left(\frac{n\,RT}{V-b}\right)dV \\ &= nRT\,\ln\left(\frac{2V-b}{V-b}\right) \end{aligned}$$

Put $n = 1$,

$$\therefore\quad W = RT\,\ln\left(\frac{2V-b}{V-b}\right)$$

2. **For *AB*** V = constant

$$\therefore\quad W = 0$$

For *BC* $W = n\,RT_2\,\ln\left(\dfrac{V_f}{V_i}\right)$

$$= RT_2 \ln\left(\frac{V_2}{V_1}\right)$$

For *CA* $V \propto T$

$$\therefore\quad p = \text{constant}$$

$$\begin{aligned} W &= nR\Delta T \\ &= R(T_1 - T_2) \end{aligned}$$

3. $\because\quad Q = n\,C_p\,\Delta T$

$$\frac{7}{2}RT_0 = (10)\left(\frac{7}{2}R\right)(T_f - T_0)$$

$$T_f = 1.1\,T_0$$

If p = constant, then $V \propto T$

If temperature becomes 1.1 times, then volume will also become 1.1 times.

4. W_{net} = area under the cycle

$$= 2p_0V_0$$

$$Q_{+\text{ve}} = Q_{ABC} = W_{ABC} + \Delta U_{ABC}$$

$$= \text{ area under the graph } + nC_V\,\Delta T$$

$$= (3p_0V_0) + n\left(\frac{3}{2}R\right)(T_C - T_A)$$

$$= (3p_0V_0) + \frac{3}{2}(p_CV_C - p_AV_A)$$

$$= 10.5\ p_0V_0$$

$$\eta = \frac{W_{\text{net}}}{Q_{+\text{ve}}} = \frac{4}{21}$$

5. $Q = W + \Delta U$

$$= \text{area under the graph} + \Delta U$$

$$= -\,p_0V_0 - \frac{3p_0V_0}{2}$$

$$= -\frac{5}{2}p_0V_0$$

6. pV^{-1} = constant

In the process, pV^x = constant

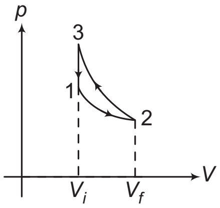
$$C = C_V + \frac{R}{1-x}$$

Here, $C = \frac{3}{2}R + \frac{R}{1+1} = 2R$

$$Q = nC\Delta T = (1)(2R)(2T_0 - T_0)$$

$$= 2RT_0$$

7.



In process 3 – 1

V = constant

$\therefore$ $T \propto p$

If p is decreasing, then T will also decrease.

8. $TV^{\gamma-1}$ = constant

$$\therefore \quad \frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

$$= \left(\frac{L_2}{L_1}\right)^{5/3-1} = \left(\frac{L_2}{L_1}\right)^{2/3}$$

9. $T \propto U$

$$T_C = T_D = T_0 = 300\text{ K}$$

$$T_A = T_B = 2T_0 = 600\text{ K}$$

$$Q_{\text{net}} = Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA}$$

$$= nRT_A \ln\left(\frac{V_f}{V_i}\right) + nC_V(T_C - T_B)$$

$$+ nRT_C \ln\left(\frac{V_f}{V_i}\right) + nC_V(T_A - T_D)$$

$$= (1)(R)(600)\ln(2) + (1)(C_V)(-300)$$

$$+ (1)(R)(300)\ln\left(\frac{1}{2}\right) + (1)(C_V)(300)$$

$$= 300\,R\ln 2$$

10. **1-2** $T \propto V$

$\therefore$ p = constant

$$Q_{\text{net}} = W_{\text{net}} = W_{12} + W_{23} + W_{31}$$

$\therefore$ $W_{23} = Q_{\text{net}} - W_{12}$ (as $W_{31} = 0$)

$$= -\,300 - nR\Delta T$$

$$= -\,300 - 2 \times 8.31 \times 300$$

$$= -\,5286\text{ J}$$

11. $Q_A = Q_B$

$$nC_p\,\Delta T_A = nC_V\Delta T_B$$

$$\therefore \quad \Delta T_B = \frac{C_p}{C_V}\cdot\Delta T_A = \gamma\Delta T_A$$

$$= (1.4)(30) = 42\text{ K}$$

12. TV^{n-1} = constant

$\therefore$ $(pV)V^{n-1}$ = constant (as $T \propto pV$)

$\therefore$ pV^n = constant

Bulk modulus in the process pV^γ = constant is γp.

Hence, in the given process Bulk modulus is np.

13. pV^{-1} = constant

In the process, pV^x = constant

$$W = \left(\frac{R}{1-x}\right)\Delta T$$

Here, $x = -1$

$$\therefore \quad W = \frac{R}{2}\Delta T = \frac{R}{2}(T_2 - T_1)$$

14. $\Delta U = Q - W = 450\text{ J} = nC_V\Delta T$

$$= n\left(\frac{3}{2}R\right)\Delta T$$

$$\therefore \quad n\,\Delta T = \left(\frac{300}{R}\right)$$

Now, $$C = \frac{Q}{n\,\Delta T} = \frac{600}{300/R} = 2R$$

15. $$W = p\Delta V = p_0(2V_0 - V_0) = p_0V_0$$

$$\begin{aligned}\Delta U &= U_f - U_i\\ &= 2p_0(2V_0) - 2p_0V_0\\ &= 2p_0V_0\end{aligned}$$

∴ $$Q = W + \Delta U = 3p_0V_0$$

16. $$C = \frac{Q}{n\,\Delta T}$$

or $$C \propto Q$$

ΔU is same. But $W_1 < W_2$ as area under p-V graph is less.

Hence, $$Q_1 < Q_2$$

or $$C_1 < C_2.$$

∴ $$\frac{C_1}{C_2} < 1$$

17. $$Q_{\text{net}} = W_{\text{net}}$$

$$Q_{AB} = Q_{CD} = 0$$

$$Q_{BC} = n\,C_p\,\Delta T = (1)\left(\frac{5}{2}R\right)(4T - T) = 7.5RT$$

$$Q_{DA} = n\,C_p\,\Delta T = (1)\left(\frac{5}{2}R\right)(3T - 5T) = -5RT$$

∴ $$W_{\text{net}} = Q_{\text{net}} = 7.5RT - 5RT = 2.5RT$$

18. $U\rho$ = constant

∴ $$T\left(\frac{1}{V}\right) = \text{constant}$$

or $$T \propto V$$

∴ $$p = \text{constant}$$

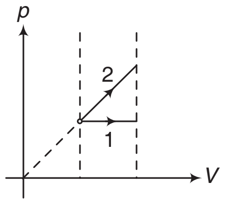
$$\frac{\Delta U}{W} = \frac{\Delta U}{Q - \Delta U} = \frac{n\,C_V\,\Delta T}{n\,C_p\Delta T - n\,C_V\Delta T} = \frac{1}{\gamma - 1} = \frac{1}{\frac{5}{3} - 1} = \frac{3}{2}$$

19. W = Area under F-x graph

$= 17.5$ J

Now, $$Q = W + \Delta U = 20 \text{ J}$$

20. In second process, $p \propto V$ i.e. p-V graph is straight line passing through origin.



Area under graph-2 is more.

∴ $$W_2 > W_1$$

Further, $$T \propto pV$$

$$(pV)_2 > (pV)_1$$

∴ $$T_2 > T_1$$

21. $dQ = -dU$ or $dU + dW = -dU$

or $$2dU + dW = 0$$

∴ $$2\,[n\,C_V dT] + p dV = 0$$

or $$2\left[n\left(\frac{R}{\gamma - 1}\right)dT\right] + p dV = 0$$

or $$\frac{2nRdT}{\gamma - 1} + \left(\frac{nRT}{V}\right)dV = 0$$

or $$\left(\frac{2}{\gamma - 1}\right)\frac{dT}{T} + \frac{dV}{V} = 0$$

Integrating we get,

$$\left(\frac{2}{\gamma - 1}\right)\ln(T) + \ln(V) = \ln(C)$$

Solving we get

$$TV^{\frac{\gamma - 1}{2}} = \text{constant}$$

22. In a cyclic process,

$$Q_{\text{net}} = W_{\text{net}}$$

∴ $Q_{AB} + Q_{BC} + Q_{CA} =$ area under the graph

∴ $$600 + 200 + Q_{CA} = \frac{1}{2}(3 \times 10^{-4})(5 \times 10^{5}) = 75 \text{ J}$$

$$\begin{aligned}\therefore\ Q_{CA} &= -725 \text{ J} = W_{CA} + \Delta U_{CA}\\ &= -725 = (-\text{Area under the graph}) + \Delta U_{CA}\\ &= -\left(\frac{1}{2} \times 11 \times 10^{5}\right)(3 \times 10^{-4}) + \Delta U_{CA}\end{aligned}$$

Solving we get, $\Delta U_{CA} = -560$ J

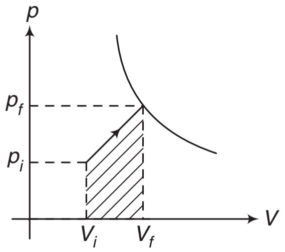
23. $T_2 = T_3$ (lying on same isotherm)

∴ $$\Delta U_1 = \Delta U_2$$

W = Hatched area

$$= \frac{1}{2}(p_i + p_f)(V_f - V_i)$$

$$= \frac{1}{2}(p_iV_f - p_iV_i + p_fV_f - p_fV_i)$$



Now, p_iV_i and p_fV_f are same for both processes. Further p_i and V_i, area also same for both processes.

For $1 \to 3$ V_f is more and p_f is less. Hence, W will be more.

Now, $$Q = W + \Delta U$$

$$W_{13} > W_{12}$$

$\therefore$ $$Q_{13} > Q_{12} \qquad \text{(as } \Delta U \text{ is same)}$$

24. $\Delta U_{\text{net}} = 0$

$\therefore$ $$\Delta U_1 + \Delta U_2 + \Delta U_3 = 0$$

$\therefore$ $$Q_1 + (Q_2 - W_2) + \Delta U_3 = 0$$

$$(200 \text{ kJ}) + (-100 + 50) \text{ kJ} + \Delta U_3 = 0$$

$\therefore$ $$\Delta U_3 = -150 \text{ kJ}$$

In adiabatic process,

$$W = -\Delta U$$

$\therefore$ $$W_3 = -\Delta U_3 = 150 \text{ kJ}$$

25. $Q_{BC} = 0$

$$\begin{aligned} Q_{CD} &= nC_p\Delta T \\ &= n\left(\frac{5}{2}R\right)(T_D - T_C) \\ &= \frac{5}{2}(p_DV_D - p_CV_C) \\ &= \frac{5}{2}(10^5 - 2\times10^5) \\ &= -25\times10^4 \text{ J} \end{aligned}$$

$$\begin{aligned} Q_{DA} &= nC_V\,\Delta T \\ &= n\left(\frac{3}{2}R\right)(T_A - T_D) \\ &= \frac{3}{2}(p_AV_A - p_DV_D) \\ &= \frac{3}{2}(2.4\times10^5 - 10^5) \\ &= 21\times10^4 \text{ J} \end{aligned}$$

Now, $$W_{\text{net}} = Q_{\text{net}}$$

$$= (9 - 25 + 21)\times10^4 \text{ J} = 5\times10^4 \text{J}$$

26. Pressure becomes half. So, temperature is doubled. $\left(\text{as } T \propto \frac{1}{p}\right)$

$$\Delta U = nC_V\,\Delta T$$

$$T_i = \frac{p_0V_0}{nR}$$

$$T_f = \frac{2p_0V_0}{nR}$$

$\therefore$ $$\Delta T = T_f - T_i = \frac{p_0V_0}{nR}$$

$\therefore$ $$\Delta U = n\left(\frac{3}{2}R\right)\left(\frac{p_0V_0}{nR}\right) = \frac{3p_0V_0}{2}$$

27. $W = nRT_0 \ln\left(\frac{p_i}{p_f}\right)$

$$W_{BC} = 2\,W_{AB}$$

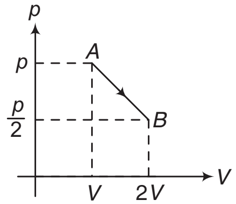
$\therefore$ $$nRT_0 \ln\left(\frac{p_C}{p_0/2}\right) = 2nRT_0 \ln\left(\frac{p_0/2}{p_0}\right)$$

Solving this equation we get,

$$p_C = \frac{p_0}{8}$$

More than One Correct Options

1.



Now, see the hint of Q-No 9 (d) of subjective questions for Level 1.

2. Temperature is increased. So, internal energy will also increase.

$\therefore$ $$\Delta U = +\text{ ve}$$

Further,

$$pV^2 = \text{constant}$$

$\therefore$ $$\left(\frac{T}{V}\right)V^2 = \text{constant} \quad \text{or} \quad V \propto \frac{1}{T}$$

Temperature is increased. So, volume will decrease and work done will be negative.

In the process pV^x = constant, molar heat capacity is given by

$$C = C_V + \frac{R}{1-x}$$

Here, $x = 2$

$\therefore$ $$C = C_V - R$$

C_V of any gas is greater than R.

So, C is positive. Hence, from the equation,

$$Q = nC\Delta T$$

Q is positive if T is increased.

or ΔT is positive.

3. ***ab*** $W = 0$ (as $V =$ constant)

$$Q_1 = \Delta U_1 = nC_V\Delta T = (2)\left(\frac{3}{2}R\right)(2T_0 - T_0) = 3RT_0$$

bc $\Delta U = 0$ (as $T =$ constant)

$$\therefore \quad Q_2 = W_2 = nRT \ln\left(\frac{V_f}{V_i}\right) = (2)(R)(2T_0)\ln(2) = 4RT_0\ln(2)$$

cd $W = 0$

$$Q_3 = \Delta U_3 = nC_V\Delta T = (2)\left(\frac{3}{2}R\right)(T_0 - 2T_0) = -3RT_0$$

da $\Delta U = 0$

$$Q_4 = W_4 = nRT \ln\left(\frac{V_f}{V_i}\right) = (2)(R)(T_0)\ln\left(\frac{1}{2}\right) = -2RT_0\ln(2)$$

Now in complete cycle,

$$\Delta U_{\text{net}} = 0$$

$$Q_{\text{net}} = W_{\text{net}} = 2RT_0\ln(2) = +\text{ ve}$$

4. ***ab*** $\rho =$ constant

$\therefore$ $V =$ constant

$\therefore$ $W = 0$

$Q = \Delta U$

ΔU is positive, as U is increasing.

Hence, Q is also positive.

bc $\rho \propto U \Rightarrow \therefore \frac{1}{V} \propto T$

ρ is decreasing, so V is increasing. Hence, work done is positive.

Further, $\frac{1}{V} \propto T$ $(T \propto pV)$

$$\therefore \quad pV^2 = \text{constant}$$

In the process, $pV^x =$ constant,

Molar heat capacity is given by

$$C = C_V + \frac{R}{1-x}$$

Here, $x = 2$

$$\therefore \quad C = C_V - R$$

For any of the gas, $C_V \neq R$.

$$\therefore \quad C \neq 0$$

$\therefore$ $Q = nC\Delta T \neq 0$ as $\Delta U \neq 0$ and $\Delta T \neq 0$

ca

ρ is increasing. Hence, V is decreasing. So, work done is negative.

5.

$$Q_1 = nC_V\Delta T$$

$$Q_2 = nC_p\,\Delta T$$

$$Q_3 = 0$$

$$C_p > C_V$$

$$\therefore \quad Q_2 > Q_1 > Q_3$$

6. **1-2** $V =$ constant

$$\therefore \quad p \propto T$$

T is increasing. So, p will also increase.

2-3 $V \propto T$

$$\therefore \quad p = \text{constant.}$$

V and T both are increasing.

Process 4-1 is reverse of 2-3.

Comprehension Based Questions

1. $Q_{\text{net}} = W_{\text{net}} =$ area under the cycle $= \frac{pV}{2}$

2. $\frac{C_V}{C_p} = \frac{1}{\gamma} = \frac{5}{3}$

3. In adiabatic process,

$$p^{1-\gamma}T^{\gamma} = \text{constant} \quad \text{or} \quad T \propto p^{\frac{\gamma-1}{\gamma}}$$

$$\therefore \quad \frac{T_B}{T_A} = \left(\frac{p_B}{p_A}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\therefore \quad T_B = (1000)\left(\frac{2}{3}\right)^{\frac{5/3-1}{5/3}} = (1000)\left(\frac{2}{3}\right)^{0.4} = 850\text{ K}$$

4. In adiabatic process,

$$W_{AB} = -\Delta U_{AB} \qquad (\text{as } Q = 0)$$

$$= nC_V(T_A - T_B) = (1)\left(\frac{3}{2}R\right)(T_A - T_B) = (1)\left(\frac{3}{2}\times\frac{25}{3}\right)(1000 - 850) = 1875\text{ J}$$

5. $W_{BC} = 0$ (as V = constant)

$$\begin{aligned}
\therefore \quad T &\propto p \quad \text{(as } V = \text{constant)}\\
p_C &= p_{B/2}\\
\therefore \quad T_C &= T_{B/2} = 425\ \text{K}\\
Q = \Delta U &= nC_V(\Delta T)\\
&= (1)\left(\frac{3}{2}R\right)(T_C - T_B)\\
&= \frac{3}{2} \times \frac{25}{3} \times (425 - 850)\\
&= 5312.5\ \text{J}
\end{aligned}$$

Match the Columns

1. (a) $W = nR\Delta T$

$$= (2)(R)(T) = 2RT$$

(b) $\Delta U = nC_V\Delta T$

$$= (2)\left(\frac{3}{2}R\right)(T) = 3RT$$

(c) $W = -\Delta U = -3RT$

(d) $\Delta U = 3RT$ (in all processes)

2. For both processes,

$$p = \text{constant} \qquad (\text{as } V \propto T)$$

$$\therefore \quad C = C_p \text{ for same gases.}$$

For W, ΔU and Q we will require number of moles also.

3. p^2V = constant or $pV^{1/2}$ = constant

Comparing with pV^x = constant, we get

$$x = \frac{1}{2}$$

∴ Molar heat capacity,

$$\begin{aligned}
C &= C_V + \frac{R}{1-x}\\
&= \left(\frac{3}{2}R\right) + 2R = 3.5\ R\\
Q &= nC\Delta T = n(3.5R)\,(\Delta T)\\
&= 3.5\ nR\Delta T\\
\Delta U &= nC_V\Delta T\\
&= n\left(\frac{3}{2}R\right)\Delta T = 1.5\ nR\Delta T
\end{aligned}$$

∴ W by the gas $= Q - \Delta U = 2nR\Delta T$ and work done on the gas $= -2nR\Delta T$

4. (a) $Q = nC_p\Delta T$

$$\begin{aligned}
\Delta U &= nC_V\Delta T\\
W &= Q - \Delta U = nR\Delta T\\
W &< \Delta U \text{ as } C_V > R
\end{aligned}$$

(b) $W = 0$

$$\begin{aligned}
\Delta U &= -\text{ve}\\
\therefore \quad Q &= \Delta U = -\text{ve}
\end{aligned}$$

(c) $Q = 0$

$$\begin{aligned}
W &= +\text{ve} \qquad (\text{as } V \text{ is increasing})\\
\therefore \quad \Delta U &= -W = -\text{ve}
\end{aligned}$$

(d) $\Delta T = 0$

$$\Rightarrow \quad \Delta U = 0$$

$$\text{and} \quad Q = W = nRT\ \ln\left(\frac{V_f}{V_i}\right) = +\text{ve}$$

5. (a) W_{ab} = area under p-V graph

$$= \frac{3p_0V_0}{2} = 1.5\ p_0V_0$$

(b) $\Delta U_{ab} = Q_{ab} - W_{ab}$

$$= 6p_0V_0 - \frac{3p_0V_0}{2} = 4.5P_0V_0$$

(c) $C = \dfrac{Q}{n\Delta T} = \dfrac{W + \Delta U}{n\Delta T}$

$$= \frac{6p_0V_0}{n\left(\dfrac{p_bV_b}{nR} - \dfrac{p_aV_a}{nR}\right)} = 2R$$

(d) $C_V = \dfrac{\Delta U}{n\Delta T}$

$$= \frac{4.5p_0V_0}{n\left(\dfrac{p_bV_b}{nR} - \dfrac{p_aV_a}{nR}\right)} = 1.5R$$

Subjective Questions

1. In a cyclic process,

$$\begin{aligned}
\Delta U &= 0\\
Q_{\text{net}} &= W_{\text{net}}\\
Q_{AB} &= n\,C_p\,\Delta T\\
&= (2)\left(\frac{5}{2}R\right)(400 - 300)\\
&= 500R\\
Q_{BC} &= n\,R\,T_B\ \ln\left(\frac{p_i}{p_f}\right)\\
&= (2)(R)(400)\ \ln\left(\frac{2}{1}\right)\\
&= 800R\ \ln\,(2)\\
Q_{CD} &= nC_p\,\Delta T\\
&= -500R\\
Q_{DA} &= n\,RT_D\ \ln\left(\frac{p_i}{p_f}\right)
\end{aligned}$$

$$= (2)(R)(300)\ \ln\left(\frac{1}{2}\right)$$

$$= -600R\ \ln(2)$$

$$\therefore \quad Q_{\text{net}} = W_{\text{net}} = (200R)\ \ln(2)$$

$$= (200)(8.31)(0.693)$$

$$\approx 1153\ \text{J}$$

2. $T_A = \dfrac{p_A V_A}{nR} = \dfrac{(1.013\times10^5)(22.4)}{(10^3)(8.31)}$

$$= 273\ \text{K}$$

Along path *AB*

$$V = \text{constant}$$

$$\therefore \quad T \propto p$$

p is doubled. So, T is also doubled.

$$\therefore \quad T_B = 2T_A = 546\ \text{K.}$$

Further, $\quad T_B = T_C$

$$\therefore \quad p_B V_B = p_C V_C$$

$$\therefore \quad V_C = \frac{(p_B)(V_A)}{(p_A)} = \frac{(2)(22.4)}{1}$$

$$= 44.8\ \text{m}^3$$

3. **For Process 1 - 2** $\quad \rho \propto p$

$$\therefore \quad \frac{1}{V} \propto p$$

$\therefore$ Process is isothermal.

$$\Delta U_1 = 0$$

$$Q_1 = W_1 = nRT\ \ln\left(\frac{p_i}{p_f}\right)$$

$$= \frac{p_0 M}{\rho_0}\ \ln\left(\frac{1}{2}\right) \quad \left(\text{as } n = 1 \text{ and } RT = \frac{PM}{\rho}\right)$$

$$= -\frac{p_0 M}{\rho_0} \ln(2)$$

For Process 2-3

$$Q_2 = Q_p = nC_p\,\Delta T$$

$$= (1)\left(\frac{5}{2}R\right)(T_3 - T_2)$$

$$= \frac{5}{2}\left(\frac{p_3 M}{\rho_3} - \frac{p_2 M}{\rho_2}\right)$$

$$= \frac{5}{2}\left(\frac{2p_0 M}{\rho_0} - \frac{2p_0 M}{2\rho_0}\right)$$

$$= 2.5\,\frac{p_0 M}{\rho_0}$$

$$\Delta U_2 = nC_V\Delta T$$

Substituting the values like above we get,

$$\Delta U_2 = \frac{1.5 p_0 M}{\rho_0}$$

$$W_3 = Q_2 - \Delta U_2 = \frac{p_0 M}{\rho_0}$$

For Process 3-1

Density is constant. Hence,
volume is constant.

$$\therefore \quad W_3 = 0$$

$$\therefore \quad Q_3 = \Delta U_3 = nC_V\Delta T$$

$$= (1)\left(\frac{3}{2}R\right)(T_1 - T_3)$$

$$= \frac{3}{2}\left(\frac{p_0 M}{\rho_0} - \frac{2p_0 M}{\rho_0}\right)$$

$$= -\frac{1.5 p_0 M}{\rho_0}$$

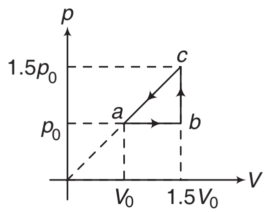
(b) $\Sigma Q_{-\text{ve}} = |Q_1 + Q_3|$

$$= \frac{p_0 M}{\rho_0}\left(\frac{3}{2} + \ln 2\right)$$

(c) $\eta = \dfrac{W_{\text{net}}}{\Sigma Q_{+\text{ve}}} = \dfrac{(+p_0 M/\rho_0) + \left(\dfrac{-p_0 M}{\rho_0}\ln 2\right)}{(2.5\ p_0 M/\rho_0)}$

$$= \frac{2}{5}(1 - \ln 2)$$

4. (a)



$$W_{\text{net}} = Q_{\text{net}} \quad \text{in a cycle.}$$

Since, Q_{net} is negative (flows out of the gas). Hence, W_{net} should also be negative. Or, cycle should be anti-clockwise as shown in figure.

From *a* to *b*

$$p = \text{constant}$$

$$\therefore \quad V \propto T$$

T has become 1.5 times. Therefore,
V will also become 1.5 times.

From *c* to *a*

$$p \propto V$$

V has become $\dfrac{1}{1.5}$ times. Therefore,

p will also become $\dfrac{1}{1.5}$ times.

In a cycle

$$|Q_{\text{net}}| = |W_{\text{net}}| = \text{Area of cycle}$$

$$\therefore \quad 800 = \frac{1}{2}(0.5p_0)(0.5V_0)$$

$$\therefore \quad p_0V_0 = 6400 \text{ J}$$

$$W_{ca} = -\text{Area under the graph}$$

$$= -\frac{1}{2}(2.5p_0)(0.5V_0)$$

$$= -0.625p_0V_0$$

$$= -0.625 \times 6400 = -4000 \text{ J}$$

5. First process is isobaric.

$$\therefore \quad \Delta Q_1 = nC_V\Delta T + p\Delta V$$

Second process is isochoric

$$\therefore \quad \Delta Q_2 = nC_V\Delta T$$

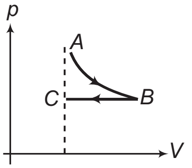
$$\Delta Q_1 - \Delta Q_2 = p\Delta V = \left[p_0 + \frac{mg}{A}\right][Ax]$$

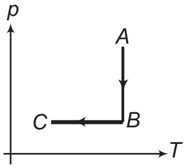
$$= (p_0A + mg)x$$

$$= [10^5 \times 60 \times 10^{-4} + 8 \times 10](0.2)$$

$$= 136 \text{ J}$$ **Ans.**

6. (a)





(b) For the process AB,

$$\frac{p_0V_0}{T_0} = \frac{p_B(2V_0)}{T_0}$$

$$\therefore \quad p_B = \frac{p_0}{2}$$

$$\Delta U = 0$$

$$Q = W + \Delta U$$

$$= nRT \ln \frac{V_B}{V_A}$$

$$= 3RT_0 \ln 2$$

For the process BC,

$$\frac{2V_0}{T_0} = \frac{V_0}{T_C}$$

$$\therefore \quad T_C = \frac{T_0}{2}$$

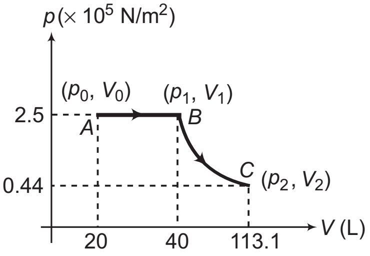
$$W = nR\Delta T = 3R\left(\frac{T_0}{2} - T_0\right) = -\frac{3}{2}RT_0$$

$$Q = nC_p\Delta T = -\frac{21}{4}RT_0$$

$$\therefore \quad W_{\text{Total}} = 3RT_0 \ln(2) - \frac{3}{2}RT_0$$

(c) $Q_{\text{Total}} = 3RT_0 \ln(2) - \frac{21}{4}RT_0$

7. (a)



(b) $p_0 = \dfrac{nRT_0}{V_0} = \dfrac{2 \times 8.31 \times 300}{20 \times 10^{-3}}$

$$= 2.5 \times 10^5 \text{ N/m}^2$$

$$p_1 = p_0 = 2.5 \times 10^5 \text{ N/m}^2$$

$$V_1 = 2V = 40 \times 10^{-3} \text{ m}^3$$

Process *AB*: $V \propto T$

$$\therefore \quad T_1 = 2T_0 = 600 \text{ K}$$

Process *BC*

Using $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$, we get

$$V_2 = 2\sqrt{2}V_1 = 113.1 \times 10^{-3} \text{ m}^3$$ **Ans.**

and $p_2 = \dfrac{nRT_2}{V_2} = \dfrac{(2)(8.31)(300)}{113.1 \times 10^{-3}}$

$$= 0.44 \times 10^5 \text{ N/m}^2$$ **Ans.**

(c) $W_{\text{Total}} = W_1 + W_2$

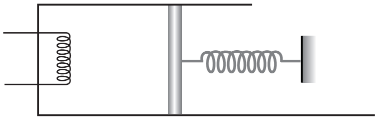
$$= p_0(V_1 - V_0) + \frac{nR}{\gamma - 1}(T_1 - T_2)$$

Substituting the values, we get

$$W_{\text{Total}} = 12479 \text{ J}$$ **Ans.**

8. (a) $\dfrac{p_1V_1}{T_1} = \dfrac{p_2V_2}{T_2}$

$$T_2 = \left(\frac{p_2V_2T_1}{p_1V_1}\right) \quad \text{...(i)}$$



Here,

$$p_1 = 1.0 \times 10^5 \text{ N/m}^2$$

$$V_1 = 2.4 \times 10^{-3} \text{ m}^3$$

$$T_1 = 300 \text{ K}$$

$$p_2 = p_1 + \frac{kx}{A}$$

$$= 1.0 \times 10^5 + \frac{8000 \times 0.1}{8 \times 10^{-3}}$$

$$= 2.0 \times 10^5 \text{ N/m}^2$$

$$V_2 = V_1 + Ax$$

$$= 2.4 \times 10^{-3} + 8 \times 10^{-3} \times 0.1$$

$$= 3.2 \times 10^{-3} \text{ m}^3$$

Substituting in Eq. (i), we get

$$T_2 = 800 \text{ K}$$ **Ans.**

(b) Heat supplied by the heater,

$$Q = W + \Delta U$$

Here,

$$\Delta U = nC_V \Delta T$$

$$= \left(\frac{p_1V_1}{RT_1}\right)\left(\frac{3}{2}R\right)(800 - 300)$$

$$= \frac{(1.0 \times 10^5)(2.4 \times 10^{-3})(1.5)(500)}{(300)}$$

$$= 600 \text{ J}$$

$$W = \frac{1}{2}kx^2 + p_1\Delta V$$

$$= \frac{1}{2} \times (8000)(0.1)^2 + (1.0 \times 10^5)(0.1)(8 \times 10^{-3})$$

$$= (40 + 80) \text{ J} = 120 \text{ J}$$

$$\therefore \quad Q = 600 + 120 = 720 \text{ J}$$ **Ans.**

9. $U \propto \sqrt{V}$

As $U \propto T$

$\therefore \quad T \propto V^{1/2}$

or $$TV^{-1/2} = \text{constant}$$

or $$pV^{1/2} = \text{constant}$$

Comparing with $pV^x = \text{constant}$, we have

$$x = \frac{1}{2}$$

$\therefore$ Molar specific heat

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$$

$$= \frac{R}{7/5 - 1} + \frac{R}{1 - 1/2} \qquad (\gamma = 7/5)$$

$$= \frac{5}{2}R + 2R = \frac{9R}{2}$$ **Ans.**

$$Q = nC\Delta T$$

$$\Delta U = nC_V\Delta T$$

$$\therefore \quad W = Q - \Delta U = n(C - C_V)\Delta T$$

$$\frac{W}{\Delta U} = \frac{C - C_V}{C_V}$$

$$\therefore \quad W = \left(\frac{C - C_V}{C_V}\right)\Delta U$$

$$= \left(\frac{9/2 - 5/2}{5/2}\right)(100) = 80 \text{ J}$$ **Ans.**

10. $dQ = dU + dW$

$$CdT = C_V\, dT + pdV$$

$$(C_V + 3aT^2)\, dT = C_V\, dT + pdV$$

$$\therefore \quad 3aT^2\, dT = pdV = \left(\frac{RT}{V}\right)dV$$

$$\therefore \quad \left(\frac{3a}{R}\right)TdT = \frac{dV}{V}$$

Integrating, we get

$$\left(\frac{3aT^2}{2R}\right) = \ln V - \ln C$$

$$V = Ce^{\frac{3aT^2}{2R}} \quad \text{or} \quad Ve^{-\frac{3aT^2}{2R}} = \text{constant}$$

11. (a) $p = \alpha T^{1/2}$

or $$pT^{-1/2} = \text{constant}$$

$$p^{1/2}V^{-1/2} = \text{constant}$$

$$pV^{-1} = \text{constant}$$

$$\therefore \quad x = -1$$

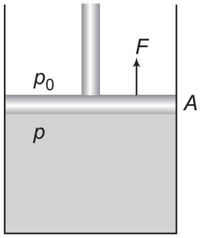
$$\Delta W = \left(\frac{R}{1 - x}\right)(\Delta T)$$

$$= \left(\frac{8.31}{2}\right)(50)$$

$$= 207.75 \text{ J}$$ **Ans.**

(b) $C = C_V + \dfrac{R}{1 - x} = \dfrac{3}{2}R + \dfrac{R}{2} = 2R$ **Ans.**

12. $F + pA = p_0A$



$$F = (p_0 - p)A$$

$$W = \int_V^{2V}(p_0 - p)A\, dx$$

$$= \int_V^{2V}p_0dV - \int_V^{2V}pdV$$

$$= p_0V - \int_V^{2V}RT\left(\frac{dV}{V}\right)$$

$$= p_0V - RT \ln(2)$$

$$= RT - RT \ln(2)$$

$$= RT(1 - \ln 2)$$ **Ans.**

13. (a) $p \propto \frac{1}{T}$ or pT = constant

$\therefore \qquad p(pV)$) = constant

or $\qquad pV^{1/2}$ = constant

In the process, pV^x = constant,

Molar heat capacity is

$$C = C_V + \frac{R}{1-x}$$

or
$$C = \frac{3}{2}R + \frac{R}{1-\frac{1}{2}}$$

$$= \frac{3}{2}R + 2R = \frac{7}{2}R \qquad \textbf{Ans.}$$

(b) $W = Q - \Delta U = nC\Delta T - nC_V\Delta T$

$$= n(C - C_V)\Delta T$$

$$= 2\left[\frac{7}{2}R - \frac{3}{2}R\right](T_2 - T_1)$$

$$= 4R(T_2 - T_1) \qquad \textbf{Ans.}$$

14. $V = \frac{a}{T}$

or $\qquad VT$ = constant

or $\qquad V(pV)$ = constant

$\therefore \qquad pV^2$ = constant

In the process pV^x = constant, molar heat capacity is

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-x}$$

Here, $\qquad x = 2$

$$\therefore \qquad C = \frac{R}{\gamma-1} + \frac{R}{1-2} = \left(\frac{2-\gamma}{\gamma-1}\right)R$$

Now, $\qquad Q = nC\Delta T$

$$= (1)\left(\frac{2-\gamma}{\gamma-1}\right)R\Delta T$$

$$= \left(\frac{2-\gamma}{\gamma-1}\right)R\Delta T \qquad \textbf{Ans.}$$

15. Process AB is isochoric (V = constant).

Hence,

$$\Delta W_{AB} = 0$$

$$\Delta W_{BCD} = p_0V_0 + \frac{\pi}{2}(p_0)\left(\frac{V_0}{2}\right)$$

$$= \left(\frac{\pi}{4} + 1\right)p_0V_0$$

$$\Delta W_{DA} = -\frac{1}{2}\left(\frac{p_0}{2} + p_0\right)(2V_0 - V_0)$$

$$= -\frac{3}{4}p_0V_0$$

$$\Delta U_{AB} = nC_V\Delta T = (2)\left(\frac{3}{2}R\right)(T_B - T_A)$$

$$\left(n = 2, C_V = \frac{3}{2}R\right)$$

$$= 3R\left(\frac{p_0V_0}{2R} - \frac{p_0V_0}{4R}\right)$$

$$= \frac{3}{4}p_0V_0 = \Delta Q_{AB} \qquad \left(T = \frac{pV}{nR}\right)$$

$$\Delta U_{BCD} = nC_V\Delta T = (2)\left(\frac{3}{2}R\right)(T_D - T_B)$$

$$= (3R)\left(\frac{2p_0V_0}{2R} - \frac{p_0V_0}{2R}\right) = \frac{3}{2}p_0V_0$$

Hence, $\quad \Delta Q_{BCD} = \Delta U_{BCD} + \Delta W_{BCD}$

$$= \left(\frac{\pi}{4} + \frac{5}{2}\right)p_0V_0$$

$$\Delta U_{DA} = nC_V\Delta T$$

$$= (2)\left(\frac{3}{2}R\right)(T_A - T_D)$$

$$= (3R)\left(\frac{p_0V_0}{4R} - \frac{2p_0V_0}{2R}\right)$$

$$= -\frac{9}{4}p_0V_0$$

$\therefore \qquad \Delta Q_{DA} = \Delta U_{DA} + \Delta W_{DA}$

$$= -\frac{9}{4}p_0V_0 - \frac{3}{4}p_0V_0$$

$$= -3p_0V_0$$

Net work done is,

$$W_{\text{net}} = \left(\frac{\pi}{4} + 1 - \frac{3}{4}\right)p_0V_0$$

$$= 1.04\, p_0V_0$$

and heat absorbed is

$$Q_{ab} = \Delta Q_{+\text{ve}}$$

$$= \left(\frac{3}{4} + \frac{\pi}{4} + \frac{5}{2}\right)p_0V_0 = 4.03p_0V_0$$

Hence, efficiency of the cycle is

$$\eta = \frac{W_{\text{net}}}{Q_{ab}} \times 100$$

$$= \frac{1.04\, p_0V_0}{4.03\, p_0V_0} \times 100$$

$$= 25.8\% \qquad \textbf{Ans.}$$

16. $p = \dfrac{\alpha T - \beta T^2}{V}$ (p = constant)

Hence $V = \dfrac{\alpha T - \beta T^2}{p}$

or $dV = \left(\dfrac{\alpha - 2\beta T}{p}\right) dT$

$$W = \int p dV = \int_{T_1}^{T_2} p \left(\frac{\alpha - 2\beta T}{p}\right) dT$$

or $W = [\alpha T - \beta T^2]_{T_1}^{T_2}$

$= \alpha (T_2 - T_1) - \beta (T_2^2 - T_1^2)$ **Ans.**

17. (a) First law of thermodynamics for the given process from state 1 to state 2

$$Q_{12} - W_{12} = U_2 - U_1$$

Here, $Q_{12} = +\, 10\, p_0 V_0$ joule

$W_{12} = 0$ (Volume remains constant)

$$U_2 - U_1 = nC_V\,(T_2 - T_1)$$

$$nC_V\,(T_2 - T_1) = 10\; p_0 V_0$$

For an ideal gas,

$$p_0 V_0 = nRT_0$$

and $C_p - C_V = R$

$\therefore$ $C_V = C_p - R$

$= \dfrac{5R}{2} - R = \dfrac{3R}{2}$

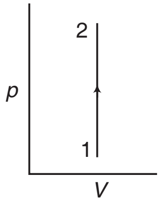
$\therefore$ $n\left(\dfrac{3R}{2}\right)(T_2 - T_0) = 10\, nRT_0$

$$T_2 = \frac{23}{3} T_0$$

As $p \propto T$ for constant volume

$$p_2 = \frac{23}{3} p_0$$

(b)



18. **Process *A*-*B*** is an isothermal process

i.e. T = constant

Hence, $p \propto \dfrac{1}{V}$

or p - V graph will be a rectangular hyperbola with increasing p and decreasing V.

$\rho \propto \dfrac{1}{V}$. Hence, ρ - V graph is also a rectangular hyperbola with decreasing V and hence increasing ρ.

$$\rho \propto p \qquad \left[\rho = \frac{pM}{RT}\right]$$

Hence, ρ - p graph will be a straight line passing through origin, with increasing ρ and p.

Process *B* -*C* is an isochoric process, because p - T graph is a straight line passing through origin i.e. V = constant

Hence, p - V graph will be a straight line parallel to p - axis with increasing p.

Since, V = constant hence ρ will also be constant

Hence ρ - V graph will be a dot.

ρ-p graph will be a straight line parallel to p -axis with increasing p, because

$$\rho = \text{constant}$$

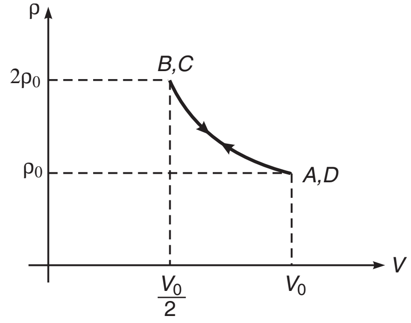
Process *C*-*D* is inverse of A-B and D -A is inverse of B-C.

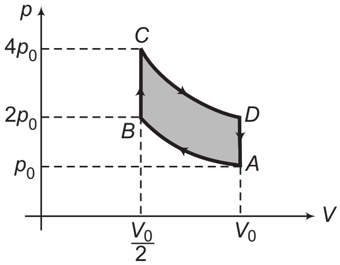
Different values of p, V, T and ρ in tabular form are shown below

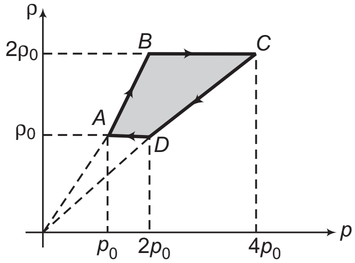
	p	V	T	ρ
A	p_0	V_0	T_0	ρ_0
B	$2p_0$	$\dfrac{V_0}{2}$	T_0	$2\rho_0$
C	$4p_0$	$\dfrac{V_0}{2}$	$2T_0$	$2\rho_0$
D	$2p_0$	V_0	$2T_0$	ρ_0

Here, $V_0 = nR\left(\dfrac{T_0}{P_0}\right)$ and $\rho_0 = \dfrac{P_0 M}{RT_0}$

The corresponding graphs are as follows







19. First process

$$T = \text{constant}$$

$$\therefore \qquad p \propto \frac{1}{V}$$

V is made 5 times. Therefore, p will become $\frac{1}{5}$ th.

Second process

$$V = \text{constant} \quad \Rightarrow \quad T \propto p$$

Pressure again becomes first times. So, temperature will also become 5 times.

$$Q_1 + Q_2 = 80 \times 10^3$$

$$\therefore \; nRT \, \ln\left(\frac{V_f}{V_i}\right) + nC_V \, \Delta T = 80 \times 10^3$$

$$\therefore \; (3)(8.31)(273)\ln(5) + (3)\left(\frac{8.31}{\gamma - 1}\right)(5 \times 273 - 273)$$

$$= 80 \times 10^3$$

Solving we get, $\gamma = 1.4$

22. Calorimetry and Heat Transfer

INTRODUCTORY EXERCISE 22.1

1. $$Q = mL_f + ms_w\,\Delta\theta + mL_f$$
$$= m\,(L_f + s_w\Delta\theta + L_f)$$
$$= 10\,[80 + 1 \times 100 + 540]$$
$$= 7200 \text{ cal}$$

2. Let $\theta\ (> 60°)$ is the mixture temperature in equilibrium.

Heat given = Heat taken

$$0 = m(1)(\theta - 20) + m(0.5)(\theta - 40) + m\,(0.25)(\theta - 60)$$

Solving the equation, we get

$$\theta = 31.43°\text{C}$$

3. $$mL = ms\,(\theta - 0°)$$

$\therefore$ $$\theta = \frac{L}{s} = \frac{80}{1} = 80°\text{C}$$

4. 75% heat is retained by bullet

$$\frac{3}{4}\left[\frac{1}{2}mv^2\right] = ms\Delta\theta + mL$$

or $$v = \sqrt{\frac{(8s\Delta\theta + 8L)}{3}}$$

Substituting the values, we have

$$v = \sqrt{\frac{(8 \times 0.03 \times 4.2 \times 300) + (8 \times 6 \times 4.2)}{3}}$$
$$= 12.96 \text{ m/s}$$

5. Let m be the mass of the steam required to raise the temperature of 100 g of water from 24°C to 90°C.

Heat lost by steam = Heat gained by water

$\therefore$ $$m\,(L + s\Delta\theta_1) = 100 s\Delta\theta_2$$

or $$m = \frac{(100)(s)(\Delta\theta_2)}{L + s(\Delta\theta_1)}$$

Here, s = specific heat of water = 1 cal/g-°C,

L = latent heat of vaporization = 540 cal/g

$$\Delta\theta_1 = (100 - 90) = 10°\text{C}$$

and $$\Delta\theta_2 = (90 - 24) = 66°\text{C}$$

Substituting the values, we have

$$m = \frac{(100)(1)(66)}{(540) + (1)(10)} = 12 \text{ g}$$

$\therefore$ $$m = 12 \text{ g}$$

6. Heat liberated by 10 g water (at 40°C) when it converts into water at 0°C

$$Q = m_1 s\Delta\theta$$
$$= 10 \times 1 \times 40$$
$$= 400 \text{ cal}$$

Mass of ice melted by this heat,

$$m_2 = \frac{Q}{L} = \frac{400}{80} = 5 \text{ g} < 15 \text{ g}$$

Therefore, whole ice is not melted.

Temperature of mixture is 0°C.

Mass of water $= m_1 + m_2 = 15$ g

Mass of ice $= 15 - m_2 = 10$ g

7. Similar to Example 22.4

INTRODUCTORY EXERCISE 22.2

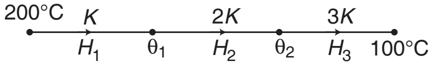
2. $$Q = \frac{KA\,(\theta_1 - \theta_2)t}{l}$$

$\therefore$ $$K = \frac{Ql}{A\,(\theta_1 - \theta_2)\,t} = \frac{\text{J - m}}{\text{m}^2\text{ K - s}}$$
$$= \frac{\text{W}}{\text{m-K}} \qquad \text{(as J/s = W)}$$

3. $$R = \frac{l}{KA} = \frac{\text{m}}{(\text{W/m -K})\,\text{m}^2}$$
$$= \frac{\text{K}}{\text{W}} = \text{KW}^{-1}$$

4. In convection, liquid is heated from the bottom.

5.



$$H_1 = H_2 = H_3$$

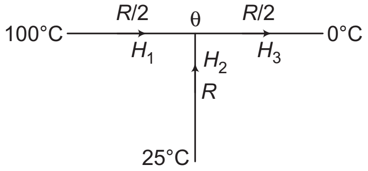
$\therefore$ $$\frac{200 - \theta_1}{(l/KA)} = \frac{\theta_1 - \theta_2}{(l/2KA)} = \frac{\theta_2 - 100}{(l/3KA)}$$

Solving these equations we get,

$$\theta_1 = 145.5°\text{C}$$

and $$\theta_2 = 118.2°\text{C}$$

6.



$$H_1 + H_2 = H_3$$
$$\frac{100 - \theta}{(R/2)} + \frac{25 - \theta}{R} = \frac{\theta - 0°}{R/2}$$

Solving we get, $\theta = 45°C$

$$H_{CD} = \frac{\text{TD}}{R} = \frac{45°\text{C} - 25°\text{C}}{5}$$
$$= 4\,\text{W}$$

7. Let us start with the assumption that

$$\left|\frac{\Delta\theta}{\Delta}\right| \propto \text{Temperature difference}$$

$$\therefore \quad \left(\frac{80-50}{5}\right) = \alpha\left[\frac{80+50}{2} - 20\right]$$

$$\text{and} \quad \left(\frac{60-30}{t}\right) = \alpha\left[\frac{60+30}{2} - 20\right]$$

Solving these two equations, we get

$$t = 9 \text{ min}$$

Exercises

LEVEL 1

Assertion and Reason

1. Specific heat is a function of temperature. But for most of the substances variation with temperature is almost negligible.

4. $R_1 = 2R$ and $R_2 = \frac{R}{2}$

Thermal resistance becomes $\frac{1}{4}$th. Therefore heat current becomes four times.

5. If temperature of a normal body is more than a perfectly black body, it can radiate more energy.

6. This law was given by Kirchhoff.

9. In steady state, temperature of different sections (perpendicular to the direction of heat flow) becomes constant but not same.

10. $-\frac{dT}{dt} = \frac{eA\sigma}{mc}(T^4 - T_0^4)$ or $-\frac{dT}{dt} \propto \frac{1}{m}$

Objective Questions

1. $\lambda_m \propto \frac{1}{T}$

$$\therefore \quad \frac{(\lambda_m)_2}{(\lambda_m)_1} = \frac{T_1}{T_2} = \frac{2000}{3000} = \frac{2}{3}$$

$$\therefore \quad (\lambda_m)_2 = \frac{2}{3}(\lambda_m)_1$$

2. At higher temperature radiation is more. So, cooling is fast.

3. $Q_1 = Q_2$

$$\therefore \quad ms_1(32-20) = ms_2(40-32)$$

$$\therefore \quad \frac{s_1}{s_2} = \frac{8}{12} = \frac{2}{3}$$

4. $Q = \int dQ = \int_1^2 ms\,dT$

$$= \int_1^2 (1)(aT^3)\,dT = \frac{15a}{4}$$

5. $\lambda_m \propto \frac{1}{T}$

$$\therefore \quad \frac{(\lambda_m)_1}{(\lambda_m)_2} = \frac{T_2}{T_1} \quad \text{or} \quad \frac{T_1}{T_2} = \frac{(\lambda_m)_2}{(\lambda_m)_1}$$
$$= \frac{350}{510} = 0.69$$

6.

$$dQ = C\,dT$$

$$\therefore \quad \frac{dT}{dQ} = \frac{1}{C} = \text{Slope of } T\text{-}Q \text{ graph.}$$

7. Thermal resistance $= \frac{l}{KA} = \frac{l}{K(\pi R^2)} \propto \frac{l}{R^2}$

The rod for which $\frac{l}{R^2}$ is minimum will conduct maximum heat.

8. Heat taken by 1 g ice in transformation from ice at 0°C to water at 100°C is

$$Q = mL + ms\Delta\theta$$
$$= (1)(80) + (1)(1)(100)$$
$$= 180 \text{ cal}$$

Mass of steam condensed to give this much heat is

$$m = \frac{Q}{L} = \frac{180}{540} = \frac{1}{3}\text{ g}$$

This is less than 1g or total mass of steam. Therefore, whole steam is not condensed and mixture temperature is 100°C.

9.

$$H_A = H_{\text{Total}}$$

$$\therefore \quad \frac{(\text{TD})_A}{R_A} = \frac{\text{TD}}{R_A + R_B}$$

$$\therefore \quad (\text{TD})_A = \left(\frac{R_A}{R_A + R_B}\right)\text{TD}$$
$$= \left(\frac{1}{1 + R_B/R_A}\right)\text{TD} \qquad \text{...(i)}$$

$$\frac{R_B}{R_A} = \left(\frac{l_B}{l_A}\right)\left(\frac{K_A A}{K_B A}\right) = \left(\frac{20}{10}\right)(3) = 6$$

Substituting in Eq. (i), we get

$$(\text{TD})_A = \left(\frac{1}{1+6}\right) \times 35 = 5°\text{C}$$

10. Thermal resistance,

$$R = \frac{l}{KA} \propto \frac{1}{K} \quad \Rightarrow \quad K_A = 2\,K_B$$

$$\therefore \qquad R_A = \frac{R_B}{2}$$

So, let $R_A = R$ then $R_B = 2R$

Now, $\qquad H_A = H_B$

$$\therefore \quad \frac{(\text{TD})_A}{R_A} = \frac{(\text{TD})_B}{R_B} \quad \text{or} \quad (\text{TD})_A = \left(\frac{R_A}{R_B}\right)(\text{TD})_B$$

$$= \left(\frac{R}{2R}\right)(36°\text{C}) = 18°\text{C}$$

11. $R = \dfrac{l}{KA}$

$$\frac{R_2}{R_1} = \left(\frac{l_2}{l_1}\right)\left(\frac{K_1}{K_2}\cdot\frac{A_1}{A_2}\right)$$

$$= (2)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

Therefore, thermal resistance of 2nd rod is half. Hence, rate of heat flow will be twice.

Subjective Questions

1. Total rate of radiation of energy

$$= e_r\,\sigma T^4 A$$

$$= (0.6)(5.67 \times 10^{-8})(800 + 273)^4(0.1)^2(2)$$

$$\approx 900 \text{ W}$$

2. $\dfrac{1}{2}\left(\dfrac{1}{2}mv^2\right) = ms\Delta\theta + mL$

$$\therefore \qquad v = 2\sqrt{s\Delta\theta + L}$$

$$= 2\sqrt{(125)\,(327 - 27) + 2.5 \times 10^4}$$

$$= 500 \text{ m/s}$$

3. $0.4\,[mg\Delta h] = ms\Delta\theta$

$$\therefore \qquad \Delta\theta = \frac{0.4\,g\Delta h}{s} = \frac{0.4 \times 9.8 \times 0.5}{800}$$

$$= 2.5 \times 10^{-3}\,°\text{C}$$

4. $P = \left(\dfrac{dm}{dt}\right) s \cdot \Delta\theta$

$$\therefore \qquad \frac{dm}{dt} = \frac{P}{s\Delta\theta} = \frac{500 \times 10^6}{(4200) \times 10}$$

$$= 11904 \text{ kg/s}$$

$$= 1.2 \times 10^4 \text{ kg/s}$$

5. $P = e_r\,\sigma A\,(T^4 - T_0^4)$

$$= (0.4)(5.67 \times 10^{-8})(4\pi)(4 \times 10^{-2})^2[(3000)^4 - (300)^4]$$

$$= 3.7 \times 10^4 \text{ W}$$

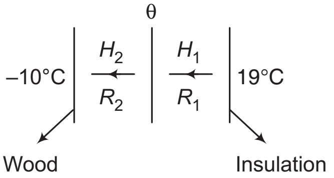
6. $L\left(\dfrac{dm}{dt}\right) = \dfrac{dQ}{dt} = H = \dfrac{\text{TD}}{R} = \dfrac{\theta - 100}{(l/KA)}$

$$\therefore \qquad \theta = 100 + \left(\frac{Ll}{KA}\right)\left(\frac{dm}{dt}\right)$$

$$= 100 + \left(\frac{2.256 \times 10^6 \times 1.2 \times 10^{-2}}{50.2 \times 0.15}\right)\left(\frac{0.44}{60 \times 5}\right)$$

$$= 105°\text{C}$$

7. (a) $H_1 = H_2$



$$\therefore \qquad \frac{19 - \theta}{R_1} = \frac{\theta + 10}{R} \qquad \left(H = \frac{\text{TD}}{R}\right)$$

$$\text{or} \qquad \frac{19 - \theta}{(l_1/K_1A)} = \frac{\theta + 10}{(l_2/K_2A)}$$

$$\therefore \qquad \frac{K_1(19 - \theta)}{l_1} = \frac{K_2(\theta + 10)}{l_2}$$

$$\therefore \qquad \frac{0.01\,(19 - \theta)}{3.5} = \frac{0.08\,(\theta + 10)}{2.0}$$

Solving, we get

$$\theta = -\,8.1°\text{C}$$

(b) $H = H_1 = \dfrac{19 - \theta}{(l_1/K_1A)}$

$$= \frac{K_1A\,(19 - \theta)}{l_1}$$

$$= \frac{(0.01)\,(19 + 8.1)\,(1)}{3.5 \times 10^{-2}}$$

$$= 7.7 \text{ W/m}^2$$

8. Hint is already given in the question

9. Let mixture is water at θ°C (where 0°C< θ <40°C)

Heat given by water = Heat taken by ice

$$\therefore \quad (200)\,(1)\,(40 - \theta) = (140)\,(0.53)\,(15) + (140)\,(80) + (140)\,(1)\,(\theta - 0)$$

Solving we get,

$$\theta = -\,12.7°\text{C}$$

Since, $\theta < 0°\text{C}$ and we have assumed the mixture to be water whose temperature can't be less than $0°\text{C}$.

Hence, mixture temperature $\theta = 0°\text{C}$.

Heat given by water in reaching upto $0°\text{C}$ is,

$$\theta = (200)(1)(40 - 0) = 8000 \text{ cal.}$$

Let m mass of ice melts by this heat, then

$$8000 = (140)(0.53)(15) + (m)(80)$$

Solving we get $m = 86$ g

∴ Mass of water = 200 + 86 = 286 g

Mass of ice = 140 − 86 = 54 g

10. Let heat is supplied at a constant rate of $(\alpha)\,\dfrac{\text{J}}{\text{min}}$.

In 4 minutes, (when temperature remains constant) ice will be melting.

$$\therefore \}\qquad mL = (4)\alpha$$

$$\therefore \qquad \frac{\alpha}{m} = \frac{L}{4} = \frac{336 \times 10^3}{4}$$

$$= 84 \times 10^3 \text{ J/min - kg}$$

Now in last two minutes,

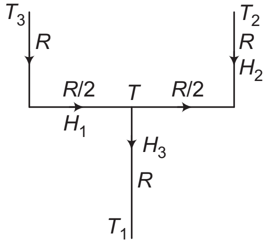
$$Q = ms\Delta\theta$$

$$\therefore \qquad (\alpha)(2) = (m)(4200)(\theta - 0°\text{C})$$

Substituting the value of $\dfrac{\alpha}{m}$ we get,

$$\theta = 40°\text{C}$$

11.

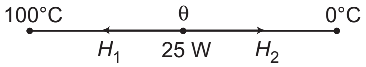


$$H_1 + H_2 = H_3$$

$$\left(\frac{T_3 - T}{3R/2}\right) + \left(\frac{T_2 - T}{3R/2}\right) = \frac{T - T_1}{R}$$

$$\therefore \qquad T = \frac{3T_1 + 2(T_2 + T_3)}{7}$$

12.



$$H_1 + H_2 = 25$$

$$\therefore \quad \frac{\theta - 100}{0.5/(400 \times 10^{-4})} + \frac{\theta - 0}{0.5/(400 \times 10^{-4})} = 25$$

Solving this equation, we get

$$\theta \approx 206°\text{C}$$

Now, temperature gradient on

$$\text{LHS} = \frac{\theta - 100}{0.5} = 212°\text{C/m}$$

$$\text{and on} \qquad \text{RHS} = \frac{\theta - 0}{0.5} = 424°\text{C/m}$$

13. Net power $= e_r\sigma(T^4 - T_0^4)A$

In first case,

$$210 = e_r\sigma\,[(500)^4 - (300)^4]\,A \qquad \text{...(i)}$$

In second case,

$$700 = \sigma\,[(500)^4 - (300)^4]\,A \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$e_r = \frac{210}{700} = 0.3$$

LEVEL 2

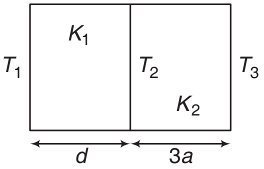
Single Correct Option

1. $R = \dfrac{l}{KA}$

l is halved, A is four times and K is $\dfrac{1}{4}$ times.

∴ R will become half. Hence, heat current will become two times. Therefore, rate of melting of ice will also become two times or 0.2 g/s.

2.



$$H_1 = H_2$$

$$\therefore \qquad \frac{T_1 - T_2}{(d/K_1A)} = \frac{T_2 - T_3}{(3d/K_2A)}$$

$$\therefore \qquad \frac{K_1}{K_2} = \left(\frac{T_2 - T_3}{T_1 - T_2}\right)\cdot\frac{1}{3}$$

But $T_1 : T_2$ and T_3 are in AP

$$\therefore \qquad T_2 - T_3 = T_1 - T_2$$

$$\therefore \qquad \frac{K_1}{K_2} = \frac{1}{3}$$

3. $H = \dfrac{\text{TD}}{R}$

$$\therefore \qquad R = \frac{\text{TD}}{H} = \frac{100 - 0}{1}$$

$$= 100 \text{ kW}^{-1}$$

Now, $R = \int_0^x dR = \int_0^x \frac{dx}{K_0(1+ax)A}$

or $100 = \int_0^x \frac{dx}{10^2\,(1+x)\,(10^{-4})}$

Solving this equation we get,

$$x = 1.7 \text{ m}$$

4. Let R = thermal resistance of each rod.

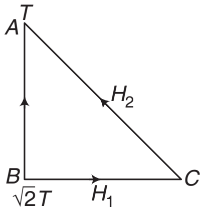
In first case, $R_{\text{net}} = R + R = 2R$ (in series)

In second case, $R_{\text{net}} = \frac{R}{2}$ (in parallel)

In second case thermal resistance has become $\frac{1}{4}$th.

So, heat current will become 4 times. So time taken to flow same amount of heat will be reduced to $\frac{1}{4}$th.

5.



$$H_1 = H_2$$

$$\therefore \quad \frac{\sqrt{2}T - T_C}{(l/KA)} = \frac{T_C - T}{(\sqrt{2}l/KA)}$$

Solving we get,

$$T_C = \frac{3T}{\sqrt{2}+1}$$

6. Net power available per second,

$$P = 1000 - 160 = 840 \text{ J/s}$$

Heat required,

$$Q = ms\Delta\theta = (2)\,(4200)\,(77-27)$$
$$= 420000 \text{ J}$$

$\therefore$ Time required $= \frac{Q}{P} = 500 \text{ s} = 8 \text{ min } 20 \text{ s}$

7. $R_{\text{net}} = R_1 + R_2 = \left(\frac{X}{KA}\right) + \frac{4X}{2KA} = \frac{3X}{KA}$

Now, $H = \frac{dQ}{dt} = \frac{\text{TD}}{R_{\text{net}}}$

$$= \frac{(T_2 - T_1)}{(3X/KA)} = \left[\frac{KA\,(T_2 - T_1)}{x}\right]\left(\frac{1}{3}\right)$$

$$\therefore \quad f = \frac{1}{3}$$

More than One Correct Options

1. Total radiation per second is given by

$$P = e_r \sigma T^4 A$$

Here, $e_r : \sigma : T$ and A all are same.

Hence, P will be same.

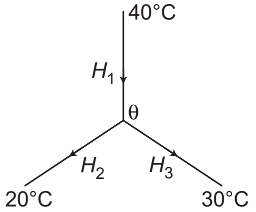
Same is the case with absorption per second.

Now, $P = ms\left(-\frac{dT}{dt}\right)$

or Rate of cooling $\left(-\frac{dT}{dt}\right) = \frac{P}{ms} \propto \frac{1}{m}$

Mass of hollow sphere is less. So, its initial rate of cooling will be more.

2.



$$H_1 = H_2 + H_3$$

$$\therefore \quad \frac{40-\theta}{R} = \frac{\theta - 20}{R} + \frac{\theta - 30}{R}$$

where, R = thermal resistance of each rod

Solving this equation, we get

$$\theta = 30°\text{C}$$

Heat flows from higher temperature to lower temperature.

3. $ms\,(2\theta - \theta') = m\,(2s)\,(\theta' - \theta)$

Solving this equation we get,

$$\theta' = \frac{4}{3}\theta$$

Further, heat capacity

$$C = ms \quad \text{or} \quad C \propto s \qquad (\text{as } m \text{ is same})$$

$$\therefore \quad \frac{C_1}{C_2} = \frac{s_1}{s_2} = \frac{1}{2}$$

4. $q_1 = \frac{\text{TD}}{R_1}$

$$\therefore \quad R_1 = \frac{\text{TD}}{q_1}$$

Similarly, $R_2 = \frac{\text{TD}}{q_2}$

In series,

$$q_s = \frac{\text{TD}}{R_1 + R_2} = \frac{\text{TD}}{\frac{\text{TD}}{q_1} + \frac{\text{TD}}{q_2}}$$

$$= \frac{q_1 q_2}{q_1 + q_2}$$

In parallel, $q_p = \frac{\text{TD}}{R_{\text{net}}} = \text{TD}\left(\frac{1}{R_{\text{net}}}\right)$

$$= \text{TD}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= \text{TD}\left(\frac{q_1}{\text{TD}} + \frac{q_2}{\text{TD}}\right) = q_1 + q_2$$

5. (b) If temperature difference is small, the rate of cooling is proportional to TD.

Match the Columns

1. (a) $E = \sigma T^4$ = energy radiated per unit surface are per unit time by a black body

$$\therefore \quad \sigma = \left[\frac{E}{T^4}\right] = \left[\frac{\text{ML}^2\text{T}^{-2}}{\text{L}^2\text{T}\theta^4}\right]$$

$$= [\text{MT}^{-3}\,\theta^{-4}]$$

(b) $$b = \lambda_m T$$

$$\therefore \quad [b] = [\text{L}\theta]$$

(c) Emissive power is energy radiated per unit time per unit surface area.

$$\therefore \quad [\text{E}] = \left[\frac{\text{ML}^2\,\text{T}^{-2}}{\text{L}^2\,\text{T}}\right] = [\text{MT}^{-3}]$$

(d) $H = \frac{dQ}{dt} = \frac{\text{TD}}{R}$

$$\therefore \quad R = \frac{\text{TD}}{(dQ/dt)} = \frac{\theta}{[\text{ML}^2\text{T}^{-2}/\text{T}]}$$

$$= [\text{M}^{-1}\,\text{L}^{-2}\,\text{T}^3\theta]$$

2. (a) $$\frac{dQ}{dt} = ms\frac{d\theta}{dt}$$

$$\therefore \quad \left(\frac{d\theta}{dt}\right) = \frac{dQ/dt}{ms}$$

$\therefore$ Slope of θ - t graph $\propto \frac{1}{m}$

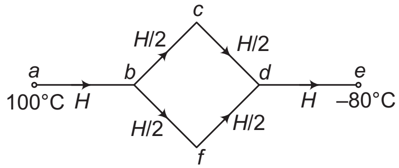
(b) $$Q_{\text{Total}} = mL$$

or $\left(\frac{dQ}{dt}\right)(\text{time}) = mL$

or time $\propto m$ or length of line $bc \propto m$

(d) *ab* : only solid state
bc : solid + liquid state
cd : only liquid state
de : liquid + gaseous state
ef : only gaseous state.

3. If R = thermal resistance of one rod



Then, $R_{\text{net}} = 3R$

$$\therefore \quad H = \frac{100 - (-80)}{3R} = \frac{60}{R}$$

(a) ***ab*** $$H = \frac{60}{R} = \frac{100 - \theta_b}{R}$$

$$\therefore \quad \theta_b = 40°\text{C}$$

(b) ***bc*** $$\frac{H}{2} = \frac{30}{R} = \frac{40 - \theta_c}{R}$$

$$\therefore \quad \theta_C = 10°\text{C}$$

(c) Same as above

(d) ***de***

$$H = \frac{60}{R} = \frac{\theta_d - (-80)}{R}$$

$$\therefore \quad \theta_d = -20°\text{C}$$

4. (a) $m\,(s)(\theta' - \theta) = 2m\,(s)\,(2\theta - \theta')$

Solving we get,

$$\theta' = \frac{5}{3}\theta$$

(b) $ms\,(\theta' - \theta) = 3m\,(s)\,(3\theta - \theta')$

Solving we get,

$$\theta' = \frac{5}{2}\theta$$

(c) $2m\,(s)\,(\theta' - 2\theta) = 3m\,(s)\,(3\theta - \theta')$

Solving we get,

$$\theta' = \frac{13}{5}\theta$$

(d) $ms\,(\theta' - \theta) + 2\,ms\,(\theta' - 2\theta) + 3ms\,(\theta' - 3\theta)$

$$= 0$$

Solving we get,

$$\theta' = \frac{7}{3}\theta$$

5. Unit of heat capacity is J/°C.

Unit of latent heat is J/kg.

Subjective Questions

1. (a) Between t = 1 min to t = 3 min, there is no rise in the temperature of substance. Therefore, solid melts in this time.

$$L=\frac{Q}{m}=\frac{Ht}{m}=\frac{10\times 2}{0.5}$$
$$=40\text{ kJ/kg}$$

(b) From $Q=ms\Delta T$ or $s=\dfrac{Q}{m\Delta T}$

Specific heat in solid state

$$s=\frac{10\times 1}{0.5\times 15}=1.33\text{ kJ/kg-}^\circ\text{C}$$

2. Let T_0 be the temperature of surrounding and T be the temperature of hot body at some instant. Then,

$$-\frac{dT}{dt}=K(T-T_0)$$

or $$\int_{T_m}^{T}\frac{dT}{T-T_0}=-K\int_0^t dT$$

(T_m = temperature at $t=0$)

Solving this equation, we get

$$T=T_0+(T_m-T_0)e^{-Kt} \qquad \text{(i)}$$

Maximum temperature it can lose is (T_m-T_0)

From Eq. (i),

$$T-T_0=(T_m-T_0)e^{-Kt}$$

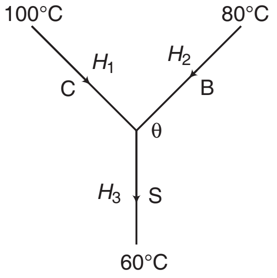
Given that

$$T-T_0=\frac{T_m-T_0}{2}=(T_m-T_0)e^{-Kt}$$

Solving this equation we get,

$$t=\frac{\ln(2)}{K}$$

3. Let θ be the temperature of junction. Then,



$$H_1+H_2=H_3$$

$$\therefore\quad \frac{100-\theta}{(46/0.92A)}+\frac{80-\theta}{(13/0.26A)}=\frac{\theta-60}{(12/0.12A)}$$

Solving this equation we have,

$$\theta=84^\circ\text{C}$$

(b) Heat current in copper rod,

$$H_1=\frac{100-\theta}{(46/0.92\times 4)}$$
$$=1.28\text{ cal/s} \qquad \textbf{Ans.}$$

4. Let heat capacity of flask is C and latent heat of fusion of ice is L.

Then,

$$C(70-40)+200\times 1\times(70-40)=50L+50\times 1\times(40-0)$$

or $$3C-5L=-400 \qquad \ldots\text{(i)}$$

Further, $$C(40-10)+250\times 1\times(40-10)=80L+80\times 1\times(10-0)$$

or $$3C-8L=-670 \qquad \ldots\text{(ii)}$$

Solving Eq. (i) and (ii), we have

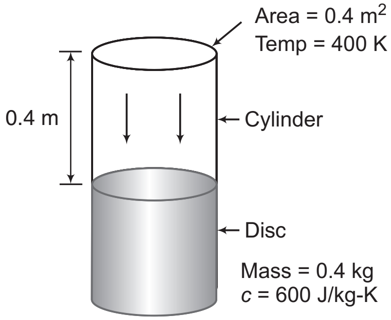
$$L=90\text{ cal/g} \qquad \textbf{Ans.}$$

5. $ms\Delta\theta$ = work done against friction

$=(\mu mg\cos\theta)d$ (but $\mu mg\cos\theta=mg\sin\theta$)

$$\Delta\theta=\frac{(\mu g\cos\theta)d}{s}=\frac{(g\sin\theta)d}{s}$$
$$=\frac{(10)\left(\frac{3}{5}\right)(0.6)}{420}$$
$$=8.57\times 10^{-3}\ ^\circ\text{C}$$
$$\approx 8.6\times 10^{-3}\ ^\circ\text{C} \qquad \textbf{Ans.}$$

6. Using $$\frac{dQ}{dt}=mc\frac{d\theta}{dt}=\frac{KA(\theta_1-\theta_2)}{l}$$



We have,

$$mc\left(\frac{d\theta}{dt}\right)=\frac{KA(400-\theta)}{0.4}$$

$$(0.4)(600)\frac{d\theta}{dt}=\frac{(10)(0.04)(400-\theta)}{0.4}$$

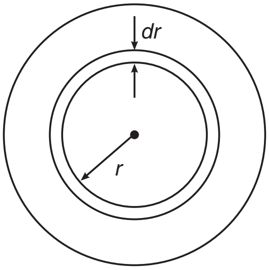
$$\therefore\quad \left(\frac{d\theta}{400-\theta}\right)=\frac{1}{240}dt$$

$$\therefore\quad \int_0^t dt=240\int_{300}^{350}\frac{d\theta}{400-\theta}$$

Solving this, we get

$$t\approx 166\text{ s} \qquad \textbf{Ans.}$$

7. Consider a differential cylinder



$$H=\frac{dQ}{dt}=KA\frac{d\theta}{dr}=(2\pi Krl)\frac{d\theta}{dr}$$

∴ $$\frac{H}{2\pi Kl}\int_{r_1}^{r_2}\frac{dr}{r}=\int_0^{50}d\theta$$

or $$\frac{H}{2\pi Kl}\ln\left(\frac{r_2}{r_1}\right)=50$$

∴ $$H=\frac{100\pi Kl}{\ln(r_2/r_1)}$$

Now, $$Ht=mL$$

∴ $$t=\frac{mL}{H}=\frac{mL\ln(r_2/r_1)}{100\pi Kl}$$ **Ans.**

8. Three thermal resistances are in series. $\left(R=\frac{l}{KA}\right)$

∴ $$R=R_1+R_2+R_3$$
$$=\frac{2.5\times10^{-2}}{0.125\times137}+\frac{1.0\times10^{-2}}{1.5\times137}+\frac{2.5\times10^{-2}}{1.0\times137}$$
$$=0.0017\ {}^\circ\text{C-s/J}$$

Now, heat current $H=\dfrac{\text{Temperature difference}}{\text{Net thermal resistance}}$

$$=\frac{30}{0.0017}=17647\ \text{W}$$

9. Thermal resistance of plastic coating $\left(R_t=\frac{l}{KA}\right)$

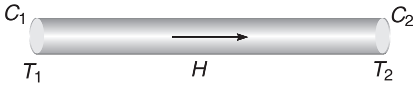
$$R_t=\frac{t}{K(\pi d)l}\qquad (d=\text{diameter})$$

$$=\frac{0.06\times10^{-3}}{(0.16\times10^{-2}\times4.18\times10^{2})(\pi)(0.64\times10^{-3})(2)}$$
$$=0.0223\ {}^\circ\text{C-s/J}$$

Now, $$i^2R_e=\frac{\text{TD}}{R_t}$$

∴ $$\text{TD}=i^2R_eR_t$$
$$=(5)^2(4)(0.0223)$$
$$=2.23^\circ\text{C}$$ **Ans.**

10. Let T_1 be the temperature of C_1 and T_2 the temperature of C_2 at some instant of time. Further let T be the temperature difference at that instant.



Then,

$$C_1\left(-\frac{dT_1}{dt}\right)=H=\frac{T}{l/KA}=\frac{KA}{l}(T)$$

$$(\because T=T_1-T_2)$$

and $$C_2\left(+\frac{dT_2}{dt}\right)=H=\frac{T}{l/KA}=\frac{KA}{l}(T)$$

∴ $$-\frac{dT}{dt}=-\frac{dT_1}{dt}+\frac{dT_2}{dt}$$

∴ $$-\frac{dT_1}{dt}=\frac{KA}{lC_1}(T)$$

and $$+\frac{dT_2}{dt}=\frac{KA}{lC_2}$$

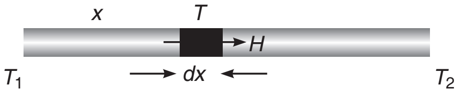
Further, $$-\frac{dT}{dt}=-\frac{dT_1}{dt}+\frac{dT_2}{dt}$$
$$=\frac{KA(C_1+C_2)}{lC_1C_1}T$$

or $$\int_{\Delta T_0}^{T}\frac{dT}{T}=-\frac{KA(C_1+C_2)}{lC_1C_2}\int_0^t dt$$

Solving, we get $T=\Delta T_0e^{-\alpha t}$

where, $$\alpha=\frac{KA(C_1+C_2)}{lC_1C_2}$$

11.



$$H=\frac{\text{TD}}{R}=\frac{(-dT)}{(dx)/KA}$$
$$=-\left(\frac{dT}{dx}\right)\left(\frac{a}{T}A\right)=\text{constant}\quad\ldots(\text{i})$$

∴ $$\int_{T_1}^{T_2}-\frac{dT}{T}=\frac{H}{aA}\int_0^l dx$$

$$\ln\left(\frac{T_1}{T_2}\right)=\frac{Hl}{aA}$$

or $$H=\frac{aA}{l}\ln(T_1/T_2)$$

Substituting in Eq. (i), we have

$$\frac{aA}{l}\ln\left(\frac{T_1}{T_2}\right) = -\left(\frac{dT}{dx}\right)\frac{aA}{T}$$

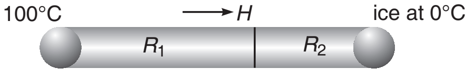
or $$\int_{T_1}^{T}\frac{dT}{T} = -\frac{\ln\left(\frac{T_1}{T_2}\right)}{l}\int_0^x dx$$

∴ $$\ln\left(\frac{T}{T_1}\right) = -\frac{x}{l}\ln(T_1/T_2) = \ln\left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}}$$

or $$\frac{T}{T_1} = \left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}}$$

or $$T = T_1\left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}} = T_1\left(\frac{T_2}{T_1}\right)^{\frac{x}{l}}$$ **Ans.**

12. $$L\left(\frac{dm}{dt}\right) = \frac{\text{TD}}{R_1 + R_2}$$

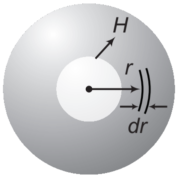


or $$\frac{80\times 360}{3600} = \frac{100}{\frac{20}{0.25\times 10} + \frac{10}{K\times 10}}$$

Solving this equation, we get

$$K = 0.222 \text{ cal/cm-s-°C}$$ **Ans.**

13. $$H = \frac{-d\theta}{dr/K(4\pi r^2)} = \left(-\frac{d\theta}{dr}\right)(4\pi K r^2) \qquad \ldots\text{(i)}$$



or $$\int_{R_1}^{R_2}\frac{dr}{r^2} = -\frac{4\pi K}{H}\int_{\theta_1}^{\theta_2} d\theta$$

or $$\frac{R_2 - R_1}{R_1R_2} = \frac{4\pi K}{H}(\theta_1 - \theta_2)$$

∴ $$H = \frac{4\pi K(R_1R_2)(\theta_1 - \theta_2)}{R_2 - R_1}$$

Substituting this value of H in Eq. (i), we have

$$\frac{R_1R_2}{R_2 - R_1}(\theta_1 - \theta_2) = -\frac{d\theta}{dr}(r^2)$$

∴ $$\int_{\theta_1}^{\left(\frac{(\theta_1+\theta_2)}{2}\right)} d\theta = -\frac{R_1R_2}{R_2 - R_1}(\theta_1 - \theta_2)\int_{R_1}^{r}\frac{dr}{r^2}$$

or $$\frac{\theta_1 + \theta_2}{2} - \theta_1 = \frac{R_1R_2}{R_2 - R_1}(\theta_1 - \theta_2)\left[\frac{1}{r} - \frac{1}{R_1}\right]$$

or $$\frac{\theta_1 - \theta_2}{2} = \frac{R_1R_2}{R_2 - R_1}(\theta_1 - \theta_2)\left(\frac{1}{R_1} - \frac{1}{r}\right)$$

$$\frac{1}{R_1} - \frac{1}{r} = \frac{R_2 - R_1}{2R_1R_2}$$

$$\frac{1}{r} = \frac{1}{R_1} - \frac{R_2 - R_1}{2R_1R_2} = \frac{R_1 + R_2}{2R_1R_2}$$

or $$r = \frac{2R_1R_2}{R_1 + R_2}$$ **Ans.**

14. See the extra points just before solved examples. Growth of ice on ponds. We have already derived that

$$t = \frac{1}{2}\frac{\rho L}{K\theta}y^2$$

∴ $$\frac{dt}{dy} = \frac{\rho L y}{K\theta}$$

∴ $$\frac{dy}{dt} = \frac{K\theta}{L\rho y}$$

JEE Main and Advanced

Previous Years' Questions (2018-13)

JEE Main

1. Two moles of an ideal monoatomic gas occupies a volume V at 27°C. The gas expands adiabatically to a volume $2V$. Calculate (i) the final temperature of the gas and (ii) change in its internal energy. **(2018)**

(a) (i) 189 K (ii) 2.7 kJ (b) (i) 195 K (ii) –2.7 kJ
(c) (i) 189 K (ii) –2.7 kJ (d) (i) 195 K (ii) 2.7 kJ

2. A silver atom in a solid oscillates in simple harmonic motion in some direction with a frequency of 10^{12} per second. What is the force constant of the bonds connecting one atom with the other? (Take, molecular weight of silver = 108 and Avogadro number = 6.02×10^{23} g mol^{-1}) **(2018)**

(a) 6.4 N/m (b) 7.1 N/m
(c) 2.2 N/m (d) 5.5 N/m

3. A granite rod of 60 cm length is clamped at its middle point and is set into longitudinal vibrations. The density of granite is 2.7×10^3 kg/m^3 and its Young's modulus is 9.27×10^{10} Pa. What will be the fundamental frequency of the longitudinal vibrations? **(2018)**

(a) 5 kHz (b) 2.5 kHz
(c) 10 kHz (d) 7.5 kHz

4. An observer is moving with half the speed of light towards a stationary microwave source emitting waves at frequency 10 GHz. What is the frequency of the microwave measured by the observer? (speed of light = 3×10^8 ms^{-1}) **(2017)**

(a) 12.1 GHz (b) 17.3 GHz
(c) 15.3 GHz (d) 10.1 GHz

5. C_p and C_V are specific heats at constant pressure and constant volume, respectively. It is observed that $C_p - C_V = a$ for hydrogen gas $C_p - C_V = b$ for nitrogen gas. The correct relation between a and b is **(2017)**

(a) $a = b$ (b) $a = 14b$
(c) $a = 28b$ (d) $a = \frac{1}{14}b$

6. A copper ball of mass 100 g is at a temperature T. It is dropped in a copper calorimeter of mass 100 g, filled with 170 g of water at room temperature. Subsequently, the temperature of the system is found to be 75°C. T is (Given, room temperature = 30°C, specific heat of copper = 0.1 cal/g°C) **(2017)**

(a) 885°C (b) 1250°C
(c) 825°C (d) 800°C

7. An external pressure p is applied on a cube at 0°C so that it is equally compressed from all sides. K is the bulk modulus of the material of the cube and α is its coefficient of linear expansion. Suppose we want to bring the cube to its original size by heating. The temperature should be raised by **(2017)**

(a) $\frac{p}{\alpha K}$ (b) $\frac{3\alpha}{pK}$
(c) $3pK\alpha$ (d) $\frac{p}{3\alpha K}$

8. The temperature of an open room of volume 30 m^3 increases from 17°C to 27°C due to the sunshine. The atmospheric pressure in the room remains 1×10^5 Pa. If n_i and n_f are the number of molecules in the room before and after heating, then $n_f - n_i$ will be **(2017)**

(a) 1.38×10^{23} (b) 2.5×10^{25}
(c) -2.5×10^{25} (d) -1.61×10^{23}

9. A uniform string of length 20 m is suspended from a rigid support. A short wave pulse is introduced at its lowest end. It starts moving up the string. The time taken to reach the support is (Take, $g = 10\ \text{ms}^{-2}$) **(2016)**

(a) $2\pi\sqrt{2}$ s (b) 2 s

(c) $2\sqrt{2}$ s (d) $\sqrt{2}$ s

10. A pipe open at both ends has a fundamental frequency f in air. The pipe is dipped vertically in water, so that half of it is in water. The fundamental frequency of the air column is now **(2016)**

(a) $\frac{f}{2}$ (b) $\frac{3f}{4}$

(c) $2f$ (d) f

11. A pendulum clock loses 12 s a day if the temperature is 40°C and gains 4 s a day if the temperature is 20°C. The temperature at which the clock will show correct time, and the coefficient of linear expansion α of the metal of the pendulum shaft are, respectively. **(2016)**

(a) 25°C, $\alpha = 1.85 \times 10^{-5}/°\text{C}$

(b) 60°C, $\alpha = 1.85 \times 10^{-4}/°\text{C}$

(c) 30°C, $\alpha = 1.85 \times 10^{-3}/°\text{C}$

(d) 55°C, $\alpha = 1.85 \times 10^{-2}/°\text{C}$

12. An ideal gas undergoes a quasi static, reversible process in which its molar heat capacity C remains constant. If during this process the relation of pressure p and volume V is given by pV^n = constant, then n is given by (Here, C_p and C_V are molar specific heat at constant pressure and constant volume, respectively) **(2016)**

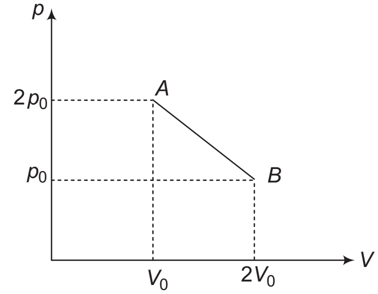
(a) $n = \frac{C_p}{C_V}$

(b) $n = \frac{C - C_p}{C - C_V}$

(c) $n = \frac{C_p - C}{C - C_V}$

(d) $n = \frac{C - C_V}{C - C_p}$

13. n moles of an ideal gas undergoes a process A and B as shown in the figure. The maximum temperature of the gas during the process will be **(2016)**



(a) $\frac{9}{4}\frac{p_0V_0}{nR}$ (b) $\frac{3}{2}\frac{p_0V_0}{nR}$ (c) $\frac{9}{2}\frac{p_0V_0}{nR}$ (d) $\frac{9p_0V_0}{nR}$

14. A train is moving on a straight track with speed 20 ms^{-1}. It is blowing its whistle at the frequency of 1000 Hz. The percentage change in the frequency heard by a person standing near the track as the train passes him is close to (speed of sound = 320 ms^{-1}) **(2015)**

(a) 12% (b) 6% (c) 18% (d) 24%

15. Consider a spherical shell of radius R at temperature T. The black body radiation inside it can be considered as an ideal gas of photons with internal energy per unit volume $u = \frac{U}{V} \propto T^4$ and pressure $p = \frac{1}{3}\left(\frac{U}{V}\right)$. If the shell now undergoes an adiabatic expansion, the relation between T and R is **(2015)**

(a) $T \propto e^{-R}$ (b) $T \propto \frac{1}{R}$ (c) $T \propto e^{-3R}$ (d) $T \propto \frac{1}{R^3}$

16. Consider an ideal gas confined in an isolated closed chamber. As the gas undergoes an adiabatic expansion, the average time of collision between molecules increases as V^q, where V is the volume of the gas. The value of q is $\left(\gamma = \frac{C_p}{C_V}\right)$ **(2015)**

(a) $\frac{3\gamma + 5}{6}$ (b) $\frac{\gamma + 1}{2}$ (c) $\frac{3\gamma - 5}{6}$ (d) $\frac{\gamma - 1}{2}$

17. A solid body of constant heat capacity 1 J/°C is being heated by keeping it in contact with reservoirs in two ways

(i) Sequentially keeping in contact with 2 reservoirs such that each reservoir supplies same amount of heat.

(ii) Sequentially keeping in contact with 8 reservoirs such that each reservoir supplies same amount of heat.

In both the cases, body is brought from initial temperature 100°C to final temperature 200°C. Entropy change of the body in the two cases respectively, is **(2015)**

(a) ln2, ln2 (b) ln2, 2ln2

(c) 2ln2, 8ln2 (d) ln2, 4ln2

18. A pipe of length 85 cm is closed from one end. Find the number of possible natural oscillations of air column in the pipe whose frequencies lie below 1250 Hz. The velocity of sound in air is 340 m/s. **(2014)**

(a) 12 (b) 8 (c) 6 (d) 4

19. Parallel rays of light of intensity $I = 912\ \text{Wm}^{-2}$ are incident on a spherical black body kept in surroundings of temperature 300 K. Take Stefan constant $\sigma = 5.7 \times 10^{-8}\ \text{Wm}^{-2}\ \text{K}^{-4}$ and assume that the energy exchange with the surroundings is only through radiation. The final steady state temperature of the black body is close to **(2014)**

(a) 330 K (b) 660 K (c) 990 K (d) 1550

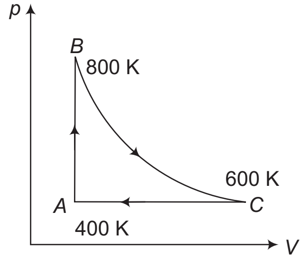
20. Three rods of copper, brass and steel are welded together to form a Y-shaped structure. Area of cross-section of each rod is $4\ \text{cm}^2$. End of copper rod is maintained at 100°C whereas ends of brass and steel are kept at 0°C. Lengths of the copper, brass and steel rods are 46, 13 and 12 cm respectively.

The rods are thermally insulated from surroundings except at ends. Thermal conductivities of copper, brass and steel are 0.92, 0.26 and 0.12 in CGS units, respectively. Rate of heat flow through copper rod is **(2014)**

(a) 1.2 cal/s (b) 2.4 cal/s

(c) 4.8 cal/s (d) 6.0 cal/s

21. One mole of diatomic ideal gas undergoes a cyclic process ABC as shown in figure. The process BC is adiabatic. The temperatures at A, B and C are 400 K, 800 K and 600 K, respectively. Choose the correct statement. **(2014)**



(a) The change in internal energy in whole cyclic process is 250 R

(b) The change in internal energy in the process CA is 700 R

(c) The change in internal energy in the process AB is –350 R

(d) The change in internal energy in the process BC is –500 R

22. A sonometer wire of length 1.5 m is made of steel. The tension in it produces an elastic strain of 1%. What is the fundamental frequency of steel if density and elasticity of steel are $7.7 \times 10^3\ \text{kg/m}^3$ and $2.2 \times 10^{11}\ \text{N/m}^2$ respectively?

(a) 188.5 Hz **(2013)**

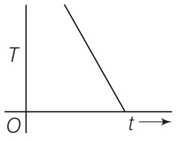
(b) 178.2 Hz

(c) 200.5 Hz

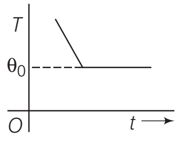
(d) 770 Hz

23. If a piece of metal is heated to temperature θ and then allowed to cool in a room which is at temperature θ_0. The graph between the temperature T of the metal and time t will be closed to **(2013)**

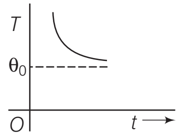
(a)



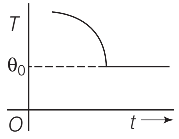
(b)



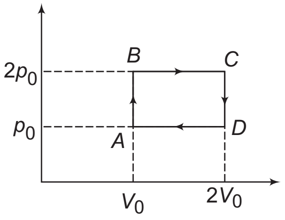
(c)



(d)



24. The shown p-V diagram represents the thermodynamic cycle of an engine, operating with an ideal monoatomic gas. The amount of heat, extracted from the source in a single cycle is **(2013)**



(a) p_0V_0 (b) $\left(\frac{13}{2}\right)p_0V_0$

(c) $\left(\frac{11}{2}\right)p_0V_0$ (d) $4p_0V_0$

25. An ideal gas enclosed in a vertical cylindrical container supports a freely moving piston of mass M. The piston and the cylinder have equal cross-sectional area A. When the piston is in equilibrium, the volume of the gas is V_0 and its pressure is p_0. The piston is slightly displaced from the equilibrium position and released. Assuming that the system is completely isolated from its surrounding, the piston executes a simple harmonic motion with frequency **(2013)**

(a) $\frac{1}{2\pi}\frac{A_\gamma p_0}{V_0M}$ (b) $\frac{1}{2\pi}\frac{V_0Mp_0}{A^2\gamma}$

(c) $\frac{1}{2\pi}\sqrt{\frac{A^2\gamma p_0}{MV_0}}$ (d) $\frac{1}{2\pi}\sqrt{\frac{MV_0}{A_\gamma p_0}}$

Answer *with* Explanations

1. **(c)** For adiabatic process relation of temperature and volume is,

$$T_2V_2^{\gamma-1} = T_1V_1^{\gamma-1}$$

$$\Rightarrow \quad T_2(2V)^{2/3} = 300(V)^{2/3}$$

$$\left[\gamma = \frac{5}{3} \text{ for monoatomic gases}\right]$$

$$\Rightarrow \quad T_2 = \frac{300}{2^{2/3}} \approx 189 \text{ K}$$

Also, in adiabatic process,

$\Delta Q = 0$, $\Delta U = -\Delta W$

$$\text{or} \quad \Delta U = \frac{-nR(\Delta T)}{\gamma - 1} = -2 \times \frac{3}{2} \times \frac{25}{3}(300 - 189)$$

$$\approx -2.7 \text{ kJ}$$

$$T_2 \approx 189 \text{ K}, \Delta U \approx 2.7 \text{ kJ}$$

2. **(b)** For a harmonic oscillator,

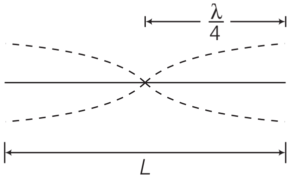
$T = 2\pi\sqrt{\frac{m}{k}}$, where k = force constant and $T = \frac{1}{f}$

$$\therefore \quad k = 4\pi^2f^2m$$

$$= 4 \times \left(\frac{22}{7}\right)^2 \times (10^{12})^2 \times \frac{108 \times 10^{-3}}{6.02 \times 10^{23}}$$

$$\Rightarrow \quad k = 7.1 \text{ N/m}$$

3. **(a)**



From vibration mode,

$$\frac{\lambda}{2} = L \quad \Rightarrow \quad \lambda = 2L$$

$$\therefore \text{Wave speed, } v = \sqrt{\frac{Y}{\rho}}$$

$$\text{So, frequency } f = \frac{v}{\lambda}$$

$$\Rightarrow \quad f = \frac{1}{2L}\sqrt{\frac{Y}{\rho}}$$

$$= \frac{1}{2 \times 60 \times 10^{-2}}\sqrt{\frac{9.27 \times 10^{10}}{2.7 \times 10^3}}$$

$$\approx 5000 \text{ Hz}$$

$$f = 5 \text{ kHz}$$

4. **(b)** As observer is moving with relativistic speed; formula $\frac{\Delta f}{f} = \frac{v_{\text{radial}}}{c}$, does not apply here.

Relativistic doppler's formula is

$$f_{\text{observed}} = f_{\text{actual}} \cdot \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}$$

$$\text{Here,} \quad \frac{v}{c} = \frac{1}{2}$$

$$\text{So,} \quad f_{\text{observed}} = f_{\text{actual}}\left(\frac{3/2}{1/2}\right)^{1/2}$$

$$\therefore \quad f_{\text{observed}} = 10 \times \sqrt{3}$$

$$= 17.3 \text{ GHz}$$

5. **(b)** By Mayor's relation, for 1 g mole of a gas,

$$C_p - C_V = R$$

So, when n gram moles are given,

$$C_p - C_V = \frac{R}{n}$$

As per given question,

$$a = C_p - C_V = \frac{R}{2}; \text{ for } H_2$$

$$b = C_p - C_V = \frac{R}{28}; \text{ for } N_2$$

$$a = 14b$$

6. (a) Heat gained (water + calorimeter) = Heat lost by copper ball

$$\Rightarrow \quad m_w s_w \Delta T + m_c s_c \Delta T = m_B s_B \Delta T$$

$$\Rightarrow \quad 170 \times 1 \times 45 + 100 \times 0.1 \times 45 = 100 \times 0.1 \times (T - 75)$$

$$\therefore \quad T = 885°\text{C}$$

7. (d) $K = \dfrac{p}{(-\Delta V / V)} \Rightarrow \dfrac{\Delta V}{V} = \dfrac{p}{K}$

$$\Rightarrow \quad -\Delta V = \frac{pV}{K} \Rightarrow \frac{pV}{K} = V(3\alpha)\,\Delta T \Rightarrow \Delta T = \frac{p}{3\alpha K}$$

8. (c) From $pV = nRT = \dfrac{N}{N_A} RT$

We have, $n_f - n_i = \dfrac{pVN_A}{RT_f} - \dfrac{pVN_A}{RT_i}$

$$\Rightarrow \quad n_f - n_i = \frac{10^5 \times 30}{8.3} \times 6.02 \times 10^{23} \cdot \left(\frac{1}{300} - \frac{1}{290}\right)$$

$$= -2.5 \times 10^{25}$$

$$\therefore \quad \Delta n = -2.5 \times 10^{25}$$

9. (c) At distance x from the bottom

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{\left(\frac{mgx}{L}\right)}{\left(\frac{m}{L}\right)}} = \sqrt{gx} \Rightarrow \frac{dx}{dt} = \sqrt{x}\sqrt{g}$$

$$\Rightarrow \quad \int_0^L x^{-1/2} dx = \sqrt{g}\int_0^t dt \Rightarrow \left[\frac{x^{1/2}}{(1/2)}\Bigg|_0^L\right] = \sqrt{g} \cdot t$$

$$\Rightarrow \quad t = \frac{2\sqrt{L}}{\sqrt{g}} \Rightarrow t = 2\sqrt{\frac{20}{10}} = 2\sqrt{2}\,\text{s}$$

10. (d) Fundamental frequency of open pipe.

$$f = \frac{v}{2l}$$

Now, after half filled with water it becomes a closed pipe of length $\dfrac{l}{2}$.

Fundamental frequency of this closed pipe,

$$f' = \frac{v}{4(l/2)} = \frac{v}{2l} = f$$

11. (a) $T_0 = 2\pi\sqrt{\dfrac{L}{g}}$

$$T' = T_0 + \Delta T = 2\pi\sqrt{\frac{L + \Delta L}{g}} \Rightarrow T' = T_0 + \Delta T$$

$$= 2\pi\sqrt{\frac{L(1 + \alpha\,\Delta\theta)}{g}} = \left\{2\pi\sqrt{\frac{L}{g}}\right\}(1 + \alpha\,\Delta\theta)^{\frac{1}{2}}$$

$$\approx T_0\left(1 + \frac{\alpha\,\Delta\theta}{2}\right)$$

$$\therefore \quad \Delta T = T' - T_0 = \frac{\alpha\,\Delta\theta T_0}{2} \qquad \ldots\text{(i)}$$

or $\quad \dfrac{\Delta T_1}{\Delta T_2} = \dfrac{\alpha\,\Delta\theta_1 T_0}{\alpha\,\Delta\theta_2 T_0} \Rightarrow \dfrac{12}{4} = \dfrac{40 - \theta}{\theta - 20}$

$$\Rightarrow \quad 3(\theta - 20) = 40 - \theta \Rightarrow 4\theta = 100$$

$$\Rightarrow \quad \theta = 25°\text{C}$$

Time gained or lost is given by

$$\Delta T = \left(\frac{\Delta T}{T_0 + \Delta T}\right) t \approx \frac{\Delta t}{T_0} t$$

From Eq. (i),

$$\frac{\Delta T}{T_0} = \frac{\alpha\Delta\theta}{2} \Rightarrow \Delta t = \frac{\alpha\,(\Delta\theta) t}{2}$$

$$12 = \frac{\alpha\,(40 - 25)(24 \times 3600)}{2} \Rightarrow \alpha = 1.85 \times 10^{-5}/°\text{C}$$

12. (b) $\Delta Q = \Delta U + \Delta W$

In the process $pV^n =$ constant, molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - n} = C_V + \frac{R}{1 - n}$$

$$C - C_V = \frac{R}{1 - n} \Rightarrow 1 - n = \frac{C_p - C_V}{C - C_V}$$

$$\Rightarrow n = 1 - \left(\frac{C_p - C_V}{C - C_V}\right) = \frac{(C - C_V) - (C_p - C_V)}{C - C_V} = \frac{C - C_p}{C - C_V}$$

13. (a) p-V equation for path AB

$$p = -\left(\frac{p_0}{V_0}\right)V + 3p_0 \Rightarrow pV = 3p_0 V - \frac{p_0}{V_0}V^2$$

or $\quad T = \dfrac{pV}{nR} = \dfrac{1}{nR}\left(3p_0 V - \dfrac{p_0}{V_0}V^2\right)$

For maximum temperature,

$$\frac{dT}{dV} = 0 \Rightarrow 3p_0 - \frac{2p_0 V}{V_0} = 0$$

$$\Rightarrow \quad V = \frac{3}{2}V_0 \text{ and } p = 3p_0 - \frac{p_0}{V_0} = \frac{3p_0}{2}$$

Therefore, at these values :

$$\therefore \quad T_{\max} = \frac{\left(\frac{3p_0}{2}\right)\left(\frac{3V_0}{2}\right)}{nR} = \frac{9p_0 V_0}{4nR}$$

14. (a) Observer is stationary and source is moving.

During approach, $f_1 = f\left(\dfrac{v}{v - v_s}\right)$

$$= 1000\left(\frac{320}{320 - 20}\right) = 1066.67\text{ Hz}$$

During recede, $f_2 = f\left(\dfrac{v}{v + v_s}\right)$

$$= 1000\left(\frac{320}{320 + 20}\right) = 941.18 \text{ Hz}$$

$$|\% \text{ change in frequency}| = \left(\frac{f_1 - f_2}{f_1}\right) \times 100 \approx 12\%$$

15. **(b)** Given, $\dfrac{U}{V} \propto T^4$

$$\frac{U}{V} = \alpha T^4 \qquad \text{...(i)}$$

It is also given that, $P = \dfrac{1}{3}\left(\dfrac{U}{V}\right) \Rightarrow \dfrac{nR_0T}{V} = \dfrac{1}{3}(\alpha T^4)$

(R_0 = Gas constant)

or $\qquad VT^3 = \dfrac{3nR_0}{\alpha} = \text{constant}$

$\therefore \quad \left(\dfrac{4}{3}\pi R^3\right)T^3 = \text{constant or } RT = \text{constant}$

$\therefore \qquad T \propto \dfrac{1}{R}$

16. **(b)** Average time between two collisions is given by

$$\tau = \frac{1}{\sqrt{2}\,\pi n v_{\text{rms}} d^2} \qquad \text{...(i)}$$

Here, n = number of molecules per unit volume $= \dfrac{N}{V}$

and $\qquad v_{\text{rms}} = \sqrt{\dfrac{3RT}{M}}$

Substituting these values in Eq.(i) we have,

$$\tau \propto \frac{V}{\sqrt{T}} \qquad \text{...(ii)}$$

For adiabatic process, $TV^{\gamma - 1}$ = constant

Substituting in Eq. (ii), we have $\tau \propto \dfrac{V}{\sqrt{\left(\dfrac{1}{V^{\gamma - 1}}\right)}}$

or $\qquad \tau \propto V^{1 + \left(\frac{\gamma - 1}{2}\right)}$ or $\tau \propto V^{\left(\frac{1+\gamma}{2}\right)}$

17. **(a)** Entropy is a state functions. Therefore in both cases answer should be same.

18. **(c)** For closed organ pipe $= \dfrac{(2n + 1)v}{4l}$ $[n = 0, 1, 2 \text{ ......}]$

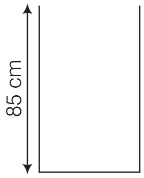
$$\frac{(2n + 1)v}{4l} < 1250$$

$$(2n + 1) < 1250 \times \frac{4 \times 0.85}{340}$$

$$(2n + 1) < 12.52 \, 2n < 11.50 \Rightarrow n < 5.25$$

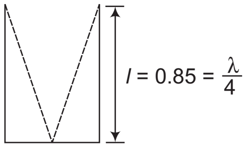
So, $\qquad n = 0, 1, 2, 3, \ldots, 5$

So, we have 6 possibilities.



Alternate method

In closed organ pipe, fundamental node



i.e. $\qquad \dfrac{\lambda}{4} = 0.85$

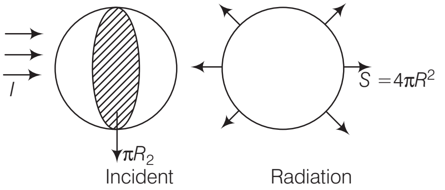
$\Rightarrow \qquad \lambda = 4 \times 0.85$

As we know, $\nu = \dfrac{c}{\lambda}$

$\Rightarrow \quad \dfrac{340}{4 \times 0.85} = 100 \text{ Hz}$

$\therefore$ Possible frequencies = 100 Hz, 300 Hz, 500 Hz, 700 Hz, 900 Hz, 1100 Hz below 1250 Hz.

19. **(a)** In steady state



Energy incident per second = Energy radiated per second

$\therefore \qquad I\pi R^2 = \sigma\,(T^4 - T_0^4)\, 4\pi R^2$

$\Rightarrow \qquad I = \sigma\,(T^4 - T_0^4)\, 4$

$\Rightarrow \qquad T^4 - T_0^4 = 40 \times 10^8$

$\Rightarrow T^4 - 81 \times 10^8 = 40 \times 10^8$

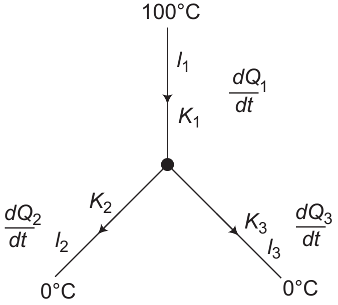
$\Rightarrow \qquad T^4 = 121 \times 10^8$

$\Rightarrow \qquad T \approx 330 \text{ K}$

20. **(c)** In thermal conduction, it is found that in steady state the heat current is directly proportional to the area of cross-section A which is proportional to the change in temperature $(T_1 - T_2)$.

Then, $\qquad \dfrac{\Delta Q}{\Delta t} = \dfrac{KA(T_1 - T_2)}{x}$

According to thermal conductivity, we get



i.e. $$\frac{dQ_1}{dt} = \frac{dQ_2}{dt} + \frac{dQ_3}{dt}$$

$$\frac{0.92\ (100 - T)}{46} = \frac{0.26\ (T - 0)}{13} + \frac{0.12\ (T - 0)}{12}$$

$$\Rightarrow \qquad T = 40°\ \text{C}$$

$$\therefore \qquad \frac{dQ_1}{dt} = \frac{0.92 \times 4\ (100 - 40)}{40} = 4.8\ \text{cal/s}$$

21. **(d)** According to first law of thermodynamics, we get

(i) Change in internal energy from A to B i.e. ΔU_{AB}

$$\Delta U_{AB} = nC_V(T_B - T_A)$$
$$= 1 \times \frac{5R}{2}(800 - 400) = 1000\,R$$

(ii) Change in internal energy from B to C

$$\Delta U_{BC} = nC_V(T_C - T_B)$$
$$= 1 \times \frac{5R}{2}(600 - 800)$$
$$= -\,500\,R$$

(iii) $\Delta U_{\text{isothermal}} = 0$

(iv) Change in internal energy from C to A i.e. ΔU_{CA}

$$\Delta U_{CA} = nC_V(T_A - T_C)$$
$$= -\,500\,R$$

22. **(b)** Fundamental frequency of sonometer wire

$$f = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T}{\mu}} = \frac{1}{2l}\sqrt{\frac{T}{Ad}}$$

Here, μ = mass per unit length of wire.

Also, Young's modulus of elasticity $Y = \frac{Tl}{A\Delta l}$

$$\Rightarrow \qquad \frac{T}{A} = \frac{Y\Delta l}{l} \Rightarrow f = \frac{1}{2l}\sqrt{\frac{Y\Delta l}{ld}}$$

$$\Rightarrow \qquad l = 1.5\ \text{m},\ \frac{\Delta l}{l} = 0.01$$

$$d = 7.7 \times 10^3\ \text{kg}/\text{m}^3$$

$$\Rightarrow \qquad Y = 2.2 \times 10^{11}\ \text{N}/\text{m}^2$$

After substituting the values we get,

$$f \approx 178.2\ \text{Hz}$$

23. **(c)** According to Newton's cooling law, option (c) is correct answer.

24. **(b)** Heat is extracted from the source means heat is given to the system (or gas) or Q is positive. This is positive only along the path ABC.

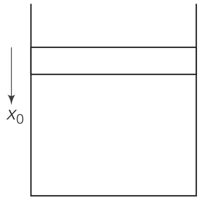
Heat supplied

$$\therefore\ Q_{ABC} = \Delta U_{ABC} + W_{ABC}$$
$$= nC_V\,(T_f - T_i) + \text{Area under } p\text{-}V \text{ graph}$$
$$= n\left(\frac{3}{2}R\right)(T_C - T_A) + 2p_0V_0$$
$$= \frac{3}{2}(nRT_C - nRT_A) + 2p_0V_0$$
$$= \frac{3}{2}(p_CV_C - p_AV_{A)} + 2p_0V_0$$
$$= \frac{3}{2}(4p_0V_0 - p_0V_0) + 2p_0V_0$$
$$= \frac{13}{2}\,p_0V_0$$

25. **(c)** In equilibrium,

$$p_0A = Mg \qquad \ldots\text{(i)}$$

when slightly displaced downwards,



$$dp = -\,\gamma\left(\frac{p_0}{V_0}\right)dV$$

$$\left(\text{As in adiabatic process, } \frac{dp}{dV} = -\,\gamma\,\frac{p}{V}\right)$$

$\therefore$ Restoring force,

$$F = (dp)A$$
$$= -\left(\frac{\gamma\,p_0}{V_0}\right)(A)\,(Ax)$$
$$F \propto -\,x$$

Therefore, motion is simple harmonic comparing with $F = -\,kx$ we have

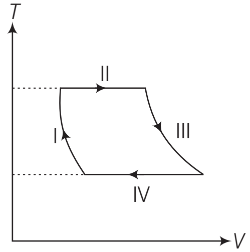
$$k = \frac{\gamma\,p_0A^2}{V_0}$$

$$\therefore \qquad f = \frac{1}{2\,\pi}\sqrt{\frac{k}{m}}$$
$$= \frac{1}{2\,\pi}\sqrt{\frac{\gamma\,p_0A^2}{MV_0}}$$

JEE Advanced

1. One mole of a monoatomic ideal gas undergoes a cyclic process as shown in the figure (where, V is the volume and T is the temperature). Which of the statements below is (are) true ?

(More than One Correct Option, 2018)



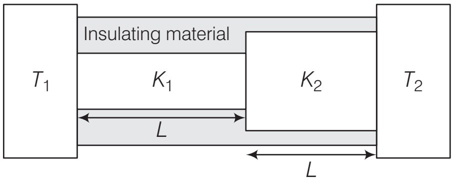
(a) Process I is an isochoric process
(b) In process II, gas absorbs heat
(c) In process IV, gas releases heat
(d) Processes I and III are not isobaric

2. Two men are walking along a horizontal straight line in the same direction. The main in front walks at a speed 1.0ms^{-1} and the man behind walks at a speed 2.0 ms^{-1}. A third man is standing at a height 12 m above the same horizontal line such that all three men are in a vertical plane. The two walking men are blowing identical whistles which emit a sound of frequency 1430 Hz. The speed of sound in air 330 ms^{-1}. At the instant, when the moving men are 10 m apart, the stationary man is equidistant from them. The frequency of beats in Hz, heard by the stationary man at this instant, is ....

(Numerical Value, 2018)

3. Two conducting cylinders of equal length but different radii are connected in series between two heat baths kept at temperatures $T_1 = 300\text{K}$ and $T_2 = 100\text{K}$, as shown in the figure. The radius of the bigger cylinder is twice that of the smaller one and the thermal conductivities of the materials of the smaller and the larger cylinders are K_1 and K_2, respectively. If the temperature at the junction of the cylinders in the steady state is 200K, then $K_1/K_2 =$........... .

(Numerical Value, 2018)



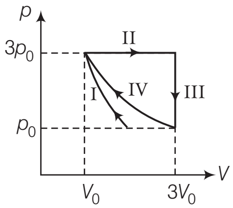
4. In an experiment to measure the speed of sound by a resonating air column, a tuning fork of frequency 500 Hz is used. The length of the air column is varied by changing the level of water in the resonance tube. Two successive resonances are heard at air columns of length 50.7 cm and 83.9 cm. Which of the following statements is (are) true?**(More than One Correct Option, 2018)**

(a) The speed of sound determined from this experiment is 332 m s^{-1}
(b) The end correction in this experiment is 0.9 cm
(c) The wavelength of the sound wave is 66.4 cm
(d) The resonance at 50.7 cm corresponds to the fundamental harmonic

5. One mole of a monoatomic ideal gas undergoes an adiabatic expansion in which its volume becomes eight times its initial value. If the initial temperature of the gas is 100 K and the universal gas constant $R = 8.0 \text{ j mol}^{-1} \text{ K}^{-1}$, the decrease in its internal energy in joule, is ....

(Numerical Value, 2018)

6. One mole of a monoatomic ideal gas undergoes four thermodynamic processes as shown schematically in the pV-diagram below.



Among these four processes, one is isobaric, one is isochoric, one is isothermal and one is adiabatic. Match the processes mentioned in List-l with the corresponding statements in List-II.

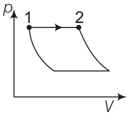
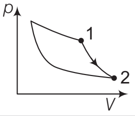
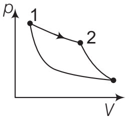
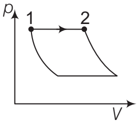
(Matching Type Question, 2018)

	List-I		List-II
P.	In process I	1.	Work done by the gas is zero
Q.	In process II	2.	Temperature of the gas remains unchanged
R.	In process III	3.	No heat is exchanged between the gas and its surroundings
S.	In process IV	4.	Work done by the gas is $6p_0V_0$

(a) P → 4; Q → 3; R → 1; S → 2
(b) P → 1; Q → 3; R → 2; S → 4
(c) P → 3; Q → 4; R → 1; S → 2
(d) P → 3; Q → 4; R → 2; S → 1

Directions (Q.Nos. 7-8) Matching the information given in the three columns of the following table.

An ideal gas is undergoing a cyclic thermodynamic process in different ways as shown in the corresponding *p*-*V* diagrams in column 3 of the table. Consider only the path from state 1 to state 2. *W* denotes the corresponding work done on the system. The equations and plots in the table have standards notations and used in thermodynamic processes. Here γ is the ratio of heat capacities at constant pressure and constant volume. The number of moles in the gas is *n*. **(Matching Type, 2017)**

Column 1	Column 2	Column 3
(I) $W_{1\to 2}=\dfrac{1}{\gamma-1}(p_2V_2-p_1V_1)$	(i) Isothermal	(P) 
(II) $W_{1\to 2}=-pV_2+pV_1$	(ii) Isochoric	(Q) 
(III) $W_{1\to 2}=0$	(iii) Isobaric	(R) 
(IV) $W_{1\to 2}=-nRT\ln\left(\dfrac{V_2}{V_1}\right)$	(iv) Adiabatic	(S) 

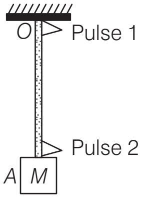
7. Which of the following options is the only correct representation of a process in which $\Delta U=\Delta Q-p\Delta V$?

(a) (II) (iii) (S)
(b) (II) (iii) (P)
(c) (III) (iii) (P)
(d) (II) (iv) (R)

8. Which one of the following options is the correct combination?

(a) (II) (iv) (P)
(b) (III) (ii) (S)
(c) (II) (iv) (R)
(d) (IV) (ii) (S)

9. A block *M* hangs vertically at the bottom end of a uniform rope of constant mass per unit length. The top end of the rope is attached to a fixed rigid support at *O*. A transverse wave pulse (Pulse 1) of wavelength λ_0 is produced at point *O* on the rope. The pulse takes time T_{OA} to reach point *A*. If the wave pulse of wavelength λ_0 is produced at point *A* (Pulse 2) without disturbing the position of *M* it takes time T_{AO} to reach point *O*. Which of the following options is/are correct? **(More than One Correct Option, 2017)**



(a) The time $T_{AO}=T_{OA}$
(b) The wavelength of Pulse 1 becomes longer when it reaches point *A*
(c) The velocity of any pulse along the rope is independent of its frequency and wavelength
(d) The velocities of the two pulses (Pulse 1 and Pulse 2) are the same at the mid-point of rope

10. A stationary source emits sound of frequency $f_0=492$ Hz. The sound is reflected by a large car approaching the source with a speed of 2 ms^{-1}. The reflected signal is received by the source and superposed with the original. What will be the beat frequency of the resulting signal in Hz? (Given that the speed of sound in air is 330 ms^{-1} and the car reflects the sound at the frequency it has received). **(Single Integer Type, 2017)**

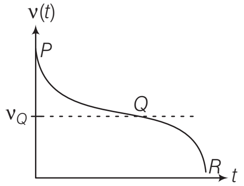
11. Two loudspeakers M and N are located 20 m apart and emit sound at frequencies 118 Hz and 121 Hz, respectively. A car in initially at a point P, 1800 m away from the midpoint Q of the line MN and moves towards Q constantly at 60 km/h along the perpendicular bisector of MN.
It crosses Q and eventually reaches a point R, 1800 m away from Q. Let $v(t)$ represent the beat frequency measured by a person sitting in the car at time t. Let ν_P, ν_Q and ν_R be the beat frequencies measured at locations P, Q and R respectively.

The speed of sound in air is 330 ms^{-1}.

Which of the following statement(s) is (are) true regarding the sound heard by the person?

(More than One Correct Option, 2016)

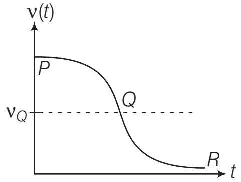
(a) The plot below represents schematically the variation of beat frequency with time



(b) The rate of change in beat frequency is maximum when the car passes through Q

(c) $\nu_P + \nu_R = 2\nu_Q$

(d) The plot below represents schematically the variations of beat frequency with time



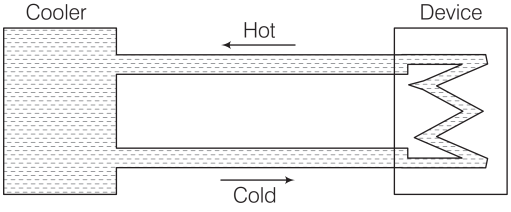
12. A gas is enclosed in a cylinder with a movable frictionless piston. Its initial thermodynamic state at pressure $P_i = 10^5$ Pa and volume $V_i = 10^{-3}\ \text{m}^3$ changes to a final state at $P_f = (1/32) \times 10^5$ Pa and $V_f = 8 \times 10^{-3}\ \text{m}^3$ in an adiabatic quasi-static process, such that $P^3V^5 =$ constant. Consider another thermodynamic process that brings the system from the same initial state to the same final state in two steps : an isobaric expansion at P_i followed by an isochoric (isovolumetric) process at volume V_f. The amount of heat supplied to the system in the two-step process is approximately

(Single Correct Option, 2016)

(a) 112 J (b) 294 J (c) 588 J (d) 813 J

13. A water cooler of storage capacity 120 litres can cool water at a constant rate of P watts. In a closed circulation system (as shown schematically in the figure), the water from the cooler is used to cool an external device that generates constantly 3 kW of heat (thermal load). The temperature of water fed into the device cannot exceed 30°C and the entire stored 120 litres of water is initially cooled to 10°C.

The entire system is thermally insulated. The minimum value of P (in watts) for which the device can be operated for 3 hours is **(Single Correct Option, 2016)**



(Specific heat of water is 4.2 kJ $\text{kg}^{-1}\text{K}^{-1}$ and the density of water is 1000 kg m^{-3})

(a) 1600 (b) 2067 (c) 2533 (d) 3933

14. A metal is heated in a furnace where a sensor is kept above the metal surface to read the power radiated (P) by the metal. The sensor has a scale that displays $\log_2(P/P_0)$, where P_0 is a constant. When the metal surface is at a temperature of 487°C, the sensor shows a value 1. Assume that the emissivity of the metallic surface remains constant. What is the value displayed by the sensor when the temperature of the metal surface is raised to 2767°C? **(Single Integer Type, 2016)**

15. The ends Q and R of two thin wires, PQ and RS, are soldered (joined) together. Initially, each of the wire has a length of 1 m 10°C. Now, the end P is maintained at 10°C, while the end S is heated and maintained at 400°C. The system is thermally insulated from its surroundings. If the thermal conductivity of wire PQ is twice that of the wire RS and the coefficient of linear thermal expansion of PQ is $1.2 \times 10^{-5}\ \text{K}^{-1}$, the change in length of the wire PQ is

(Single Correct Option, 2016)

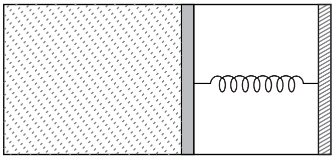
(a) 0.78 mm (b) 0.90 mm
(c) 1.56 mm (d) 2.34 mm

16. A container of fixed volume has a mixture of one mole of hydrogen and one mole of helium in equilibrium at temperature T. Assuming the gases are ideal, the correct statements is/are

(a) The average energy per mole of the gas mixture is $2RT$
(b) The ratio of speed of sound in the gas mixture to that in helium gas is $\sqrt{\frac{6}{5}}$
(c) The ratio of the rms speed of helium atoms to that of hydrogen molecules is $\frac{1}{2}$
(d) The ratio of the rms speed of helium atoms to that of hydrogen molecules is $\frac{1}{\sqrt{2}}$

(More than One Correct Option, 2015)

17. An ideal monoatomic gas is confined in a horizontal cylinder by a spring loaded piston (as shown in the figure). Initially the gas is at temperature T_1, pressure P_1 and volume V_1 and the spring is in its relaxed state. The gas is then heated very slowly to temperature T_2, pressure P_2 and volume V_2. During this process the piston moves out by a distance x.



Ignoring the friction between the piston and the cylinder, the correct statements is/are **(More than One Correct Option, 2015)**

(a) If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the energy stored in the spring is $\frac{1}{4}P_1V_1$
(b) If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the change in internal energy is $3P_1V_1$
(c) If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the work done by the gas is $\frac{7}{3}P_1V_1$
(d) If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the heat supplied to the gas is $\frac{17}{6}P_1V_1$

18. Two spherical stars A and B emit blackbody radiation. The radius of A is 400 times that of B and A emits 10^4 times the power emitted from B. The ratio $\left(\frac{\lambda_A}{\lambda_B}\right)$ of their wavelengths λ_A and λ_B at which the peaks occur in their respective radiation curves is **(Single Integer Type, 2015)**

19. Four harmonic waves of equal frequencies and equal intensities I_0 have phase angles 0, $\frac{\pi}{3}$, $\frac{2\pi}{3}$ and π. When they are superposed, the intensity of the resulting wave is nI_0. The value of n is

(Single Integer Type, 2015)

20. Two vehicles, each moving with speed u on the same horizontal straight road, are approaching each other. Wind blows along the road with velocity w. One of these vehicles blows a whistle of frequency f_1. An observer in the other vehicle hears the frequency of the whistle to be f_2. The speed of sound in still air is v. The correct statement(s) is (are)

(More than One Correct Option, 2014)

(a) If the wind blows from the observer to the source, $f_2 > f_1$.
(b) If the wind blows from the source to the observer, $f_2 > f_1$
(c) If the wind blows from the observer to the source, $f_2 < f_1$
(d) If the wind blows from the source to the observer, $f_2 < f_1$

21. One end of a taut string of length 3 m along the x-axis is fixed at $x = 0$. The speed of the waves in the string is $100\,\text{ms}^{-1}$. The other end of the string is vibrating in the y-direction so that stationary waves are set up in the string. The possible waveform(s) of these stationary wave is (are)

(More than One Correct Option, 2014)

(a) $y(t) = A \sin \dfrac{\pi x}{6} \cos \dfrac{50\pi t}{3}$

(b) $y(t) = A \sin \dfrac{\pi x}{3} \cos \dfrac{100\pi t}{3}$

(c) $y(t) = A \sin \dfrac{5\pi x}{2} \cos \dfrac{250\pi t}{3}$

(d) $y(t) = A \sin \dfrac{5\pi x}{2} \cos 250\pi t$

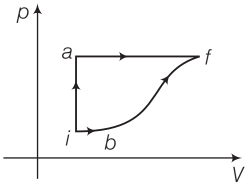
22. A student is performing an experiment using a resonance column and a tuning fork of frequency $244\,\text{s}^{-1}$. He is told that the air in the tube has been replaced by another gas (assume that the column remains filled with the gas). If the minimum height at which resonance occurs is (0.350 ± 0.005) m, the gas in the tube is (Useful information : $\sqrt{167RT} = 640\ \text{J}^{1/2}\ \text{mole}^{-1/2}$; $\sqrt{140RT} = 590\ \text{J}^{1/2}\ \text{mole}^{-1/2}$. The molar masses M in grams are given in the options. Take the value of $\sqrt{10/M}$ for each gas as given there.) **(Single Correct Option, 2014)**

(a) Neon ($M = 20$, $\sqrt{10/20} = 7/10$)

(b) Nitrogen ($M = 28$, $\sqrt{10/28} = 3/5$)

(c) Oxygen ($M = 32$, $\sqrt{10/32} = 9/16$)

(d) Argon ($M = 36$, $\sqrt{10/36} = 17/32$)



23. A thermodynamic system is taken from an initial state i with internal energy $U_i = 100$ J to the final state f along two different paths iaf and ibf, as schematically shown in the figure. The work done by the system along the paths af, ib and bf are $W_{af} = 200$ J, $W_{ib} = 50$ J and $W_{bf} = 100$ J respectively. The heat supplied to the system along the path iaf, ib and bf are Q_{iaf}, Q_{ib} and Q_{bf} respectively. If the internal energy of the system in the state b is $U_b = 200$ J and $Q_{iaf} = 500$ J, the ratio Q_{bf} / Q_{ib} is

(Single Integer Type, 2014)

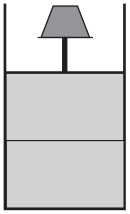
24. Parallel rays of light of intensity $I = 912\ \text{Wm}^{-2}$ are incident on a spherical black body kept in surroundings of temperature 300 K. Take Stefan constant $\sigma = 5.7 \times 10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$ and assume that the energy exchange with the surroundings is only through radiation. The final steady state temperature of the black body is close to

(Single Correct Option 2014)

(a) 330 K	(b) 660 K
(c) 990 K	(d) 1550 K

Passage (Q. Nos. 25-26)

In the figure a container is shown to have a movable (without friction) piston on top. The container and the piston are all made of perfectly insulating material allowing no heat transfer between outside and inside the container. The container is divided into two compartments by a rigid partition made of a thermally conducting material that allows slow transfer of heat.



The lower compartment of the container is filled with 2 moles of an ideal monoatomic gas at 700 K and the upper compartment is filled with 2 moles of an ideal diatomic gas at 400 K. The heat capacities per mole of an ideal monoatomic gas are $C_V = \dfrac{3}{2}R$, $C_p = \dfrac{5}{2}R$, and those for an ideal diatomic gas are $C_V = \dfrac{5}{2}R$, $C_p = \dfrac{7}{2}R$.

(Passage Type, 2014)

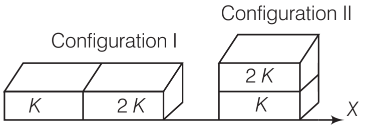
25. Consider the partition to be rigidly fixed so that it does not move. When equilibrium is achieved, the final temperature of the gases will be

(a) 550 K (b) 525 K (c) 513 K (d) 490 K

26. Now consider the partition to be free to move without friction so that the pressure of gases in both compartments is the same. Then total work done by the gases till the time they achieve equilibrium will be

(a) 250 R (b) 200 R (c) 100 R (d) –100 R

27. Two rectangular blocks, having indentical dimensions, can be arranged either in configuration I or in configuration II as shown in the figure. One of the blocks has thermal conductivity K and the other $2K$. The temperature difference between the ends along the x-axis is the same in both the configurations. It takes 9 s to transport a certain amount of heat from the hot end to the cold end in the configuration I. The time to transport the same amount of heat in the configuration II is **(Single Correct Option, 2013)**



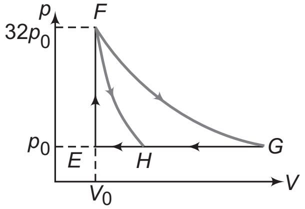
(a) 2.0 s (b) 3.0 s (c) 4.5 s (d) 6.0 s

28. Two non-reactive monoatomic ideal gases have their atomic masses in the ratio 2 : 3. The ratio of their partial pressures, when enclosed in a vessel kept at a constant temperature, is 4 : 3. The ratio of their densities is **(Single Correct Option, 2013)**

(a) 1 : 4 (b) 1 : 2 (c) 6 : 9 (d) 8 : 9

29. One mole of a monatomic ideal gas is taken along two cyclic processes $E \to F \to G \to E$ and $E \to F \to H \to E$ as shown in the p-V diagram.

The processes involved are purely isochoric, isobaric, isothermal or adiabatic.



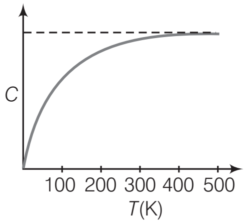
Match the paths in Column I with the magnitudes of the work done in Column II and select the correct answer using the codes given below the lists. **(Matching Type, 2013)**

Column I	Column II
P. $G \to E$	1. $160\ p_0V_0 \ln 2$
Q. $G \to H$	2. $36\ p_0V_0$
R. $F \to H$	3. $24\ p_0V_0$
S. $F \to G$	4. $31\ p_0V_0$

Codes

	P	Q	R	S		P	Q	R	S
(a)	2	3	1	4	(b)	1	2	4	3
(c)	4	3	2	1	(d)	2	3	4	1

30. The figure below shows the variation of specific heat capacity (C) of a solid as a function of temperature (T). The temperature is increased continuously from 0 to 500 K at a constant rate. Ignoring any volume change, the following statement(s) is (are) correct to reasonable approximation.



(More than One Correct Option, 2013)

(a) the rate at which heat is absorbed in the range 0-100 K varies linearly with temperature T

(b) heat absorbed in increasing the temperature from 0-100 K is less than the heat required for increasing the temperature from 400-500 K

(c) there is no change in the rate of heat absorbtion in the range 400-500 K

(d) the rate of heat absorption increases in the range 200-300 K

31. A horizontal stretched string, fixed at two ends, is vibrating in its fifth harmonic according to the equation, $y(x, t) = (0.01 \text{ m}) [\sin(62.8 \text{ m}^{-1})x] \cos[(628 \text{ s}^{-1})t]$.

Assuming $\pi = 3.14$, the correct statement(s) is (are)

(a) the number of nodes is 5

(b) the length of the string is 0.25 m

(c) the maximum displacement of the mid-point of the string from its equilibrium position is 0.01 m

(d) the fundamental frequency is 100 Hz

(More than One Correct Option, 2013)

Answer *with* Explanations

1. **(b,c,d)** (b) Process-II is isothermal expansion,

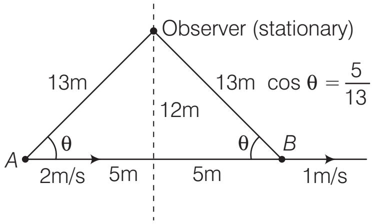
$$\Delta U = 0, W > 0$$
$$\Delta Q = W > 0$$

(c) Process-IV is isothermal compression,

$$\Delta U = 0, W < 0$$
$$\Delta Q = W < 0$$

(d) Process-I and III are not isobaric because in isobaric process $T \propto V$ hence, T-V graph should be a straight line passing through origin.

2. (5 **Hz**)



$$f_A = 1430\left[\frac{330}{330 - 2\cos\theta}\right]$$

$$= 1430\left[\frac{1}{1 - \frac{2\cos\theta}{330}}\right] \approx 1430\left[1 + \frac{2\cos\theta}{330}\right]$$

[from binomial expansion]

$$f_B = 1430\left[\frac{330}{330 + 1\cos\theta}\right]$$

$$\approx 1430\left[1 - \frac{\cos\theta}{330}\right]$$

$$\text{Beat frequency} = f_A - f_B = 1430\left[\frac{3\cos\theta}{330}\right] = 13\cos\theta$$

$$= 13\left(\frac{5}{13}\right) = 5.00 \text{ Hz}$$

3. **(4)** Rate of heat flow will be same,

$$\therefore \quad \frac{300 - 200}{R_1} = \frac{200 - 100}{R_2} \qquad \left(\text{as } H = \frac{dQ}{dt} = \frac{T \cdot D}{R}\right)$$

$$\therefore \quad R_1 = R_2 \Rightarrow \frac{L_1}{K_1A_1} = \frac{L_2}{K_2A_2}$$

$$\Rightarrow \quad \frac{K_1}{K_2} = \frac{A_2}{A_1} = 4$$

4. **(a, c)** Let n th harmonic is corresponding to 50.7 cm and $(n + 1)$th harmonic is corresponding 83.9 cm.

$\therefore$ Their difference is $\frac{\lambda}{2}$.

$$\therefore \quad \frac{\lambda}{2} = (83.9 - 50.7) \text{ cm}$$

$$\text{or} \quad \lambda = 66.4 \text{ cm}$$

$$\therefore \quad \frac{\lambda}{4} = 16.6 \text{ cm}$$

Length corresponding to fundamental mode must be close to $\frac{\lambda}{4}$ and 50.7 cm must be an odd multiple of this length. $16.6 \times 3 = 49.8$ cm. Therefore, 50.7 is 3rd harmonic.

If end correction is e, then

$$e + 50.7 = \frac{3\lambda}{4}$$

$$e = 49.8 - 50.7 = -0.9 \text{ cm}$$

$\therefore$ Speed of sound, $v = f\lambda$

$$\Rightarrow \quad v = 500 \times 66.4 \text{ cm/s} = 332 \text{ m/s}$$

5. **(900)** Given, $n = 1$, $\gamma = \frac{5}{3}$

T-V equation in adiabatic process is

$$TV^{\gamma - 1} = \text{constant}$$

$$\therefore \quad T_1V_1^{\gamma - 1} = T_2V_2^{\gamma - 1}$$

$$\Rightarrow \quad T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma - 1} = 100 \times \left(\frac{1}{8}\right)^{\frac{2}{3}}$$

$$\Rightarrow \quad T_2 = 25 \text{ K}$$

$C_V = \frac{3}{2}R$ for monoatomic gas

$$\therefore \quad \Delta U = nC_V\Delta T = n \times \left(\frac{3R}{2}\right)(T_2 - T_1)$$

$$= 1 \times \frac{3}{2} \times 8 \times (25 - 100) = -900 \text{ J}$$

$\therefore$ Decrease in internal energy = 900 J

6. (c) $\left(\frac{dp}{dV}\right)_{\text{adiabatic}} = \gamma\left(\frac{dp}{dV}\right)_{\text{isothermal}}$

List-I

(P) Process I $\Rightarrow$ Adiabatic $\Rightarrow Q = 0$

(Q) Process II $\Rightarrow$ Isobaric

$\therefore \quad W = p\Delta V = 3p_0[3V_0 - V_0] = 6p_0V_0$

(R) Process III $\Rightarrow$ Isochoric $\Rightarrow W = 0$

(S) Process (IV) $\Rightarrow$ Isothermal

$\Rightarrow$ Temperature = Constant

7. **(b)** $\Delta U = \Delta Q - p\Delta V$

$$\Delta U + p\Delta V = \Delta Q$$

As $\Delta U \neq 0, W \neq 0, \Delta Q \neq 0$. The process represents, isobaric process

$$W_{\text{gas}} = -p(\Delta V) = -p(V_2 - V_1) = -pV_2 + pV_1$$

Graph 'P' satisfies isobaric process.

8. **(b)** Work done in isochoric process is zero.

$$W_{12} = 0 \text{ as } \Delta V = 0$$

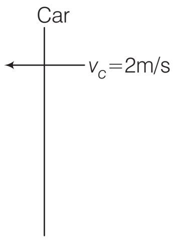
Graph 'S' represents isochoric process.

9. **(a,c,d** or **a,c)** $v = \sqrt{\frac{T}{\mu}}$, so speed at any position will be same for both pulses, therefore time taken by both pulses will be same.

$$\lambda f = v \quad \Rightarrow \quad \lambda = \frac{v}{f}$$

since when pulse 1 reaches at A tension and hence speed decreases therefore λ decreases.

10. **(6)**



Frequency observed at car

$$f_1 = f_0\left(\frac{v + v_C}{v}\right) \quad (v = \text{speed of sound})$$

Frequency of reflected sound as observed at the source

$$f_2 = f_1\left(\frac{v}{v - v_C}\right) = f_0\left(\frac{v + v_C}{v - v_C}\right)$$

Beat frequency $= f_2 - f_0$

$$= f_0\left[\frac{v + v_C}{v - v_C} - 1\right] = f_0\left[\frac{2v_C}{v - v_C}\right]$$

$$= 492 \times \frac{2 \times 2}{328} = 6 \text{ Hz}$$

$\Rightarrow \quad \lambda \propto v \propto T$

11. **(b,c,d)** Speed of car,

$$v = 60 \text{ km/h} = \frac{500}{3} \text{ m/s}$$

At a point S, between P and Q

$$\nu'_M = \nu_M\left(\frac{C + v\cos\theta}{C}\right);$$

$$\nu'_N = \nu_N\left(\frac{C + v\cos\theta}{C}\right)$$

$$\Rightarrow \quad \Delta v = (v_N - v_M)\left(1 + \frac{v\cos\theta}{C}\right)$$

Similarly, between Q and R

$$\Delta\nu = (\nu_N - \nu_M)\left(1 - \frac{v\cos\theta}{C}\right)$$

$$\frac{d(\Delta\nu)}{dt} = \pm(\nu_N - \nu_M)\frac{v}{C}\sin\theta\frac{d\theta}{dt}$$

$\theta \approx 0°$ at P and R as they are large distance apart.

$\Rightarrow$ Slope of graph is zero.

at Q, $\quad \theta = 90°$

$\sin\theta$ is maximum also value of $\frac{d\theta}{dt}$ is maximum

as $\frac{d\theta}{dt} = \frac{v}{r}$, where v is its velocity and r is the length of the line joining P and S. and r is minimum at Q.

$\Rightarrow$ Slope is maximum at Q.

At P, $\nu_P = \Delta\nu = (\nu_N - \nu_M)\left(1 + \frac{V}{C}\right)$

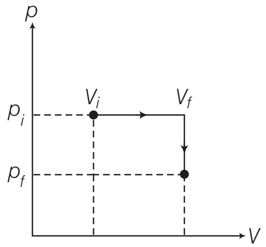
$(\theta \approx 0°)$ At R, $\nu_R = \Delta\nu = (\nu_N - \nu_M)\left(1 - \frac{V}{C}\right)$

$(\theta \approx 0°)$ At Q,

$$\nu_Q = \Delta\nu = (\nu_N - \nu_M)$$
$$(\theta = 90°)$$

From these equations, we can see that $\nu_P + \nu_R = 2\nu_Q$

12. (c) In the first process : $p_iV_i^\gamma = p_fV_f^\gamma$



$$\Rightarrow \quad \frac{p_i}{p_f} = \left(\frac{V_f}{V_i}\right)^\gamma \quad \Rightarrow \quad 32 = 8^\gamma \quad \Rightarrow \quad \gamma = \frac{5}{3} \qquad \ldots\text{(i)}$$

For the two step process

$$W = p_i(V_f - V_i)$$
$$= 10^5(7 \times 10^{-3})$$
$$= 7 \times 10^2 \text{ J}$$

$$\Delta U = \frac{f}{2}(p_f V_f - p_i V_i)$$

$$= \frac{1}{\gamma - 1}\left(\frac{1}{4} \times 10^2 - 10^2\right)$$

$$\Delta U = -\frac{3}{2}\cdot\frac{3}{4} \times 10^2 = -\frac{9}{8} \times 10^2 \text{ J}$$

$$Q - W = \Delta U$$

$$\Rightarrow \quad Q = 7 \times 10^2 - \frac{9}{8} \times 10^2$$

$$= \frac{47}{8} \times 10^2 \text{ J} = 588 \text{ J}$$

13. **(b)** Heat generated in device in 3 h

$$= \text{time} \times \text{power} = 3 \times 3600 \times 3 \times 10^3$$

$$= 324 \times 10^5 \text{ J}$$

Heat used to heat water

$$= ms\,\Delta\theta = 120 \times 1 \times 4.2 \times 10^3 \times 20 \text{ J}$$

Heat absorbed by coolant

$$= Pt = 324 \times 10^5 - 120 \times 1 \times 4.2 \times 10^3 \times 20 \text{ J}$$

$$Pt = (325 - 100.8) \times 10^5 \text{ J} = 223.2 \times 10^5 \text{ J}$$

$$P = \frac{223.2 \times 10^5}{3600} = 2067 \text{ W}$$

14. **(9)** $\log_2 \frac{p_1}{p_0} = 1,$

Therefore, $\frac{p_1}{p_0} = 2$

According to Stefan's law, $p \propto T$

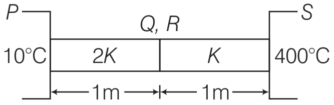
$$\Rightarrow \quad \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^4 = \left(\frac{2767 + 273}{487 + 273}\right)^4 = 4^4$$

$$\frac{p_2}{p_1} = \frac{p_2}{2p_0} = 4^4 \quad \Rightarrow \quad \frac{p_2}{p_0} = 2 \times 4^4$$

$$\log_2 \frac{p_2}{p_0} = \log_2 [2 \times 4^4] = \log_2 2 + \log_2 4^4$$

$$= 1 + \log_2 2^8 = 1 + 8 = 9$$

15. **(a)**



Rate of heat flow from P to Q

$$\frac{dQ}{dt} = \frac{2KA(T - 10)}{1}$$

Rate of heat flow from Q to S

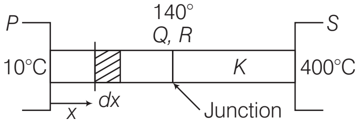
$$\frac{dQ}{dt} = \frac{KA(4000 - T)}{1}$$

At steady state, state rate of heat flow is same

$$\therefore \quad \frac{2KA(T - 10)}{1} = KA(400 - T)$$

or $\quad 2T - 20 = 400 - T$ or $3T = 420$

$\therefore \quad T = 140°$



Temperature of junction is 140°C

Temperature at a distance x from end P is

$$T_x = (130x + 10°)$$

Change in length dx is suppose dy

Then, $\quad dy = \alpha dx\,(T_x - 10)$

$$\int_0^{\Delta y} dy = \int_0^1 \alpha dx (130x + 10 - 10)$$

$$\Delta y = \left[\frac{\alpha x^2}{2} \times 130\right]_0^1$$

$$\Delta y = 1.2 \times 10^{-5} \times 65$$

$$\Delta y = 78.0 \times 10^{-5} \text{ m} = 0.78 \text{ mm}$$

16. **(a, b, d)** (a) Total internal energy

$$U = \frac{f_1}{2} nRT + \frac{f_2}{2} nRT$$

$$(U_{\text{ave}})_{\text{per mole}} = \frac{U}{2n} = \frac{1}{4}[5RT + 3RT] = 2RT$$

(b) $$\gamma_{\text{mix}} = \frac{n_1 C_{p_1} + n_2 C_{p_2}}{n_2 C_{v_1} + n_2 C_{v_2}}$$

$$= \frac{(1)\frac{7R}{2} + (1)\frac{5R}{2}}{(1)\frac{5R}{2} + (1)\frac{3R}{2}} = \frac{3}{2}$$

$$M_{\text{mix}} = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2} = \frac{M_1 + M_2}{2} = \frac{2 + 4}{2} = 3$$

Speed of sound $V = \sqrt{\frac{\gamma RT}{M}}$

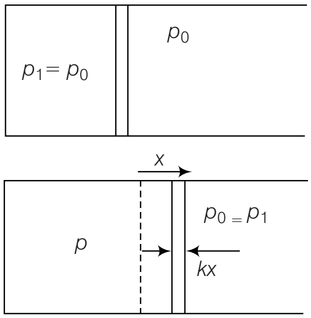
$$\Rightarrow \quad V \propto \sqrt{\frac{\gamma}{M}}$$

$$\frac{V_{\text{mix}}}{V_{\text{He}}} = \sqrt{\frac{\gamma_{\text{mix}}}{\gamma_{\text{He}}} \times \frac{M_{\text{He}}}{M_{\text{mix}}}} = \sqrt{\frac{3/2}{5/3} \times \frac{4}{3}} = \sqrt{\frac{6}{5}}$$

(d) $$V_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

$$\Rightarrow \quad V_{\text{rms}} \propto \frac{1}{\sqrt{M}}, \frac{V_{\text{He}}}{V_{\text{H}}} = \sqrt{\frac{M_{\text{H}}}{M_{\text{He}}}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$$

17. **(a, b, c)** **Note** *This question can be solved if right hand side chamber is assumed open, so that its pressure remains constant even if the piston shifts towards right.*



(a) $$pV = nRT$$

$$\Rightarrow \quad p \propto \frac{T}{V}$$

Temperature is made three times and volume is doubled

$$\Rightarrow \quad p_2 = \frac{3}{2}p_1$$

Further $$x = \frac{\Delta V}{A} = \frac{V_2 - V_1}{A} = \frac{2V_1 - V_1}{A} = \frac{V_1}{A}$$

$$p_2 = \frac{3p_1}{2} = p_1 + \frac{kx}{A} \quad \Rightarrow \quad kx = \frac{p_1 A}{2}$$

Energy of spring

$$\frac{1}{2}kx^2 = \frac{p_1 A}{4}x = \frac{p_1 V_1}{4}$$

(b) $$\Delta U = nc_v \Delta T = n\left(\frac{3}{2}R\right)\Delta T$$

$$= \frac{3}{2}(p_2 V_2 - p_1 V_1)$$

$$= \frac{3}{2}\left[\left(\frac{3}{2}p_1\right)(2V_1) - p_1 V_1\right] = 3\,p_1 V_1$$

(c) $$p_2 = \frac{4p_1}{3}$$

$$\Rightarrow \quad p_2 = \frac{4}{3}p_1 = p_1 + \frac{kx}{A}$$

$$\Rightarrow \quad kx = \frac{p_1 A}{3}$$

$$\Rightarrow \quad x = \frac{\Delta V}{A} = \frac{2V_1}{A}$$

$$W_{\text{gas}} = (p_0\,\Delta V + W_{\text{spring}})$$

$$= \left(p_1 Ax + \frac{1}{2}kx \cdot x\right)$$

$$= +\left(p_1 A \cdot \frac{2V_1}{A} + \frac{1}{2} \cdot \frac{p_1 A}{3} \cdot \frac{2V_1}{A}\right)$$

$$= 2p_1 V_1 + \frac{p_1 V_1}{3} = \frac{7p_1 V_1}{3}$$

(d) $$\Delta Q = W + \Delta U$$

$$= \frac{7p_1 V_1}{3} + \frac{3}{2}(p_2 V_2 - p_1 V_1)$$

$$= \frac{7p_1 V_1}{3} + \frac{3}{2}\left(\frac{4}{3}p_1 \cdot 3V_1 - p_1 V_1\right)$$

$$= \frac{7p_1 V_1}{3} + \frac{9}{2}p_1 V_1 = \frac{41 p_1 V_1}{6}$$

Note $\Delta U = \frac{3}{2}(p_2 V_2 - p_1 V_1)$ *has been obtained in part (b).*

18. **(2)** Power, $P = (\sigma T^4 A) = \sigma T^4 (4\pi R^2)$

or, $$P \propto T^4 R^2 \quad \ldots\text{(i)}$$

According to Wien's law,

$$\lambda \propto \frac{1}{T}$$

(λ is the wavelength at which peak occurs)

$\therefore$ Eq. (i) will become,

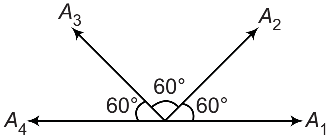
$$P \propto \frac{R^2}{\lambda^4}$$

or $$\lambda \propto \left[\frac{R^2}{P}\right]^{1/4}$$

$$\Rightarrow \quad \frac{\lambda_A}{\lambda_B} = \left[\frac{R_A}{R_B}\right]^{1/2}\left[\frac{P_B}{P_A}\right]^{1/4}$$

$$= [400]^{1/2}\left[\frac{1}{10^4}\right]^{1/4} = 2$$

19. **(3)** Let individual amplitudes are A_0 each. Amplitudes can be added by vector method.



$$A_1 = A_2 = A_3 = A_4 = A_0$$

Resultant of A_1 and A_4 is zero. Resultant of A_2 and A_3 is

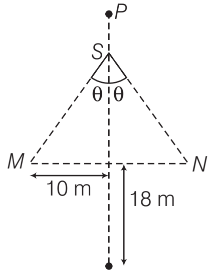
$$A = \sqrt{A_0^2 + A_0^2 + 2A_0 A_0 \cos 60°} = \sqrt{3}A_0$$

This is also the net resultant.

Now, $I \propto A^2$

$\therefore$ Net intensity will become $3I_0$.

$\therefore$ Answer is 3.



20. **(a,b)** When wind blows from S to O

$$f_2 = f_1\left(\frac{v+w+u}{v+w-u}\right)$$

or $f_2 > f_1$

when wind blows from O to S

$$f_2 = f_1\left(\frac{v-w+u}{v-w-u}\right) \Rightarrow f_2 > f_1$$

21. **(a,c,d)** There should be a node at $x = 0$ and antinode at $x = 3$ m. Also,

$$v = \frac{\omega}{k} = 100 \text{ m/s.}$$

$\therefore$ $y = 0$ at $x = 0$

and $y = \pm A$ at $x = 3$ m.

Only (a), (c) and (d) satisfy the condition.

22. **(d)** Minimum length $= \dfrac{\lambda}{4}$

$\Rightarrow$ $\lambda = 4l$

Now, $v = f\lambda = (244) \times 4 \times l$

as $l = 0.350 \pm 0.005$

$\Rightarrow$ v lies between 336.7 m/s to 346.5 m/s

Now, $v = \sqrt{\dfrac{\gamma RT}{M \times 10^{-3}}}$, here M is molecular mass in gram

$$= \sqrt{100\gamma RT} \times \sqrt{\frac{10}{M}}.$$

For monoatomic gas, $\gamma = 1.67$

$$\Rightarrow \quad v = 640 \times \sqrt{\frac{10}{M}}$$

For diatomic gas,

$\gamma = 1.4$

$$\Rightarrow \quad v = 590 \times \sqrt{\frac{10}{M}}$$

$$\therefore \quad v_{\text{Ne}} = 640 \times \frac{7}{10} = 448 \text{ m/s}$$

$$v_{\text{Ar}} = 640 \times \frac{17}{32} = 340 \text{ m/s}$$

$$v_{\text{O}_2} = 590 \times \frac{9}{16} = 331.8 \text{ m/s}$$

$$v_{\text{N}_2} = 590 \times \frac{3}{5} = 354 \text{ m/s}$$

$\therefore$ Only possible answer is Argon.

23. **(2)**

$$W_{ibf} = W_{ib} + W_{bf}$$
$$= 50 \text{ J} + 100 \text{ J} = 150 \text{ J}$$
$$W_{iaf} = W_{ia} + W_{af}$$
$$= 0 + 200 \text{ J} = 200 \text{ J}$$
$$Q_{iaf} = 500 \text{ J}$$

So $\Delta U_{iaf} = Q_{iaf} - W_{iaf}$

$$= 500 \text{ J} - 200 \text{ J} = 300 \text{ J}$$
$$= U_f - U_i$$

So, $U_f = U_{iaf} + U_i$

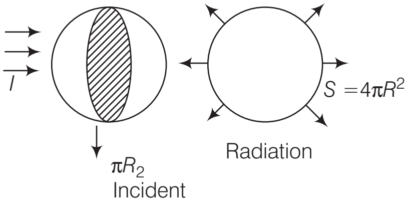
$$= 300 \text{ J} + 100 \text{ J} = 400 \text{ J}$$

$$\Delta U_{ib} = U_b - U_i$$
$$= 200 \text{ J} - 100 \text{ J} = 100 \text{ J}$$
$$Q_{ib} = \Delta U_{ib} + W_{ib}$$
$$= 100 \text{ J} + 50 \text{ J} = 150 \text{ J}$$
$$Q_{ibf} = \Delta U_{ibf} + W_{ibf}$$
$$= \Delta U_{iaf} + W_{ibf}$$
$$= 300 \text{ J} + 150 \text{ J} = 450 \text{ J}$$

So, the required ratio $\dfrac{Q_{bf}}{Q_{ib}} = \dfrac{Q_{ibf} - Q_{ib}}{Q_{ib}}$

$$= \frac{450 - 150}{150} = 2$$

24. **(a)** In steady state



Energy incident per second = Energy radiated per second

$$\therefore \quad I\pi R^2 = \sigma(T^4 - T_0^4)\,4\pi R^2$$
$$\Rightarrow \quad I = \sigma(T^4 - T_0^4)\,4$$
$$\Rightarrow \quad T^4 - T_0^4 = 40 \times 10^8$$
$$\Rightarrow \quad T^4 - 81 \times 10^8 = 40 \times 10^8$$
$$\Rightarrow \quad T^4 = 121 \times 10^8$$
$$\Rightarrow \quad T \approx 330 \text{ K}$$

25. **(d)** Let final equilibrium temperature of gases is T

Heat rejected by gas by lower compartment

$$= nC_V\Delta T$$
$$= 2 \times \frac{3}{2}R\,(700 - T)$$

Heat received by the gas in above compartment

$$= nC_p\Delta T$$
$$= 2 \times \frac{7}{2}R\,(T - 400)$$

Equating the two, we get

$$2100 - 3T = 7T - 2800 \quad \Rightarrow \mathbf{T = 490\,K}$$

26. **(d)** $\Delta W_1 + \Delta U_1 = \Delta Q_1 \quad ...(i)$

$\Delta W_2 + \Delta U_2 = \Delta Q_2 \quad ...(ii)$

$\Delta Q_1 + \Delta Q_2 = 0$

$\therefore (nC_p\Delta T)_1 + (nC_p\Delta T)_2 = 0$

But $n_1 = n_2 = 2$

$$\therefore \quad \frac{5}{2}R(T-700) + \frac{7}{2}R(T-400) = 0$$

Solving, we get $T = 525$ K

Now, from equations (i) and (ii), we get

$\Delta W_1 + \Delta W_2 = -\Delta U_1 - \Delta U_2$

as $\Delta Q_1 + \Delta Q_2 = 0$

$$\therefore \quad \Delta W_1 + \Delta W_2 = -[(nC_V\Delta T)_1 + (nC_V\Delta T)_2]$$

$$= -\left[2 \times \frac{3}{2}R \times (525-700) + 2 \times \frac{5}{2}R \times (525-400)\right]$$

$$= -100R$$

27. **(a)** $R_{\mathrm{I}} = R_1 + R_2 = \left(\frac{l}{KA}\right) + \left(\frac{l}{2KA}\right) = \frac{3}{2}\left(\frac{l}{KA}\right)$

$$\frac{1}{R_{\mathrm{II}}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{KA}{l} + \frac{2KA}{l}$$

or $$R_{\mathrm{II}} = \frac{l}{3KA} = \frac{R_{\mathrm{I}}}{4.5}$$

Since thermal resistance R_{II} is 4.5 times less than thermal resistance R_{I}.

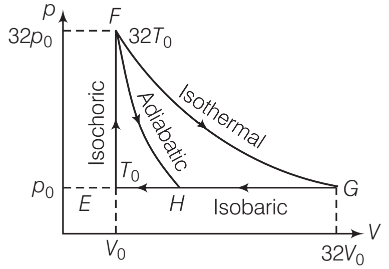
$$\therefore \quad t_{\mathrm{II}} = \frac{t_{\mathrm{I}}}{4.5} = \frac{9}{4.5}\text{ s} = 2\text{ s}$$

28. **(d)** $\rho = \frac{pM}{RT}$

$\therefore \quad \rho \propto pM$

or $$\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)\left(\frac{M_1}{M_2}\right) = \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) = \frac{8}{9}$$

29. **(c)**



In $F \to G$ work done in isothermal process is

$$nRT \ln\left(\frac{V_f}{V_i}\right) = 32\, p_0V_0 \ln\left(\frac{32V_0}{V_0}\right)$$

$$= 32\, p_0V_0 \ln 2^5$$

$$= 160p_0\, V_0 \ln 2$$

In $G \to E$,

$$\Delta W = p_0\Delta V = p_0(31V_0) = 31\, p_0V_0$$

In $G \to H$ work done is less than $31\, p_0V_0$ *i.e.*, $24\, p_0V_0$

In $F \to H$ work done is $36\, p_0V_0$

30. **(b, c, d)** $Q = mCT$

$$\frac{dQ}{dt} = mc\frac{dT}{dt}$$

$$R = \text{rate of absortion of heat} = \frac{dQ}{dt} \propto C$$

(i) in 0 – 100 K

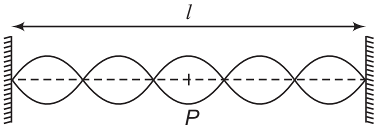
C increase, so R increases but not linearly

(ii) $\Delta Q = mC\Delta T$ as C is more in (400 K – 500 K) then (0 – 100 K) so heat is increasing

(iii) C remains constant so there no change in R from (400K – 500 K)

(iv) C is increases so R increases in range (200 K – 300K)

31. **(b,c)**



Number of nodes = 6

From the given equation, we can see that

$$k = \frac{2\pi}{\lambda} = 62.8\text{ m}^{-1}$$

$$\therefore \quad \lambda = \frac{2\pi}{62.8}\text{m} = 0.1\text{ m}$$

$$l = \frac{5\lambda}{2} = 0.25\text{ m}$$

The mid-point of the string is P, an antinode

$\therefore$ maximum displacement = 0.01 m

$$\omega = 2\pi f = 628\text{ s}^{-1}$$

$$\therefore \quad f = \frac{628}{2\pi} = 100\text{ Hz}$$

But this is fifth harmonic frequency.

$\therefore$ Fundamental frequency $f_0 = \frac{f}{5} = 20$ Hz.